Research Statement

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My research lies at the intersection of algebraic combinatorics and representation theory. The interplay between these fields is strikingly powerful: combinatorics offers concrete and intuitive structures that are easy to manipulate and ideal for gathering experimental data, while algebra offers guidance about which manipulations are natural as well as elegant methods of proof.

Canonical bases originated in the investigations of Kazhdan and Lusztig on singularities of Schubert varieties. Since their introduction in the famous paper [16] of 1979, they have played a central role at the intersection of representation theory, algebraic geometry, and combinatorics. Of particular importance for my work is their remarkable ability to connect combinatorics and representation theory. For instance, canonical bases beautifully connect the RSK correspondence with quantum Schur-Weyl duality (see [10, 5]) and give a Littlewood-Richardson rule for all types [21, 15]. Much of my research involves canonical bases, including my thesis work (§1) and my recent work on the Kronecker problem (§3).

1. Canonical bases for Garsia-Procesi modules

My thesis work [6, 4, 7] uses and develops canonical basis theory of Hecke algebras, drawing on ideas of Lusztig [22], Xi [30, 31], and Geck [13]. I apply this to understand deep combinatorial phenomena like the cyclage and catabolism operations of Lascoux and Schützenberger [20, 19], its generalizations by Shimozono and Weyman [28], and the k-atoms of Lascoux, Lapointe, and Morse [18].

The central object of my thesis work is the polynomial ring $R = \mathbb{C}[y_1, \dots, y_n]$, thought of as a graded representation of the symmetric group $\mathcal{S}_n$ with the action given by permuting variables. For each partition λ of n, the irreducible $\mathcal{S}_n$-module M_λ occurs with multiplicity one in degree $n(\lambda) = \sum_i (i-1)\lambda_i$ of R, and not at all in degree less than $n(\lambda)$. There is a unique maximal homogeneous $\mathcal{S}_n$-invariant ideal $I_\lambda \subseteq R$ not containing this copy of M_λ, and the *Garsia-Procesi module* R_λ is the corresponding quotient R/I_λ. For λ the single column 1^n, the copy of M_{1^n} in degree $\binom{n}{2}$ is the $\mathbb{C}$-span of the Vandermonde determinant, I_{1^n} is the ideal generated by symmetric polynomials without constant term, and R_{1^n} is the *ring of coinvariants*.

It has been known for some time that the characters of the Garsia-Procesi modules have a combinatorial description in terms of cyclage and catabolism, however the proof is difficult and indirect. I show that the canonical basis of the type A affine Hecke algebra directly connects this representation theory and combinatorics [4]. Specifically, I identify a subalgebra $\widehat{\mathscr{H}_n}^+$ of the extended affine Hecke algebra $\widehat{\mathscr{H}_n}$ of type A. The subalgebra $\widehat{\mathscr{H}_n}^+$ is a q-analog of the monoid algebra of $\mathcal{S}_n \ltimes \mathbb{Z}^n_{\geq 0}$ and inherits a canonical basis from that of $\widehat{\mathscr{H}_n}$. I then exhibit a cellular subquotient $\mathscr{R}_{1^n}$ of $\widehat{\mathscr{H}_n}^+$ that is a q-analog of the ring of coinvariants with left cells labeled by standard Young tableaux. I go on to prove that $\mathscr{R}_{1^n}$ has cellular quotients $\mathscr{R}_\lambda$ that are q-analogs of the Garsia-Procesi modules R_λ with left cells labeled by the λ-catabolizable tableaux.

An intriguing part of [4] are conjectures relating the canonical basis of $\widehat{\mathscr{H}_n}^+$ to other, less well-understood combinatorics. I conjecture that the k-atoms of Lascoux, Lapointe, and Morse [18] and the R-catabolizable tableaux of Shimozono and Weyman [28] have cellular counterparts in $\widehat{\mathscr{H}_n}^+$. I conjecture that $\widehat{\mathscr{H}_n}^+$ is "tiled" by dual versions of the $\mathscr{R}_\lambda$ and conjecture precisely how this would connect catabolism to the combinatorics of the cells of $\widehat{\mathscr{H}_n}$ worked out by Shi, Lusztig, and Xi.

2. Geometric complexity theory

For the last three years, my research has centered around geometric complexity theory (GCT), an approach to **P** vs. **NP** and related problems in complexity theory using algebraic geometry and representation theory. GCT originated in the 1999 paper [23], in which Mulmuley successfully applied algebraic geometry to prove a weakened version of the $\mathbf{NC} \neq \mathbf{P}$ conjecture. Geometric complexity theory has since been developed primarily by Mulmuley and Sohoni. In the last several years, it has attracted attention from researchers in algebraic geometry, complexity theory, and algebraic combinatorics including Landsberg, Weyman, Kumar, Bürgisser, and myself.

Much of algebraic combinatorics is motivated by the search for positive combinatorial formulae for quantities that are known to be nonnegative by geometry or representation theory [29]. The archetypal example of such a formula is the *Littlewood-Richardson rule*, which gives the multiplicities for decomposing a tensor product of two irreducible representations of GL_n into irreducibles.

GCT provides a new intriguing motivation for positivity problems in algebraic combinatorics. A positive combinatorial interpretation of some quantity that is already known to be nonnegative is not *only* a more elementary proof of nonnegativity, but is also at the heart of efficient computation. Computing Littlewood-Richardson coefficients is $\#\mathbf{P}$-complete (a counting analog of $\mathbf{NP}$-complete) [11]. However, deciding nonvanishing of Littlewood-Richardson coefficients can be done in polynomial time [27] by the saturation theorem of Knutson and Tao [17]. The *permanent vs. determinant problem*, which has been the main focus of GCT so far, is to show that the permanent of an $n \times n$ variable matrix cannot be expressed as the determinant of an $m \times m$ matrix with constant or single variable entries if $m = O(2^{\operatorname{polylog} n})$. Representation theoretic questions arising in the permanent vs. determinant problem, together with the Knutson-Tao result and the Flip Theorem [24], is the basis for a philosophy in GCT called the *flip* [26, 24, 25]. The flip suggests that separating $\mathbf{P}$ from $\mathbf{NP}$ will require solving a problem in representation theory in polynomial time, of the same flavor as deciding nonvanishing of Littlewood-Richardson coefficients, but much harder.

3. The Kronecker problem

The positivity problem which Mulmuley and I believe to be the current most important step for GCT is the *Kronecker problem*, which asks for a positive combinatorial formula for *Kronecker coefficients*. The Kronecker coefficient $g_{\lambda\mu\nu}$ is the multiplicity of the $GL(V) \times GL(W)$-irreducible $V_\lambda \otimes W_\mu$ in the restriction of the $GL(X)$-irreducible X_ν via the natural map $GL(V) \times GL(W) \to GL(V \otimes W)$, where $X = V \otimes W$. The bulk of my recent research has been to develop (with Mulmuley and Sohoni) an approach to this problem using quantum groups and canonical bases [2, 3, 5, 9].

In [2], we construct two quantum objects for the Kronecker problem, which we call the nonstandard quantum group and nonstandard Hecke algebra. We show that the nonstandard quantum group has a compact real form and its representations are completely reducible, that the nonstandard Hecke algebra is semisimple, and that they satisfy an analog of quantum Schur-Weyl duality. Using these nonstandard objects as a guide, we follow the approach of Adsul, Sohoni, and Subrahmanyam [1] to construct, in the case $\dim(V) = \dim(W) = 2$, a representation $\check{X}_\nu$ of the nonstandard quantum group that specializes to $\operatorname{Res}_{GL(V)\times GL(W)} X_\nu$ at $q = 1$. We then define a global crystal basis $B(\nu)$ of $\check{X}_\nu$ that solves the Kronecker problem in the case of two two-row shapes: the number of highest weight elements of $B(\nu)$ of weight (λ, μ) is the Kronecker coefficient $g_{\lambda\mu\nu}$. We go on to develop the beginnings of a graphical calculus for this basis, along the lines of the $U_q(\mathfrak{sl}_2)$ graphical calculus from [12], and use this to organize the crystal components of $B(\nu)$ into eight families. This yields the first positive formula for Kronecker coefficients in this case.

A few months ago, I happened upon an amazing computer experiment that could be a guide to a complete solution to the Kronecker problem. So far, it has led to a solution in the case of one hook shape and two arbitrary shapes [8]. The rule uses *mixed insertion*, a generalization of Schensted insertion to colored words, developed by Haiman in [14]. This is an important breakthrough since all previously known formulae for Kronecker coefficients require two of the partitions to be restricted.

References

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