Demazure crystals and the Schur positivity of Catalan functions

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Joint work with Jennifer Morse and Anna Pun

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Catalan functions

A family of symmetric functions studied by Panyushev and Chen-Haiman

- Are equal to graded Euler characteristics of vector bundles on the flag variety.
- Contain the modified Hall-Littlewood polynomials $H_\mu(\mathrm{x}; q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(\mathrm{x})$ and their parabolic generalizations.
- Contain the k-Schur functions $\{s_\mu^{(k)}(\mathrm{x}; q)\}$, which at $q = 1$ represent Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$.
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Demazure operators

- We work in the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}]$ or $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_\ell^{\pm 1}][[q]]$.
- The symmetric group $\mathcal{S}_\ell$ acts by permuting the x_i.
- $s_i \in \mathcal{S}_\ell$ denotes the simple transposition $(i \ i+1)$.

Def. The *Demazure operator* π_i is given by

$$\pi_i(f) = \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}.$$

Example.

$$\pi_1(x_1^5 x_2^2) = x_1^5 x_2^2 + x_1^4 x_2^3 + x_1^3 x_2^4 + x_1^2 x_2^5.$$

Def. For $w = s_{i_1} s_{i_2} \cdots s_{i_m} \in \mathcal{S}_\ell$ reduced, $\pi_w := \pi_{i_1} \pi_{i_2} \cdots \pi_{i_m}$.

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Schur function straightening

For $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell)$, let $s_\mu(\mathrm{x}) = s_\mu(x_1, \ldots, x_\ell)$ be the GL_ℓ character or Schur function.

Proposition (Schur function straightening)

For $\gamma \in \mathbb{Z}^\ell$,

$$\pi_{w_0}(\mathrm{x}^\gamma) = \begin{cases} \mathrm{sgn}(\gamma+\rho) s_{\mathrm{sort}(\gamma+\rho)-\rho}(\mathrm{x}) & \textit{if } \gamma+\rho \textit{ has distinct parts,} \\ 0 & \textit{otherwise,} \end{cases}$$

- $\mathrm{sort}(\beta) =$ *weakly decreasing sequence obtained by sorting* β,
- $\mathrm{sgn}(\beta) =$ *sign of the shortest permutation taking* β *to* $\mathrm{sort}(\beta)$.

Example. $\ell = 4$, $\gamma = 3125$.

$\gamma + \rho = (3,1,2,5) + (3,2,1,0) = (6,3,3,5)$ has a repeated part.

Hence $\pi_{w_0}\mathrm{x}^{3125} = 0$.

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Example. $\ell = 4$, $\gamma = 4716$.

$$\gamma + \rho = (4,7,1,6) + (3,2,1,0) = (7,9,2,6)$$

$$\mathrm{sort}(\gamma+\rho) = (9,7,6,2)$$

$$\mathrm{sort}(\gamma+\rho) - \rho = (6,5,5,2)$$

Hence $\pi_{w_0}\mathsf{x}^{4716} = s_{6552}(\mathsf{x})$.

Root ideals

- Set of positive roots $\Delta^+ := \{(i, j) \mid 1 \le i < j \le \ell\}$.
- A *root ideal* $\Psi \subseteq \Delta^+$ is an upper order ideal of positive roots.

Example. $\Psi = \{(1,3), (1,4), (1,5), (1,6), (2,5), (2,6), (3,6)\}$

		$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
				$(2,5)$	$(2,6)$
					$(3,6)$

Catalan functions

Def. (Panyushev, Chen-Haiman)

- $\Psi \subseteq \Delta^+$ is an upper order ideal of positive roots,
- $\gamma \in \mathbb{Z}^\ell$.

The *Catalan function* indexed by Ψ and γ:

$$H_\gamma^\Psi(\mathrm{x}; q) := \mathrm{poly}\left(\pi_{w_0}\Big(\prod_{(i,j)\in\Psi}(1 - qx_i/x_j)^{-1}\mathrm{x}^\gamma\Big)\right),$$

where the polynomial truncation operator poly takes s_μ to 0 if $\mu_\ell < 0$.

Example. Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a partition.

- Empty root set: $H_\mu^\varnothing(\mathrm{x}; q) = s_\mu(\mathrm{x})$.
- Full root set: $H_\mu^{\Delta^+}(\mathrm{x}; q) = H_\mu(\mathrm{x}; q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(\mathrm{x})$, the modified Hall-Littlewood polynomial.

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Example. For $\mu = 3321$, $\Psi = \{(1,3),(1,4),(2,4)\}$,

3		1, 3	1, 4
	3		2, 4
		2	
			1

$$H_\mu^\Psi(\mathrm{x};q)$$

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&H_\mu^\Psi(\mathrm{x};q)\\
&= \operatorname{poly} \pi_{w_0}\Big(\prod_{(i,j)\in\Psi}(1-qx_i/x_j)^{-1}\mathrm{x}^\mu\Big)\\
&= \operatorname{poly} \pi_{w_0}\big((1-qx_1/x_3)^{-1}(1-qx_2/x_4)^{-1}(1-qx_1/x_4)^{-1}\mathrm{x}^{3321}\big)
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&= \operatorname{poly} \pi_{w_0}\big(\mathrm{x}^{3321} + q(\mathrm{x}^{3420}+\mathrm{x}^{4311}+\mathrm{x}^{4320}) + q^2(\mathrm{x}^{4410}+\mathrm{x}^{5301}+\mathrm{x}^{5310})\\
&\quad + q^3(\mathrm{x}^{63-11}+\mathrm{x}^{5400}+\mathrm{x}^{6300}) + q^4(\mathrm{x}^{64-10}+\mathrm{x}^{73-10})\big)
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&= \operatorname{poly}\pi_{w_0}\big(\mathrm{x}^{3321}+q(\mathrm{x}^{3420}+\mathrm{x}^{4311}+\mathrm{x}^{4320})+q^2(\mathrm{x}^{4410}+\mathrm{x}^{5301}+\mathrm{x}^{5310})\\
&\quad +q^3(\mathrm{x}^{63-11}+\mathrm{x}^{5400}+\mathrm{x}^{6300})+q^4(\mathrm{x}^{64-10}+\mathrm{x}^{73-10})\big)\\
&= s_{3321}+q(s_{4320}+s_{4311})+q^2(s_{4410}+s_{5310})+q^3s_{5400}.
\end{aligned}
$$

Schur positivity and cohomology vanishing

Theorem (Lascoux-Schützenberger)

The modified Hall-Littlewood polynomials have the Schur expansion

$$H_\mu^{\Delta^+}(\mathrm{x};q) = H_\mu(\mathrm{x};q) = \sum_{U \in \mathrm{SSYT}(\mu)} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathrm{x}).$$

Def. $\Delta(\eta)$ = set of roots above the block diagonal with blocks $\eta_1, \eta_2, \ldots$.

$$\Delta(1,3,2) =$$

For partitions μ, $H_\mu^{\Delta(\eta)} = \sum_\lambda K_{\lambda\mu}^{\Delta(\eta)}(q) s_\lambda$ has been studied by many authors.

- Broer conjectured that the $H_\mu^{\Delta(\eta)}(\mathrm{x};q)$ are Schur positive and showed that this would follow from the vanishing of higher cohomology of a certain vector bundle on G/B.
- Shimozono-Weyman conjectured a combinatorial formula for the Schur expansion of $H_\mu^{\Delta(\eta)}(\mathrm{x};q)$ in terms of katabolism.

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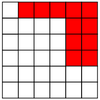
$\Delta(1,3,2) =$ 

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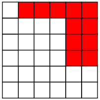
$\Delta(1,3,2) =$ 

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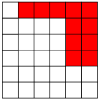
$\Delta(1,3,2) =$ 

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In many cases $H_\gamma^\Psi(x;q)$ is known or conjectured to be Schur positive.

	Higher cohomology vanishes	Combinatorial formula
$\Psi = \Delta^+, \gamma$ partition	Yes	Yes
$\Psi = \Delta(\eta)$, γ partition, constant on blocks	Yes	Yes
$\Psi = \Delta(\eta)$, γ partition	Conjectured	Conjectured
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$\Psi = \Delta^+$, $\gamma_i - \gamma_j \geq -1$ for $i < j$	Yes	Yes
$\gamma - \rho + \sum_{\beta \in \Delta^+ \setminus \Psi} \beta$ is weakly decreasing	Yes	Unknown

Broer, Lascoux-Schützenberger
Shimozono, Schilling-Warnaar, A. N. Kirillov-Schilling-Shimozono
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Tabloids

- The *diagram* of $\alpha \in \mathbb{Z}_{\geq 0}^{\ell}$ consists of a left justified array of boxes with α_i boxes in row i.
- A *tabloid* T of shape α is a filling of the diagram of α with weakly increasing rows.

Example.

1	1	1	1	4	5
2	3				
3	4				
2	2	3	4	5	

A tabloid of shape $(6, 2, 2, 0, 5)$.

Katabolism

Def. (Partial Insertion) $P_{i,\ell}(T)$ is the result of column inserting the ℓ-th row of T into rows $i, i+1, \ldots, \ell - 1$.

Example.

$T =$

1	1	1	1	4	5
2	3				
3	4				
2	2	3	4	5	

$P_{2,5}(T) =$

2	2	3	4	5

$\rightarrow$

1	1	1	1	4	5
2	3				
3	4				

Def. $\mathrm{kat}(T)$ is obtained by removing all occurrences of the smallest letter from T, then removing the first row and adding it as the new ℓ-th row.

Example.

1	1	1	1	4	4	4	6
2	2	2	5	5	5	7	
3	3	3	6	6	7		

$\xrightarrow{\mathrm{kat}}$

2	2	2	5	5	5	7
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4	4	4	6			

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Example.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&4&5 \\ \hline 2&3 \\ \cline{1-2} 3&4 \\ \cline{1-2} \\ \hline 2&2&3&4&5 \\ \hline \end{array} \qquad P_{2,5}(T) = \begin{array}{cccc} \\ \\ \\ \\ \hline \multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{4} \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|c|} \hline 1&1&1&1&4&5 \\ \hline 2&3 \\ \cline{1-2} 3&4 \\ \cline{1-2} 5 \\ \cline{1-1} \\ \end{array}$$

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$\xrightarrow{\text{kat}}$

2	2	2	5	5	5	7
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Katabolism

Def. (Partial Insertion) $P_{i,\ell}(T)$ is the result of column inserting the ℓ-th row of T into rows $i, i+1, \ldots, \ell-1$.

Example.

$T =$

1	1	1	1	4	5
2	3				
3	4				
2	2	3	4	5	

$P_{2,5}(T) =$

1	1	1	1	4	5
2	2	2	3	3	
3	4				
4	5				

Def. kat(T) is obtained by removing all occurrences of the smallest letter from T, then removing the first row and adding it as the new ℓ-th row.

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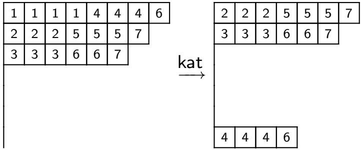

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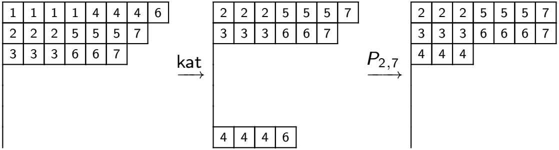

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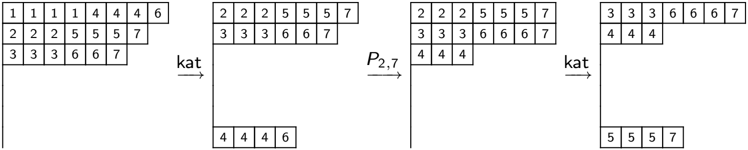

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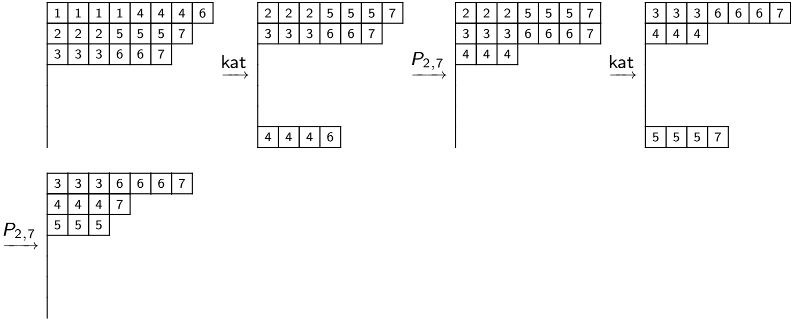

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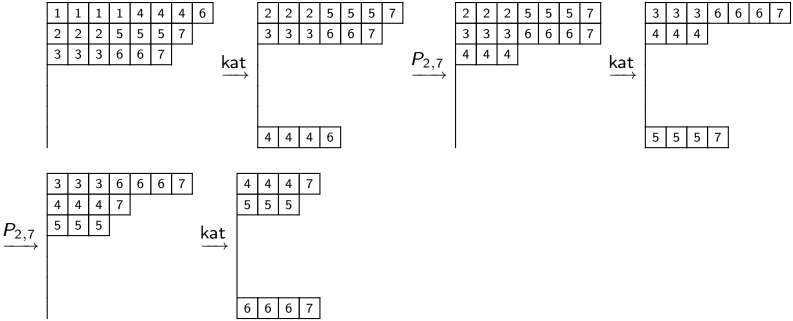

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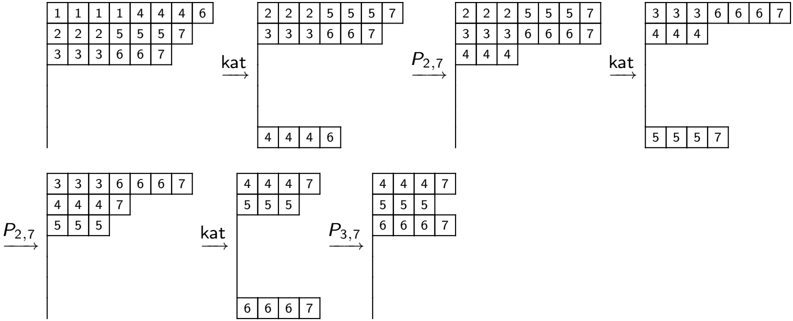

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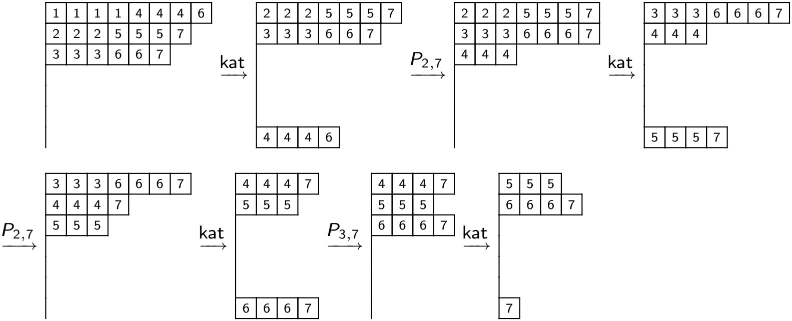

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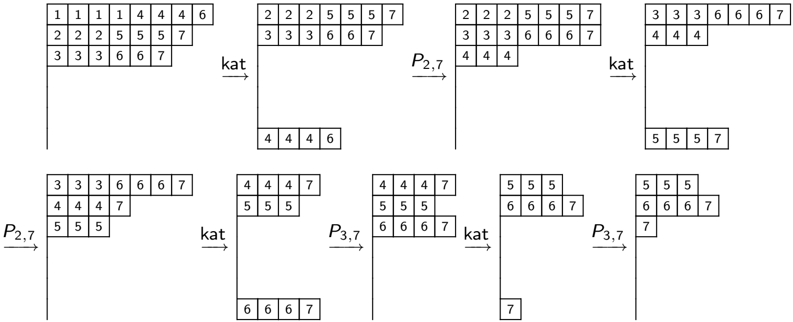

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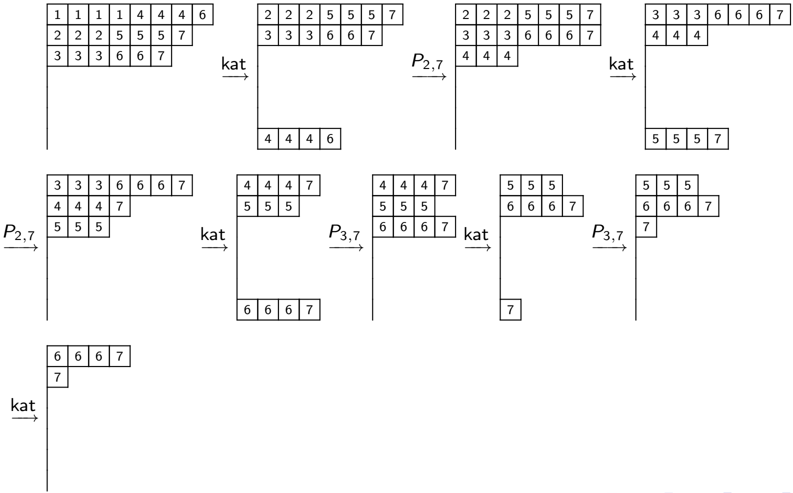

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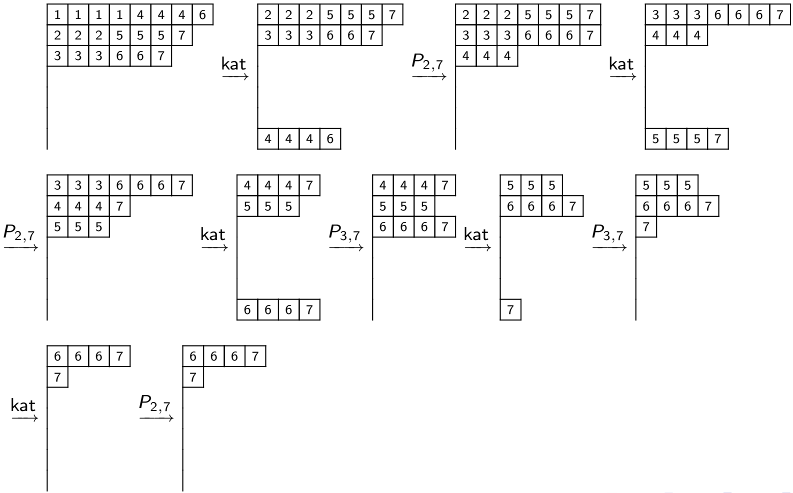

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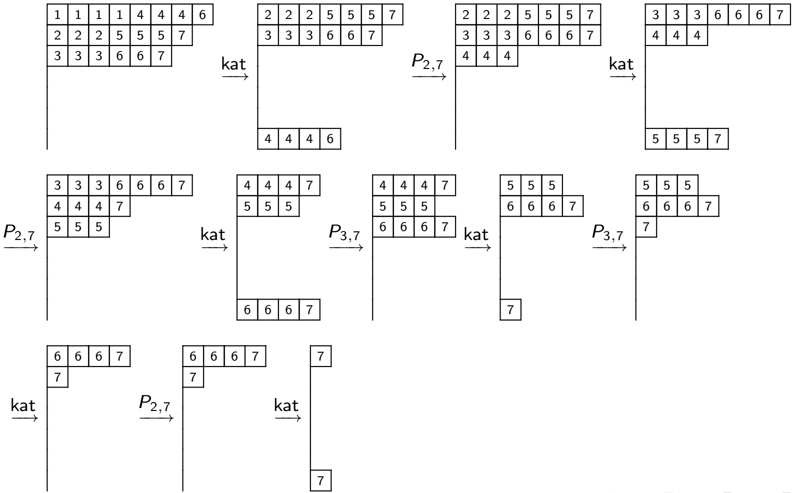

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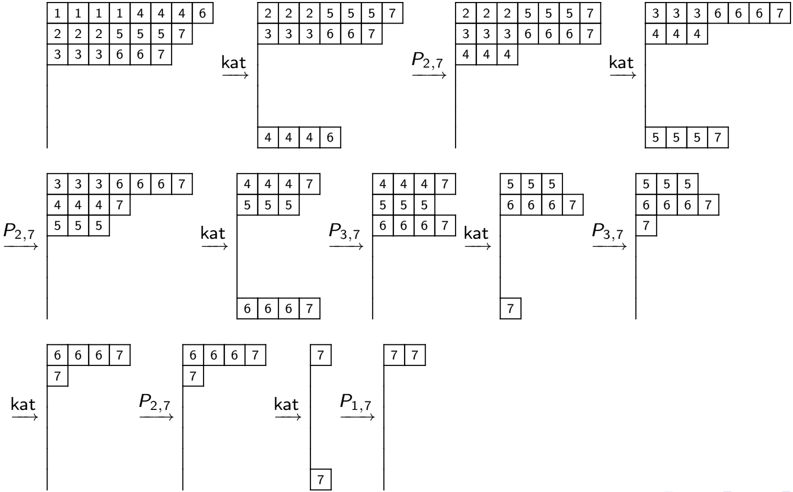

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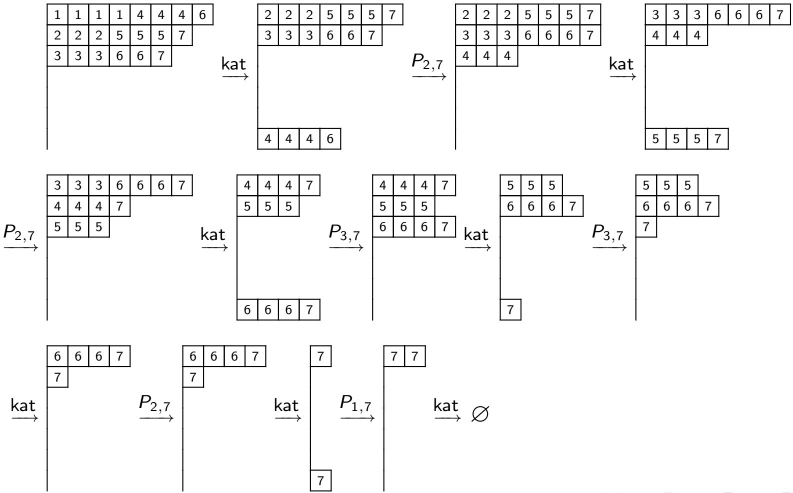

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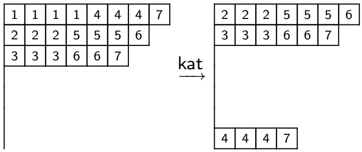

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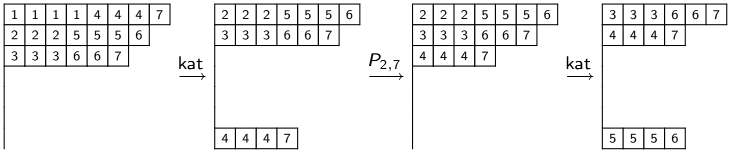

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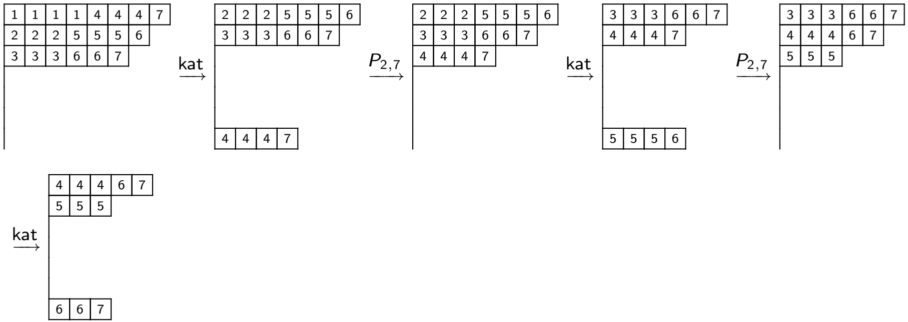

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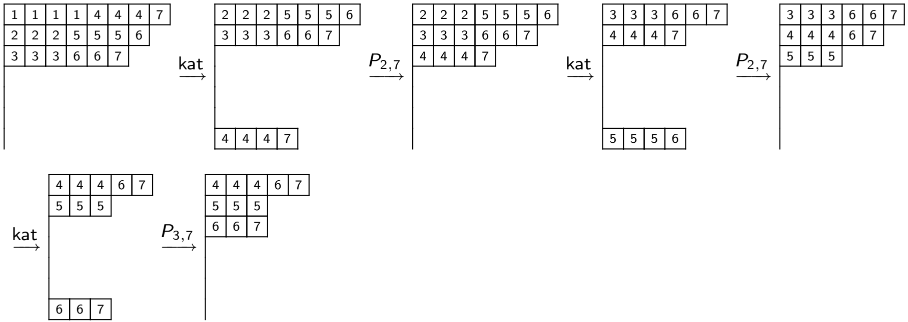

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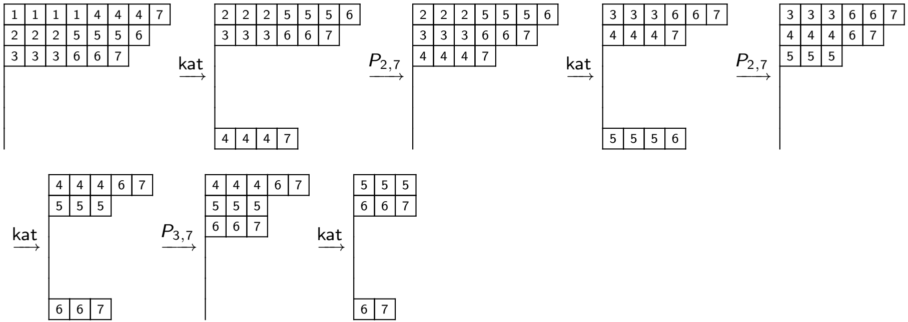

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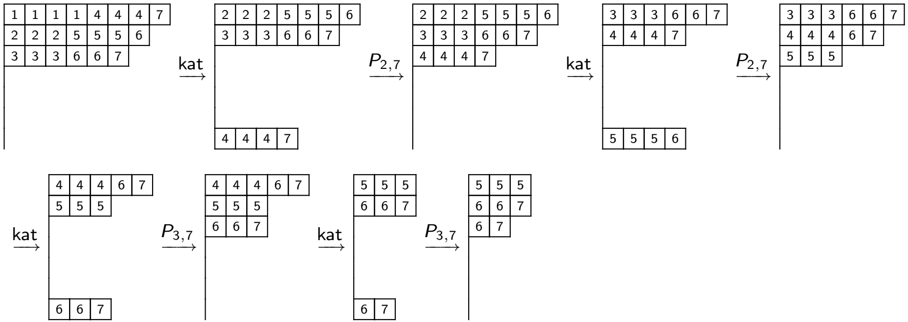

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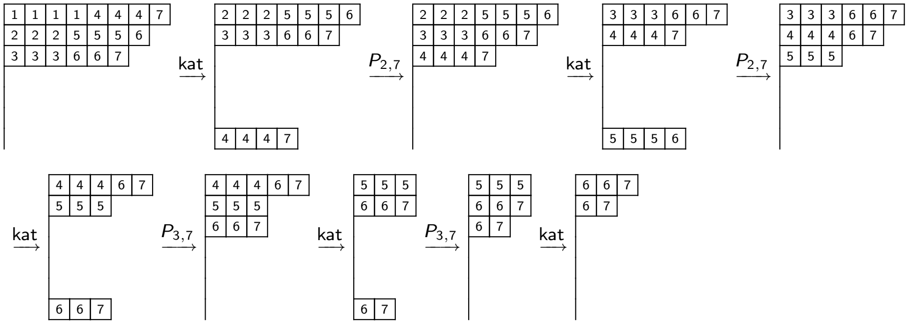

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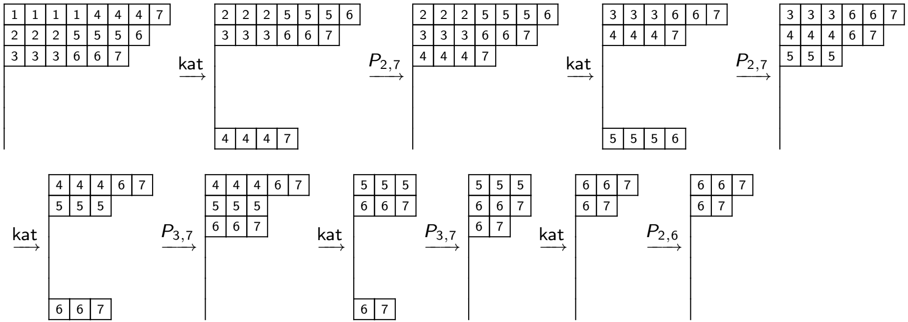

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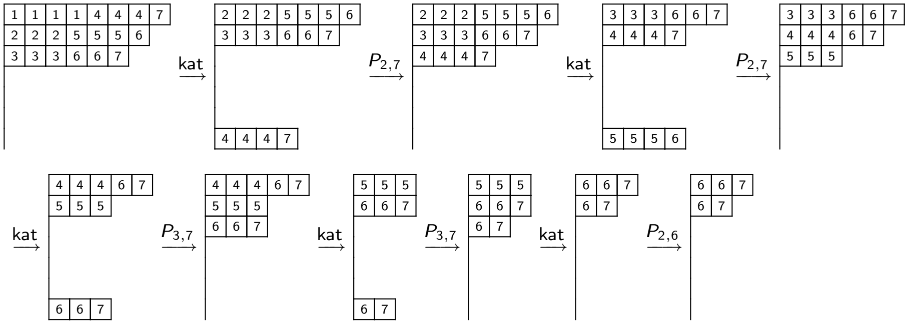
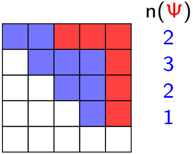


Katabolism formula for Catalan functions

Def. $n(\Psi)_i = 1 + \#$ nonroots in i-th row of Ψ.



Katabolism formula for Catalan functions

Theorem (B.-Morse-Pun)

For any root ideal Ψ *and partition* $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$, *the associated Catalan function has the following Schur positive expression:*

$$H_\mu^\Psi(\mathrm{x}; q) \;=\; \sum_{\substack{U \in \mathrm{SSYT}(\mu) \\ U \text{ is } \mathrm{n}(\Psi)\text{-katabolizable}}} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathrm{x})\,.$$

where $\mathrm{SSYT}(\mu)$ *is the set of semistandard tableaux of content* μ.

When $\Psi = \Delta^+$, we recover

Theorem (Lascoux-Schützenberger)

$$H_\mu(\mathrm{x}; q) = \sum_{U \in \mathrm{SSYT}(\mu)} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathrm{x}).$$

In the special case Ψ is parabolic, this resolves the Shimozono-Weyman katabolism conjecture.

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Rotation Theorem

Let $\Phi\colon \mathbb{Z}[q][\mathrm{x}] \to \mathbb{Z}[q][\mathrm{x}]$, $\quad \Phi(f(x_1,\ldots,x_\ell)) = f(x_2,x_3,\ldots,x_\ell,qx_1)$.

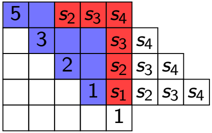
Theorem (B.-Morse-Pun)

For any root ideal Ψ *and* $\gamma \in \mathbb{Z}^\ell_{\geq 0}$,

$$H^{\Psi}_{\gamma}(\mathrm{x};q) = \pi_{w_0} x_1^{\gamma_1}\,\Phi\,\pi_{v_1} x_1^{\gamma_2}\,\Phi\,\pi_{v_2} x_1^{\gamma_3} \cdots \Phi\,\pi_{v_{\ell-1}} x_1^{\gamma_\ell},$$

where $v_i = s_{\ell-1}s_{\ell-2}\cdots s_{\mathrm{n}(\Psi)_i}$.

Example.



$$H^{\Psi}_{53211}(\mathrm{x};q) = \pi_{w_0} x_1^5\,\Phi\,\pi_{s_4s_3s_2} x_1^3\,\Phi\,\pi_{s_4s_3} x_1^2\,\Phi\,\pi_{s_4s_3s_2} x_1\,\Phi\,\pi_{s_4s_3s_2s_1} x_1.$$

Is there a geometric or representation theoretic explanation of the rotation theorem?

$U_q(\widehat{\mathfrak{sl}}_\ell)$-crystals

Data for type $A^{(1)}_{\ell-1}$:

- Dynkin nodes $I = \mathbb{Z}/\ell\mathbb{Z} = \{0, 1, \ldots, \ell - 1\}$.
- Fundamental weights $\{\Lambda_i \mid i \in I\}$.
- Weight lattice $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{\delta}{2\ell}$.
- Dominant weights $P^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\frac{\delta}{2\ell}$.
- Dynkin diagram automorphism $\tau\colon I \to I$, $i \mapsto i + 1$.
- $\tau\colon P \to P$, determined by $\tau(\Lambda_i) = \Lambda_{i+1}$ for $i \in I$ and $\tau(\delta) = \delta$,
- Extended affine symmetric group $\widetilde{S}_\ell$ generated by τ and $s_1, \ldots, s_{\ell-1}$.

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$U_q(\widehat{\mathfrak{sl}}_\ell)$-crystals

Def. A *$U_q(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal* is a set B equipped with

- a *weight function* $\mathrm{wt}\colon B \to P$, and
- *crystal operators* $\tilde{e}_i, \tilde{f}_i\colon B \to B \sqcup \{0\}$

satisfying

- $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ whenever $\tilde{e}_i b \neq 0$,
- $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ whenever $\tilde{f}_i b \neq 0$,
- $\varepsilon_i(b) := \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\} < \infty$,
- $\phi_i(b) := \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\} < \infty$,
- $\langle \alpha_i^\vee, \mathrm{wt}(b)\rangle = \phi_i(b) - \varepsilon_i(b)$,
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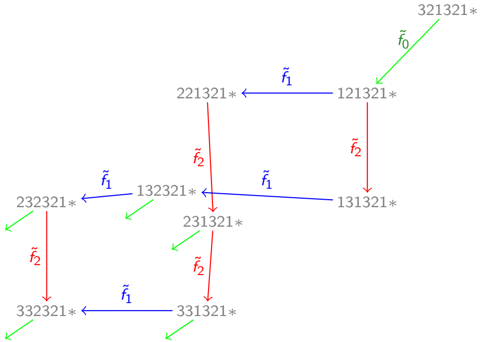
Demazure crystals

- For $\Lambda \in P^+$, $B(\Lambda) :=$ highest weight $U_q(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystal.
- $u_\Lambda \in B(\Lambda)$ denotes the highest weight element.
- For $S \subset B$ and $i \in I$, $F_i S := \{\tilde{f}_i^m b \mid b \in S, m \geq 0\} \setminus \{0\} \subset B$.

Def. Given $\Lambda \in P^+$ and a reduced expression $w = s_{i_1} \cdots s_{i_m}$ with $i_j \in I$, the associated *$U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystal* is $B_w(\Lambda) = F_{i_1} \cdots F_{i_m}\{u_\Lambda\}$.

Example. In $B(\Lambda_0)$, with $u_{\Lambda_0} = 321321321\cdots$ (abbreviate $321321*$)



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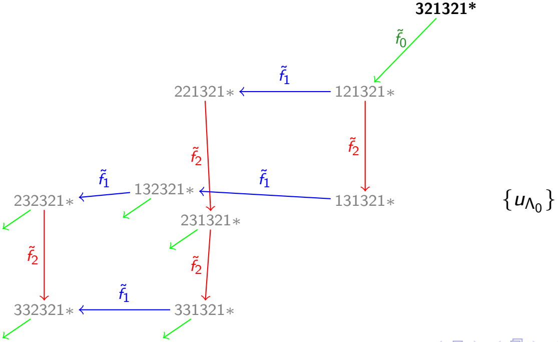

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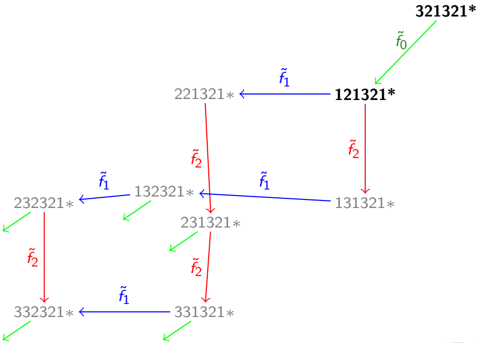

$F_0\{u_{\Lambda_0}\}$

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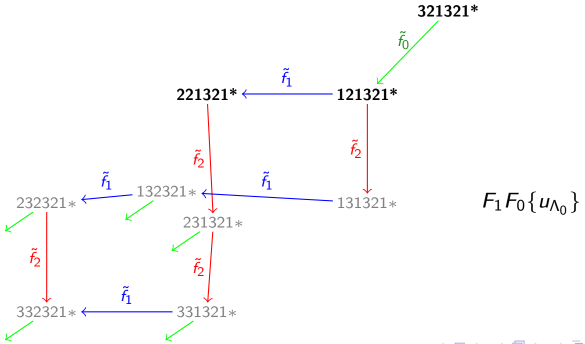

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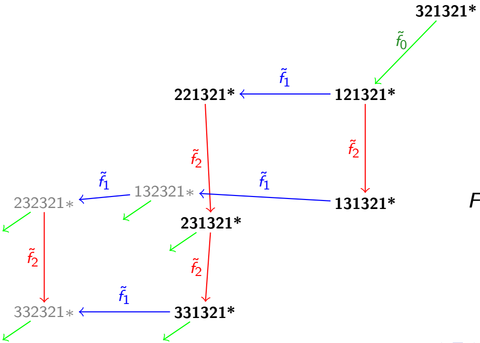

$F_2 F_1 F_0\{u_{\Lambda_0}\}$

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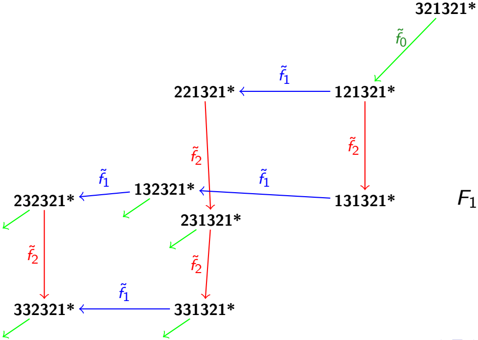

$F_1 F_2 F_1 F_0\{u_{\Lambda_0}\}$

Demazure crystals

Def. $F_\tau : B(\Lambda) \to B(\tau(\Lambda))$ is the unique bijection which takes i-edges to $i+1$-edges.

For any $w \in \widetilde{\mathcal{S}}_\ell$, write $w = \tau^j v$ with v in the affine symmetric group.

$$F_w\{u_\Lambda\} := F_\tau^j F_v\{u_\Lambda\} = B_{\tau^j v \tau^{-j}}(\tau^j(\Lambda)) \subset B(\tau^j(\Lambda)).$$

Combinatorial Excellent Filtration Theorem

Theorem (Joseph, Lakshmibai-Littelmann-Magyar)

For any $\Lambda^1, \Lambda^2 \in P^+$ and $w \in \widetilde{\mathcal{S}}_\ell$,

$$B_w(\Lambda^2) \otimes u_{\Lambda^1} \subset B(\Lambda^2) \otimes B(\Lambda^1)$$

is isomorphic to a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals.

Def. A $U_q(\widehat{\mathfrak{sl}}_\ell)$-*generalized Demazure crystal* is a subset of a tensor product of highest weight crystals of the form $F_{w_1}(F_{w_2}(\cdots F_{w_{p-1}}(F_{w_p}\{u_{\Lambda^p}\} \otimes u_{\Lambda^{p-1}}) \cdots \otimes u_{\Lambda^2}) \otimes u_{\Lambda^1})$ for $\Lambda^1, \ldots, \Lambda^p \in P^+$ and $w_1, \ldots, w_p \in \widetilde{\mathcal{S}}_\ell$.

Corollary (Lakshmibai-Littelmann-Magyar, Naoi)

Any $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystal is isomorphic to a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals.

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Any $U_q(\widehat{\mathfrak{sl}}_\ell)$-generalized Demazure crystal is isomorphic to a disjoint union of $U_q(\widehat{\mathfrak{sl}}_\ell)$-Demazure crystals.

Generalized Demazure crystals

Def. Distinguish a subclass of generalized Demazure crystals:

- $\mathbf{w} = (w_1, w_2, \ldots, w_p) \in (\mathcal{S}_\ell)^p$,
- $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$,
- $\mu^i = \mu_i - \mu_{i+1}$, with $\mu_{p+1} := 0$.

$$\mathcal{G}_\mu^{\mathbf{w}} := F_{w_1}(F_{\tau w_2}(\cdots F_{\tau w_{p-1}}(F_{\tau w_p}\{u_{\mu^p\Lambda_1}\} \otimes u_{\mu^{p-1}\Lambda_1}) \cdots \otimes u_{\mu^2\Lambda_1}) \otimes u_{\mu^1\Lambda_1})$$
$$\subset B(\mu^p\Lambda_p) \otimes \cdots \otimes B(\mu^1\Lambda_1)\,.$$

Catalan functions are generalized Demazure characters

Theorem (B.-Morse-Pun)

Catalan functions of partition weight are characters of generalized Demazure crystals.

Catalan functions are generalized Demazure characters

- $\mathbb{Z}[P]$ = group ring of P with $\mathbb{Z}$-basis $\{e^{\Lambda}\}_{\Lambda \in P}$.
- $\mathrm{char}(\mathcal{G}) := \sum_{g \in \mathcal{G}} e^{\mathrm{wt}(g)}$.

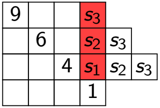
Theorem (B.-Morse-Pun)

For a root ideal Ψ and partition μ,

$$\zeta(H_\mu^\Psi) = e^{-\mu_1 \Lambda_0 + n_\ell(\mu)\delta}\, \mathrm{char}\Big(\mathcal{G}_\mu^{(w_0, v_1, \ldots, v_{\ell-1})}\Big),$$

where $v_i = s_{\ell-1} s_{\ell-2} \cdots s_{\mathrm{n}(\Psi)_i}$.

$$\zeta H^{\Psi}_{9641}(x;q) = \zeta \pi_{w_0} x_1^9 \Phi \pi_{s_3} x_1^6 \Phi \pi_{s_3 s_2} x_1^4 \Phi \pi_{s_3 s_2 s_1} x_1 \sim$$
$$\mathrm{char}\, F_{w_0}\Big(F_{\tau s_3}\Big(F_{\tau s_3 s_2}\Big(F_{\tau s_3 s_2 s_1} u_{\Lambda_1} \otimes u_{3\Lambda_1}\Big) \otimes u_{2\Lambda_1}\Big) \otimes u_{3\Lambda_1}\Big)$$



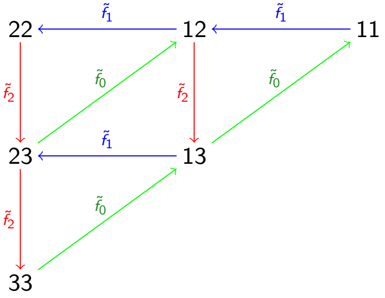
Fine print:

- $\zeta \colon \mathbb{Z}[q][x] \to \mathbb{Z}[P]$, $\quad x_i \mapsto e^{\Lambda_i - \Lambda_{i-1} + \frac{\ell+1-2i}{2\ell}\delta}$, $q \mapsto e^{-\delta}$.
- $n_\ell(\mu) = \frac{|\mu|(\ell-1)}{2\ell} - \frac{1}{\ell}\sum_{i=1}^{\ell}(i-1)\mu_i$.

Kirillov-Reshetikhin crystals

- $U'_q(\widehat{\mathfrak{sl}}_\ell) \subset U_q(\widehat{\mathfrak{sl}}_\ell)$ subalgebra.
- $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{\delta}{2\ell}$.
- $\mathrm{cl}\colon P \to P_{\mathrm{cl}} = P/\mathbb{Z}\frac{\delta}{2\ell}$ the canonical projection.
- $U'_q(\widehat{\mathfrak{sl}}_\ell)$-seminormal crystals have weight function $\mathrm{wt}\colon B \to P_{\mathrm{cl}}$.

$B^{1,s}$ = single row KR crystal, labeled by weakly increasing words of length s in the alphabet $1, 2, \ldots, \ell$.

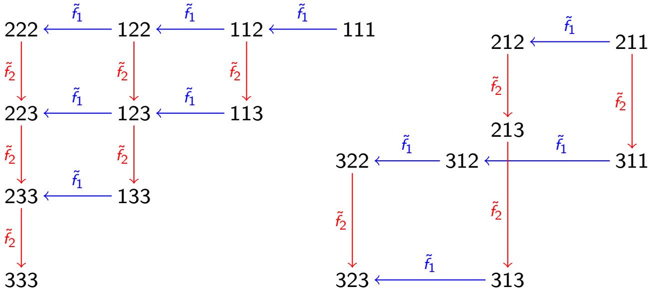


For $\ell = 3$, the single row KR crystal $B^{1,2}$.

Tensor products of KR crystals

For $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$, set

$$\mathcal{B}_\mu = B^{1,\mu_p} \otimes \cdots \otimes B^{1,\mu_1}$$



For $\ell = 3$, the tensor product of KR crystals $\mathcal{B}_{21} = B^{1,1} \otimes B^{1,2}$, restricted to $\mathfrak{sl}_\ell$.

DARK crystals

Def. $F_\tau \colon B^{1,s} \to B^{1,s}$ denotes the bijection given by adding 1 to all letters (mod ℓ) and sorting.

Example. For $\ell = 3$, $F_\tau(111233333) = 111112223$.

Def. The **K**irillov-**R**eshetikhin **A**ffine **D**emazure crystal associated to $\mu = (\mu_1 \geq \cdots \geq \mu_p \geq 0)$ and $\mathbf{w} = (w_1, \ldots, w_p) \in (\mathcal{S}_\ell)^p$ is

$$\mathcal{B}_\mu^{\mathbf{w}} := F_{w_1}\big(F_{\tau w_2}\big(\cdots F_{\tau w_{p-1}}\big(F_{\tau w_p}\{\mathrm{b}_{\mu_p}\} \otimes \mathrm{b}_{\mu_{p-1}}\big)\cdots \otimes \mathrm{b}_{\mu_2}\big) \otimes \mathrm{b}_{\mu_1}\big) \subset \mathcal{B}_\mu,$$

where $\mathrm{b}_s \in B^{1,s}$ is the element labeled by the word 1^s.

DARK crystals

$$F_{\mathrm{s}_2\mathrm{s}_1}(\mathrm{b}_1)$$

DARK crystals

$$F_{\tau s_2 s_1}(b_1)$$

DARK crystals

21

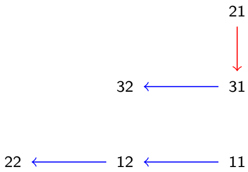
↓

31

11

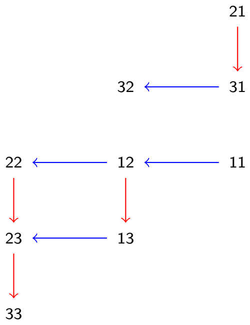
$F_{\tau s_2 s_1}(b_1) \otimes b_1$

DARK crystals



$$F_{s_1}(F_{\tau s_2 s_1}(b_1) \otimes b_1)$$

DARK crystals



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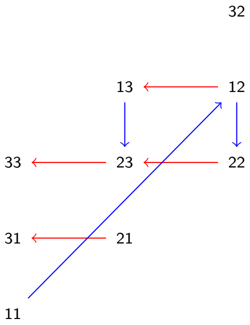

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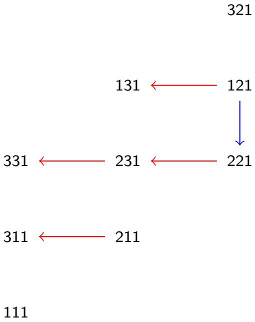

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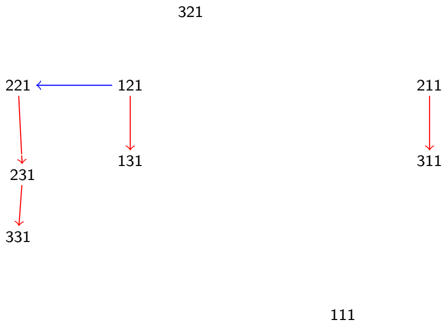

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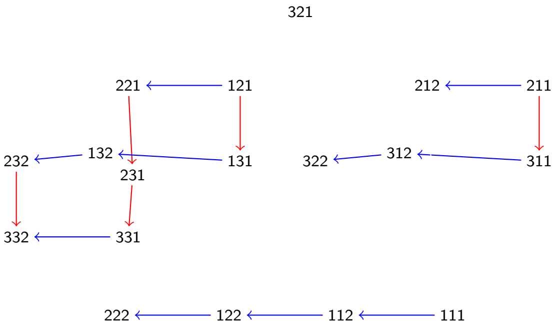

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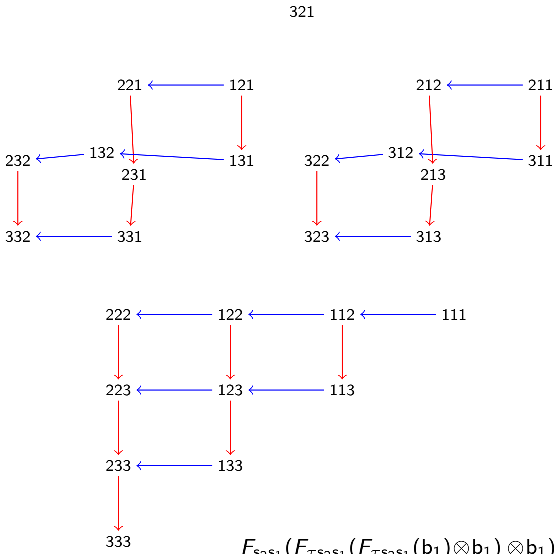
DARK crystals



$$F_{s_2 s_1}(F_{\tau s_2 s_1}(F_{\tau s_2 s_1}(b_1) \otimes b_1) \otimes b_1)$$

Naoi's Theorem

Theorem (Naoi)

There is a map

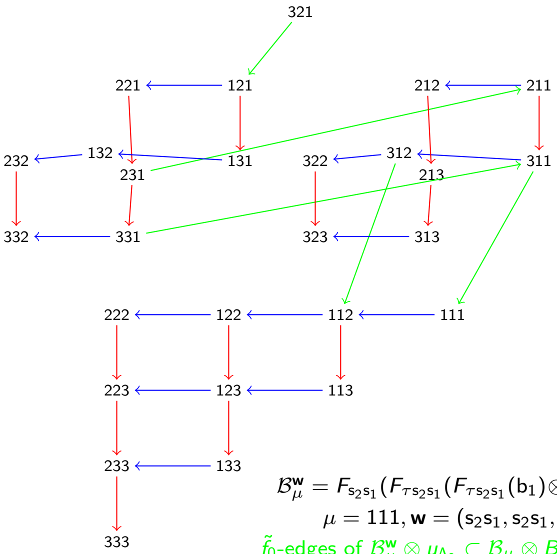
$$\Theta_\mu : \mathcal{B}_\mu \otimes B(\mu_1\Lambda_0) \hookrightarrow B(\mu^p\Lambda_p) \otimes \cdots \otimes B(\mu^1\Lambda_1),$$

which is an isomorphism from the domain onto a disjoint union of connected components of the codomain. And under this map,

$$\Theta_\mu(\mathcal{B}_\mu^{\mathrm{w}} \otimes u_{\mu_1\Lambda_0}) = \mathcal{G}_\mu^{\mathrm{w}}.$$

This builds off of and generalizes results of Kang-Kashiwara-Misra-Miwa -Nakashima-Nakayashiki, Fourier-Littelmann, Fourier-Schilling-Shimozono, and Schilling-Tingley.

DARK crystals



$$\mathcal{B}_\mu^{\mathbf{w}} = F_{s_2 s_1}(F_{\tau s_2 s_1}(F_{\tau s_2 s_1}(b_1) \otimes b_1) \otimes b_1)$$

$$\mu = 111, \mathbf{w} = (s_2 s_1, s_2 s_1, s_2 s_1)$$

$\tilde{f}_0$-edges of $\mathcal{B}_\mu^{\mathbf{w}} \otimes u_{\Lambda_0} \subset \mathcal{B}_\mu \otimes B(\Lambda_0)$ shown

DARK crystals to katabolism

- $\mathrm{Tabloids}_\ell(\mu)$ = set of tabloids with ℓ rows of content μ.
- Bijection inv: $\mathcal{B}_\mu \to \mathrm{Tabloids}_\ell(\mu)$.

Example. $b \in \mathcal{B}_{554} \quad \mapsto \quad T \in \mathrm{Tabloids}_4(554)$

$$b = \begin{pmatrix} 3333\,22222\,11111 \\ 2234\,13334\,11222 \end{pmatrix} \xrightarrow{\text{inv}} \begin{pmatrix} 44\,3333\,22222\,111 \\ 23\,2223\,11133\,112 \end{pmatrix} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 \\ \hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 & 3 \\ \hline 2 & 3 \\ \hline \end{array} = T$$

Theorem (B.-Morse-Pun)

For a partition μ and root ideal Ψ, the map inv *gives a bijection*

$$\mathcal{B}_\mu^{(w_0, v_1, \ldots, v_{\ell-1})} \xrightarrow{\text{inv}} \{T \in \mathrm{Tabloids}_\ell(\mu) \mid P(T) \text{ is } \mathrm{n}(\Psi)\text{-katabolizable}\}$$

which takes content to shape, where $v_i = s_{\ell-1} s_{\ell-2} \cdots s_{\mathrm{n}(\Psi)_i}$.

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Catalan functions to crystals to katabolism

Corollary

$$\sum_{b \in \mathcal{B}_\mu^{(w_0, v_1, \dots, v_{\ell-1})}} q^{\mathrm{charge}(\mathrm{inv}(b))} \mathrm{x}^{\mathrm{content}(b)} = \sum_{\substack{U \in \mathrm{SSYT}_\ell(\mu) \\ U \text{ is } \mathrm{n}(\Psi)\text{-katabolizable}}} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathrm{x}).$$

Combining this with Naoi's theorem and the formula for Catalan functions as generalized Demazure characters yields

Theorem (B.-Morse-Pun)

For any root ideal Ψ *and partition* $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$, *the associated Catalan function has the following Schur positive expression:*

$$H_\mu^\Psi(\mathrm{x}; q) = \sum_{\substack{U \in \mathrm{SSYT}(\mu) \\ U \text{ is } \mathrm{n}(\Psi)\text{-katabolizable}}} q^{\mathrm{charge}(U)} s_{\mathrm{shape}(U)}(\mathrm{x}) .$$

where $\mathrm{SSYT}(\mu)$ *is the set of semistandard tableaux of content* μ *and* $\mathrm{n}(\Psi)_i = 1 + \#$ *nonroots in* i*-th row of* Ψ.