Catalan functions and k-Schur positivity

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joint work with Jennifer Morse, Anna Pun, and Dan Summers

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Macdonald positivity conjecture

Theorem (Haiman)

The modified Macdonald polynomials are Schur positive:

$$H_\mu(\mathbf{x};q,t) = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda(\mathbf{x}) \qquad \textit{for } K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t].$$

$$H_{22}(\mathbf{x};q,t) = t^2 s_{(4)} + (t+qt+qt^2)s_{(3,1)} + (1+q^2t^2)s_{(2,2)} + (q+qt+q^2t)s_{(2,1,1)} + q^2 s_{(1,1,1,1)}$$

$$H_{22}(\mathbf{x};0,t) = t^2 s_{(4)} + t s_{(3,1)} + s_{(2,2)}$$

Theorem (Lascoux-Schützenberger)

The modified Hall-Littlewood polynomials have the Schur expansion

$$H_\mu(\mathbf{x};t) = H_\mu(\mathbf{x};0,t) = \sum_T t^{charge(T)} s_{\mathrm{shape}(T)}(\mathbf{x}),$$

the sum over semistandard Young tableaux T of content μ.

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Strengthened Macdonald positivity conjecture

Conjecture (Lapointe-Lascoux-Morse)

The k-Schur functions $\{A_\lambda^{(k)}(\mathbf{x};t)\}_{\lambda_1 \le k}$

- *form a basis for $\Lambda^k = \mathrm{span}_{\mathbb{Q}(q,t)}\{H_\mu(\mathbf{x};q,t)\}_{\mu_1\le k}$,*
- *are Schur positive,*
- *expansion of $H_\mu(\mathbf{x};q,t) \in \Lambda^k$ in this basis has coefficients in $\mathbb{N}[q,t]$.*

$$H_{1^4} = t^4\left(s_{(2,2)} + t s_{(3,1)} + t^2 s_{(4)}\right) + (t^2+t^3)\left(s_{(2,1,1)} + t s_{(3,1)}\right) + \left(s_{(1,1,1,1)} + t s_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

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Conjecture (Lapointe-Lascoux-Morse)

The $k+1$*-Schur expansion of a k-Schur function has coefficients in* $\mathbb{N}[t]$.

Conjecturally equivalent definitions of k-Schurs

	basis	Schur positive	branching	$H_\mu(\mathbf{x}; q, t)$ are k-Schur positive
[1998:Lapointe,Lascoux,Morse] Young tableaux and katabolism		✓		
[2001:Lapointe,Morse] Jing operators / k-split polynomials	✓			
[2006:Lam,Lapointe,Morse,Shimozono] Bruhat order on type A affine Weyl group / strong tableaux				
[2008:Chen,Haiman] Catalan functions				
[2014:Dalal,Morse and Lapointe,Pinto] Inverting affine Kostka matrix	✓			✓ $(q = 0)$
[2004:Lapointe,Morse] Weak tableaux $(t = 1)$	✓	✓	✓	✓ $(q = 0)$
[2005:Lam] Schubert classes in $H_*(\mathrm{Gr})$ $(t = 1)$	✓	✓	✓	✓ $(q = 0)$

Overview

- Part I: introduction to the k-Schur and Catalan functions.
- Part II: properties of k-Schur functions including branching, Schur positivity, and shift invariance.
- Part III: k-Schur expansions of several families of symmetric functions and relation to Gromov-Witten invariants.

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Catalan functions

A family of symmetric functions studied by Panyushev and Chen-Haiman

- Contain the modified Hall-Littlewood polynomials $H_\mu(\mathbf{x}; t)$ and their parabolic generalizations.
- Can be defined in terms of Demazure operators or raising operators on Schur functions.
- Are equal to GL_ℓ-equivariant Euler characteristics of vector bundles on the flag variety.

Schur function straightening

We work in the ring $\Lambda = \mathbb{Q}(q,t)[h_1, h_2, \dots]$ of symmetric functions in infinitely many variables $\mathbf{x} = (x_1, x_2, \dots)$.

Schur functions may be defined for any $\gamma \in \mathbb{Z}^\ell$ by

$$s_\gamma = s_\gamma(\mathbf{x}) = \det(h_{\gamma_i + j - i}(\mathbf{x}))_{1 \le i,j \le \ell} \in \Lambda.$$

Proposition (Schur function straightening)

$$s_\gamma(\mathbf{x}) = \begin{cases} \mathrm{sgn}(\gamma+\rho) s_{\mathrm{sort}(\gamma+\rho)-\rho}(\mathbf{x}) & \textit{if } \gamma+\rho \textit{ has distinct nonnegative parts,} \\ 0 & \textit{otherwise,} \end{cases}$$

- $\mathrm{sort}(\beta)$ = *weakly decreasing sequence obtained by sorting* β,
- $\mathrm{sgn}(\beta)$ = *sign of the shortest permutation taking* β *to* $\mathrm{sort}(\beta)$.

Example. $\ell = 4$, $\gamma = 3125$.

$\gamma + \rho = (3,1,2,5) + (3,2,1,0) = (6,3,3,5)$ has a repeated part.

Hence $s_{3125}(\mathbf{x}) = 0$.

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Example. $\ell = 4$, $\gamma = 4716$.

$$\gamma + \rho = (4,7,1,6) + (3,2,1,0) = (7,9,2,6)$$
$$\operatorname{sort}(\gamma+\rho) = (9,7,6,2)$$
$$\operatorname{sort}(\gamma+\rho) - \rho = (6,5,5,2)$$

Hence $s_{4716}(\mathbf{x}) = s_{6552}(\mathbf{x})$.

Root ideals

- Set of positive roots $\Delta^+ := \{(i,j) \mid 1 \leq i < j \leq \ell\}$.
- A *root ideal* $\Psi \subseteq \Delta^+$ is an upper order ideal of positive roots.

Example. $\Psi = \{(1,3),(1,4),(1,5),(1,6),(2,5),(2,6),(3,6)\}$

		(1, 3)	(1, 4)	(1, 5)	(1, 6)
				(2, 5)	(2, 6)
					(3, 6)

Def. (Panyushev, Chen-Haiman)

- $\Psi \subseteq \Delta^+$ is an upper order ideal of positive roots,
- $\gamma \in \mathbb{Z}^\ell$.

The *Catalan function* indexed by Ψ and γ:

$$H_\gamma^\Psi(\mathbf{x};t) := \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\gamma(\mathbf{x})$$

where the raising operator R_{ij} acts by $R_{ij}(s_\gamma(\mathbf{x})) = s_{\gamma+\epsilon_i-\epsilon_j}(\mathbf{x})$.

Example. Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a partition.

- Empty root set: $H_\mu^\varnothing(\mathbf{x};t) = s_\mu(\mathbf{x})$.
- Full root set: $H_\mu^{\Delta^+}(\mathbf{x};t) = H_\mu(\mathbf{x};t)$, the modified Hall-Littlewood polynomial.

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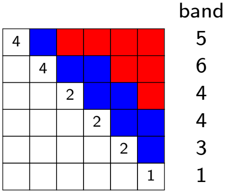
k-Schur Catalan functions

Def. $\text{band}(\Psi, \mu)_i := \mu_i + \#$ of roots in row i of $\Delta^+ \setminus \Psi$.

Example.



Def. For μ a k-bounded partition of length $\leq \ell$, define the root ideal

$$\Delta^k(\mu) = \{(i,j) \in \Delta^+ \mid k - \mu_i + i < j\}$$

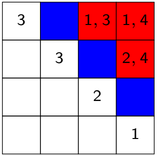
"the root ideal with band $= k$"

and the *k-Schur Catalan function*

$$\mathfrak{s}_\mu^{(k)}(\mathbf{x};t) := H_\mu^{\Delta^k(\mu)} = \prod_{i=1}^{\ell} \prod_{j=k+1-\mu_i+i}^{\ell} \left(1 - tR_{ij}\right)^{-1} s_\mu(\mathbf{x})\,.$$

Examples of Catalan functions

Example. $k = 4$, $\mu = 3321$. Then $\Delta^k(\mu) = \{(1,3),(1,4),(2,4)\}$.



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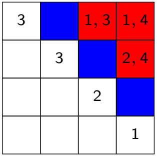

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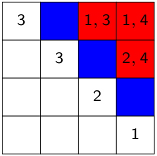

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$$= s_{3321} + t(s_{3420} + s_{4311} + s_{4320}) + t^2(s_{4410} + s_{5301} + s_{5310})$$

$$+ t^3(s_{63-11} + s_{5400} + s_{6300}) + t^4(s_{64-10} + s_{73-10})$$

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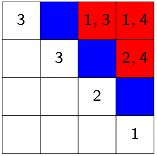

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\end{aligned}
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k-bounded partitions and $k+1$-cores

Def. A *k-bounded partition* is a partition with parts of size $\leq k$.

Def. A *$k+1$-core* is a partition whose diagram has no box with hook length $k+1$.

Proposition. There is a bijection $\kappa \mapsto \mathfrak{p}(\kappa)$ from $k+1$-cores to k-bounded partitions.

Example. $k = 4$.

κ $\qquad$ $\mathfrak{p}(\kappa)$

Def. The k-skew diagram of a $k+1$-core κ is the skew shape obtained by removing boxes of hook length $> k$.

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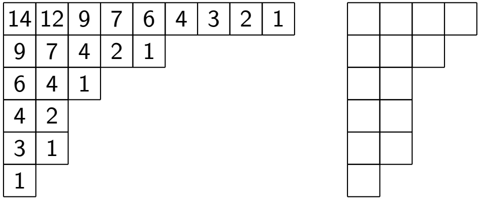

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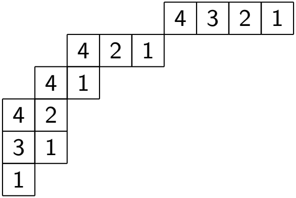
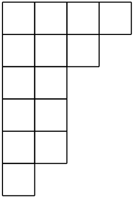


$k\text{-skew}(\kappa)$



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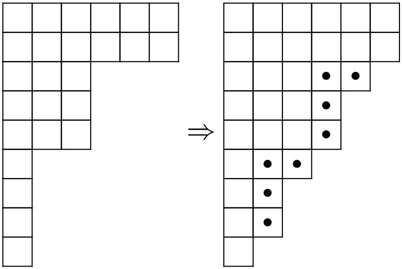
Strong covers

Def. An inclusion $\tau \subset \kappa$ of $k+1$-cores is a *strong cover*, denoted $\tau \Rightarrow \kappa$, if $|\mathfrak{p}(\tau)| + 1 = |\mathfrak{p}(\kappa)|$.

Example.

Strong cover with $k = 4$:



corresponding k-skew diagrams:

$\mathfrak{p}(\tau) = 332221111$ $\quad$ $\mathfrak{p}(\kappa) = 222222221$

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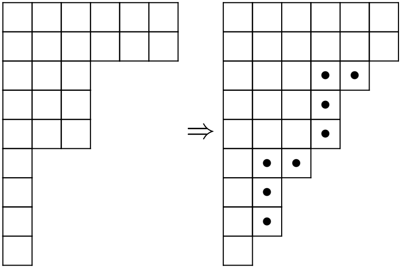
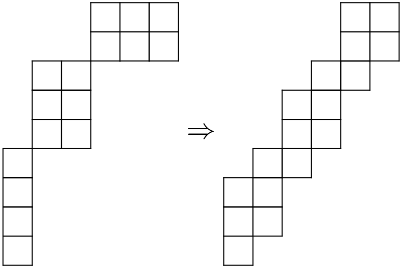
Example.

Strong cover with $k = 4$:



corresponding k-skew diagrams:

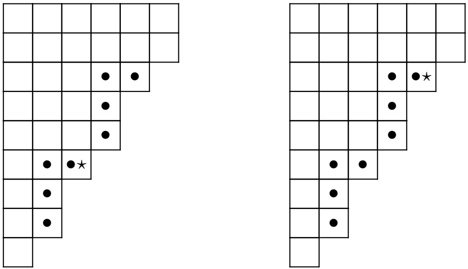


$\mathfrak{p}(\tau) = 332221111$ $\quad \mathfrak{p}(\kappa) = 222222221$

Strong marked covers

Def. A *strong marked cover* $\tau \overset{r}{\Longrightarrow} \kappa$ is a strong cover $\tau \Rightarrow \kappa$ together with a positive integer r which is allowed to be the smallest row index of any connected component of the skew shape κ/τ.

Example. The two possible markings of the previous strong cover:



$\tau \overset{6}{\Longrightarrow} \kappa$ $\qquad$ $\tau \overset{3}{\Longrightarrow} \kappa$

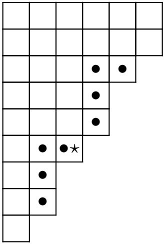
Spin

Def.

$$\text{spin}\left(\tau \stackrel{r}{\Longrightarrow} \kappa\right) = c \cdot (h-1) + N, \qquad \text{where}$$

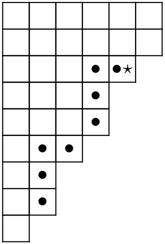
- c = number of connected components of κ/τ,
- h = height (number of rows) of each component,
- N = number of components below the marked one.

Example.



$\tau \stackrel{6}{\Longrightarrow} \kappa$

spin $= 4$



$\tau \stackrel{3}{\Longrightarrow} \kappa$

spin $= 5$

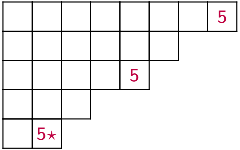
$\text{spin} = c \cdot (h-1) + N = 2 \cdot (3-1) + 0 = 4$ $\qquad$ $\text{spin} = 2 \cdot (3-1) + 1 = 5$

Strong marked tableaux

Def. For a word $w = w_1 \cdots w_m \in \mathbb{Z}_{\geq 1}^m$, a *strong tableau marked by w* is a sequence of strong marked covers of the form

$$\kappa^{(0)} \xrightarrow{w_m} \kappa^{(1)} \xrightarrow{w_{m-1}} \cdots \xrightarrow{w_1} \kappa^{(m)}.$$

Example. For $k = 4$, a strong tableau marked by 54321:



$$\kappa^{(4)} \xrightarrow{5} \kappa^{(5)}$$

- For a strong tableau T, inside$(T) := \mathfrak{p}(\kappa^{(0)})$ and outside$(T) := \mathfrak{p}(\kappa^{(m)})$.
- spin(T) $=$ sum of spins of strong covers comprising T.
- $\mathrm{SMT}^k(w\,;\mu)$ $=$ set of strong tableaux T marked by w with outside$(T) = \mu$.

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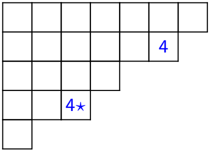

$$\kappa^{(3)} \xRightarrow{4} \kappa^{(4)}$$

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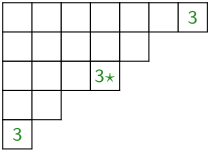

$$\kappa^{(2)} \xrightarrow{3} \kappa^{(3)}$$

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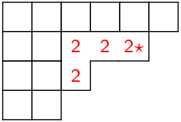

$$\kappa^{(1)} \xRightarrow{2} \kappa^{(2)}$$

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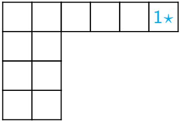

$$\kappa^{(0)} \xrightarrow{1} \kappa^{(1)}$$

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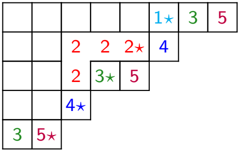

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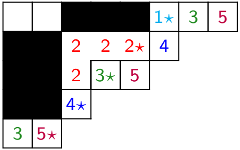

inside$(T) = 3222$

outside$(T) = 33332$

spin$(T) = 0+1+1+0+0 = 2$

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Strong Pieri operators

There is a natural t-analog of $\mathbb{Q}[h_1, \ldots, h_k] \subset \mathbb{Q}[h_1, h_2, \ldots]$ compatible with k-Schur functions.

Proposition

$$\Lambda^k := \mathrm{span}_{\mathbb{Q}(q,t)}\{H_\mu(\mathbf{x}; q, t) \mid \mu_1 \leq k\}$$
$$= \mathrm{span}_{\mathbb{Q}(q,t)}\{H_\mu(\mathbf{x}; t) \mid \mu_1 \leq k\} = \mathrm{span}_{\mathbb{Q}(q,t)}\{\mathfrak{s}^{(k)}_\mu(\mathbf{x}; t) \mid \mu_1 \leq k\}.$$

Def. The *strong Pieri operators* $u_1, u_2, \dots \in \mathrm{End}(\Lambda^k)$ are defined by

$$\mathfrak{s}^{(k)}_\mu \cdot u_p = \sum_{T \in \mathrm{SMT}^k(p\,;\mu)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)}.$$

Hence for a word $w = w_1 \cdots w_m$,

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Properties of k-Schur functions

Theorem (B.-Morse-Pun-Summers)

The k-Schur functions $\{\mathfrak{s}_\mu^{(k)} \mid \mu$ *is k-bounded of length* $\leq \ell\}$ *satisfy*

(dual Pieri rule) $$e_d^\perp \mathfrak{s}_\mu^{(k)} = \mathfrak{s}_\mu^{(k)} \cdot \left(\sum_{i_1 > \cdots > i_d} u_{i_1} \cdots u_{i_d} \right),$$

(shift invariance) $$\mathfrak{s}_\mu^{(k)} = e_\ell^\perp \mathfrak{s}_{\mu+1^\ell}^{(k+1)},$$

(Schur function stability) *if* $k \geq |\mu|$, *then* $\mathfrak{s}_\mu^{(k)} = s_\mu$.

- $e_d^\perp \in \mathrm{End}(\Lambda)$ is defined by $\langle e_d^\perp(g), h\rangle = \langle g, e_d h\rangle$ for all $g, h \in \Lambda$.
- u_i = operator for removing a strong cover marked in row i.

k-Schur branching rule

Theorem (B.-Morse-Pun-Summers)

For μ a k-bounded partition of length $\leq \ell$, the expansion of the k-Schur function $\mathfrak{s}_\mu^{(k)}$ into $k+1$-Schur functions is given by

$$\mathfrak{s}_\mu^{(k)} = \mathfrak{s}_{\mu+1^\ell}^{(k+1)} u_\ell \cdots u_1 = \sum_{T \in \mathrm{SMT}^{k+1}(\ell \cdots 2\,1\,;\,\mu+1^\ell)} t^{\mathrm{spin}(T)} \mathfrak{s}_{\mathrm{inside}(T)}^{(k+1)} \,.$$

Proof.

The shift invariance property followed by the dual Pieri rule yields

$$\mathfrak{s}_\mu^{(k)} = e_\ell^\perp \, \mathfrak{s}_{\mu+1^\ell}^{(k+1)} = \mathfrak{s}_{\mu+1^\ell}^{(k+1)} u_\ell \cdots u_1 .$$

□

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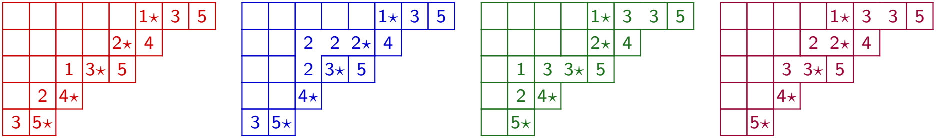
k-Schur branching rule

$$\mathfrak{s}^{(3)}_{22221} = t^3\mathfrak{s}^{(4)}_{3321} + t^2\mathfrak{s}^{(4)}_{3222} + t^2\mathfrak{s}^{(4)}_{33111} + \mathfrak{s}^{(4)}_{22221}$$



$\mathrm{SMT}^4(54321; 33332)$

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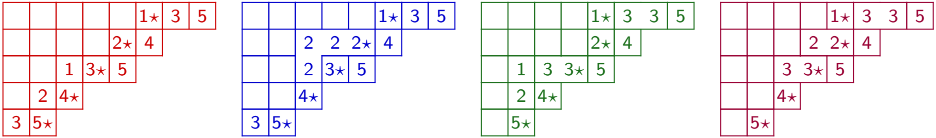
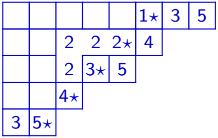
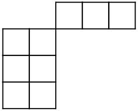
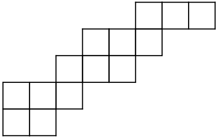


$$\mathrm{SMT}^4(54321; 33332)$$

$T =$ 





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k-Schur into Schur

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Let μ be a k-bounded partition of length $\leq \ell$ and set $m = \max(|\mu| - k, 0)$. The Schur expansion the k-Schur function $\mathfrak{s}_\mu^{(k)}$ is given by

$$\mathfrak{s}_\mu^{(k)} = \sum_{T \in \mathrm{SMT}^{k+m}(\,(\ell \cdots 1)^m\,;\,\mu + m^\ell\,)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}.$$

Proof.

Applying the shift invariance property m times followed by the dual Pieri rule, we obtain

$$\mathfrak{s}_\mu^{(k)} = (e_\ell^\perp)^m \, \mathfrak{s}_{\mu+m^\ell}^{(k+m)} = \mathfrak{s}_{\mu+m^\ell}^{(k+m)} (u_\ell \cdots u_1)^m = \sum_{T \in \mathrm{SMT}^{k+m}((\ell \cdots 1)^m; \mu + m^\ell)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}.$$

The Schur function stability property ensures this is the Schur function decomposition. □

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Schur expansion of $\mathfrak{s}^{(1)}_{111} = H_{111}$

			$1\star$	2	4	$4\star$	5	6
1	$2\star$	4	4	$5\star$	6			
$3\star$	5	$6\star$						

$t^3\, s_3$

		1	$1\star$	2	4	$4\star$	5	6
	$2\star$	4	4	$5\star$	6			
$3\star$	5	$6\star$						

$t^2\, s_{21}$

		$1\star$	3	$4\star$	5	5	5	6
	$2\star$	5	5	$5\star$	6			
1	$3\star$	$6\star$						

$t\, s_{21}$

	1	$1\star$	3	$4\star$	5	5	5	6
	$2\star$	5	5	$5\star$	6			
	$3\star$	$6\star$						

s_{111}

$$\mathfrak{s}^{(1)}_{111} = t^3 s_3 + t^2 s_{21} + t s_{21} + s_{111}$$

The Schur expansion of the 1-Schur function $\mathfrak{s}^{(1)}_{111}$ is obtained by summing $t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}$ over the set $\mathrm{SMT}^3(321321; 333)$ of strong tableaux T above.

Unifying the definitions of k-Schur functions

- $s_\mu^{(k)}(\mathbf{x}; t)$ defined as a sum of monomials over strong tableau. Equivalent to the symmetric functions satisfying the dual Pieri rule.
- $\tilde{A}_\mu^{(k)}(\mathbf{x}; t)$ defined recursively using Jing vertex operators.

Combining our results with those of Lam and Lam-Lapointe-Morse-Shimozono:

Theorem

The k-Schur functions defined from Jing vertex operators, k-Schur Catalan functions, and strong tableau k-Schur functions coincide:

$$\tilde{A}_\mu^{(k)}(\mathbf{x}; t) = \mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) = s_\mu^{(k)}(\mathbf{x}; t) \quad \text{for all } k\text{-bounded } \mu.$$

Moreover, their $t = 1$ specializations $\{s_\mu^{(k)}(\mathbf{x}; 1)\}$ match a definition using weak tableaux, and represent Schubert classes in the homology of the affine Grassmannian Gr_G of $G = SL_{k+1}$.

Shift invariance!?!!

$$\mathfrak{s}_{\mu}^{(k)} = e_{\ell}^{\perp} \mathfrak{s}_{\mu+1^{\ell}}^{(k+1)}$$

What is the geometric meaning of shift invariance?

k-Schur positive expansions

Symmetric functions known or conjectured to be k-Schur positive:

- modified Macdonald polynomials $H_\mu(\mathbf{x}; q, t)$ for k-bounded μ,
- LLT polynomials of bandwidth $\leq k$,
- products of k-Schur functions at $t = 1$,
- Catalan functions with partition weight and band $\leq k$.

(k-)Schur positivity conjectures

Conjecture (Chen-Haiman)

The Catalan function H_μ^Ψ is Schur positive for any root ideal Ψ and partition μ.

Strengthens earlier conjectures of Broer and Shimozono-Weyman.

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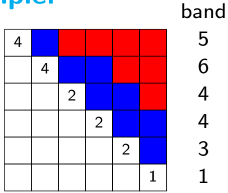
Conjecture (B.-Morse-Pun-Summers)

The Catalan function H_μ^Ψ is k-Schur positive whenever μ is a partition and $\max(\text{band}(\Psi,\mu)) \leq k$, *where*

$$\text{band}(\Psi,\mu)_i := \mu_i + \text{ \# of roots in row } i \text{ of } \Delta^+ \setminus \Psi.$$

Example.



This Catalan function is 6-Schur positive.

k-Schur positive expansions

We have obtained formulas for the k-Schur expansions of

- modified Hall-Littlewood polynomials proving the $q = 0$ case of the strengthened Macdonald positivity conjecture,
- the product of a Schur function and a k-Schur function when the indexing partitions concatenate to a partition, describing a class of Gromov-Witten invariants for the quantum cohomology of complete flag varieties,
- k-split polynomials, solving a substantial special case of a problem of Broer and Shimozono-Weyman on parabolic Hall-Littlewood polynomials.

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Yes?

Strengthened Macdonald positivity

Conjecture (Lapointe-Lascoux-Morse)

$H_\mu(\mathbf{x}; q, t)$ is k-Schur positive whenever $\mu_1 \leq k$.

Interesting even when $q = 0$. $H_\mu(\mathbf{x}; t) = H_\mu(\mathbf{x}; 0, t) = H_\mu^{\Delta^+}(\mathbf{x}; t)$ is the modified Hall-Littlewood polynomial.

Theorem (Lascoux-Schützenberger)

$$H_\mu = \sum_T t^{charge(T)} s_{\mathrm{shape}(T)},$$

the sum over semistandard Young tableaux T of content μ.

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Strengthened Macdonald positivity

- Z_θ = superstandard tableau of shape θ.
- $\mathrm{colword}(T)$ is the word obtained by concatenating the columns of T, reading each from bottom to top, starting with the leftmost.

Theorem (B.-Morse-Pun-Summers)

Let μ be a k-bounded partition of length $\leq \ell$. Set $w = \mathrm{colword}(Z_{k^\ell/\mu})$.

$$H_\mu = \mathfrak{s}^{(k)}_{k^\ell} \cdot u_w = \sum_{T \in \mathrm{SMT}^k(w\,;\,k^\ell)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(k)}_{\mathrm{inside}(T)}\,.$$

Example. $k = 3$, $\mu = 2211$.

$$Z_{(3333)/(2211)} = \begin{array}{|c|c|} \hline & 1 \\ \hline & 2 \\ \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array} \qquad \text{and} \qquad \mathrm{colword}(Z_{(3333)/(2211)}) = 434321\,.$$

$$H_{2211} = \mathfrak{s}^{(3)}_{3333} \cdot u_4 u_3 u_4 u_3 u_2 u_1 = \sum_{\mathrm{SMT}^3(434321\,;\,3333)} t^{\mathrm{spin}(T)} \mathfrak{s}^{(3)}_{\mathrm{inside}(T)}$$

Strengthened Macdonald positivity

- Z_θ = superstandard tableau of shape θ.
- $\mathrm{colword}(T)$ is the word obtained by concatenating the columns of T, reading each from bottom to top, starting with the leftmost.

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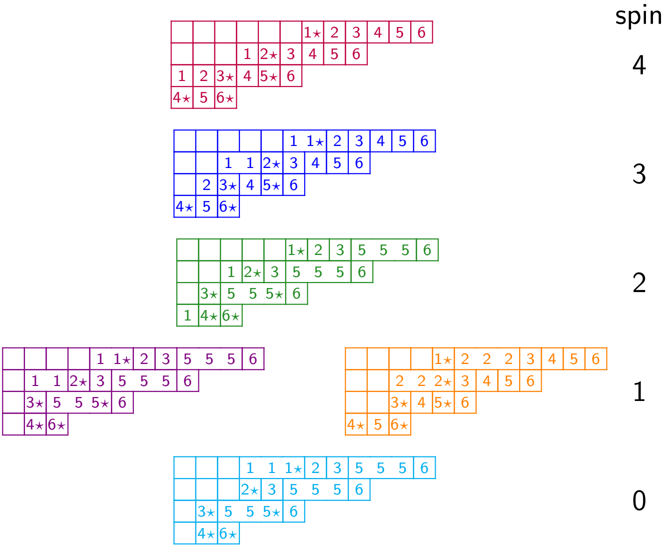
The 3-Schur expansion of H_{2211}



$$H_{2211} = t^4\,\mathfrak{s}^{(3)}_{33} + t^3\,\mathfrak{s}^{(3)}_{321} + t^2\,\mathfrak{s}^{(3)}_{321} + t\,\mathfrak{s}^{(3)}_{3111} + t\,\mathfrak{s}^{(3)}_{222} + \mathfrak{s}^{(3)}_{2211}.$$

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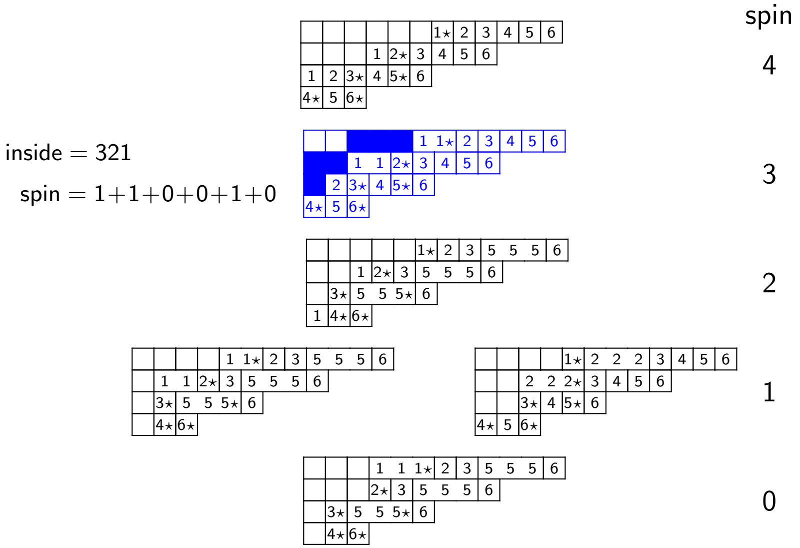

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k-Littlewood Richardson coefficients

Def. The *k-Littlewood Richardson coefficients* $c_{\mu\nu}^{\lambda}$ are defined by

$$\mathfrak{s}_{\mu}^{(k)}(\mathbf{x};1)\mathfrak{s}_{\nu}^{(k)}(\mathbf{x};1) = \sum_{\lambda} c_{\mu\nu}^{\lambda}\, \mathfrak{s}_{\lambda}^{(k)}(\mathbf{x};1).$$

Theorem (Lam)

The k-Schur functions at $t = 1$ represent Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$. Hence the structure constants for $H_*(\mathrm{Gr}_{SL_{k+1}})$ in the Schubert basis are the k-Littlewood Richardson coefficients.

Quantum equals Affine

Theorem (Peterson)

There is a ring isomorphism between a localization of $H_*(\mathrm{Gr}_{SL_{k+1}})$ and a localization of the quantum cohomology ring $QH^*(\mathrm{Fl}_{k+1})$, which matches the Schubert bases.

The 3-point Gromov-Witten invariants of genus 0 are the structure constants for $QH^*(\mathrm{Fl}_{k+1})$ in the Schubert basis. They contain the Schubert structure constants as a special case.

Corollary

The 3-point Gromov-Witten invariants of genus 0 agree with the k-Littlewood Richardson coefficients, and these nonnegative integers.

Open Problem. Find a positive combinatorial formula for the k-Littlewood Richardson coefficients and Gromov-Witten invariants.

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Schur times k-Schur into k-Schur

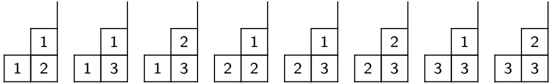
- $\mathrm{SSYT}_\theta(r)$ = semistandard Young tableaux of shape θ with entries from $\{1, \ldots, r\}$.
- $\mathbf{B}_\mu$ = Shimozono-Zabrocki generalized Hall-Littlewood vertex operator, which is multiplication by s_μ at $t = 1$.

Theorem (B.-Morse-Pun-Summers)

Let μ be a partition of length r with $\mu_1 \leq k - r + 1$, and ν a partition such that $\mu\nu$ is a partition. Set $R = (k - r + 1)^r$. Then

$$\mathbf{B}_\mu \, \mathfrak{s}_\nu^{(k)} = \sum_{T \in \mathrm{SSYT}_{R/\mu}(r)} \mathfrak{s}_{R\nu}^{(k)} \cdot u_{\mathrm{colword}(T)} \, .$$

Example. Let $k = 6$, $r = 3$, $\mu = 432$, $\nu = 22$. Then $R = 444$.

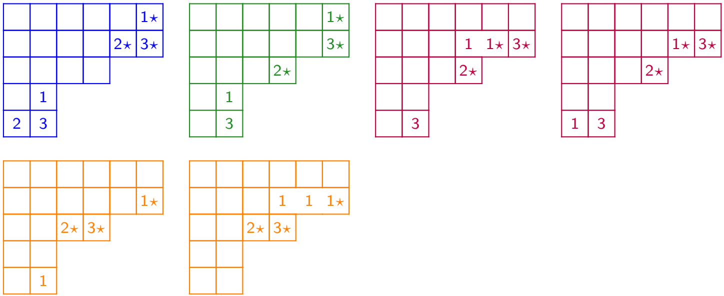
$\mathrm{SSYT}_{R/\mu}(r) = \Bigg\{$  $\Bigg\}.$

$$\mathbf{B}_\mu \, \mathfrak{s}_\nu^{(k)} = \mathfrak{s}_{R\nu}^{(k)} \cdot \big(u_{121} + u_{131} + u_{132} + u_{221} + u_{231} + u_{232} + u_{331} + u_{332}\big).$$

Schur times k-Schur into k-Schur

Example. 6-Schur expansion of a t-analog of $s_{432}\, s_{22}$.

$$\mathbf{B}_{432}\, \mathfrak{s}^{(6)}_{22} = \mathfrak{s}^{(6)}_{44422} \cdot \big(u_{121} + u_{131} + u_{132} + u_{221} + u_{231} + u_{232} + u_{331} + u_{332} \big).$$



$$\mathbf{B}_{432}\, \mathfrak{s}^{(6)}_{22} = t^3 \mathfrak{s}^{(6)}_{4441} + t^2 \mathfrak{s}^{(6)}_{44311} + t^2 \mathfrak{s}^{(6)}_{4432} + t^1 \mathfrak{s}^{(6)}_{43321} + t^1 \mathfrak{s}^{(6)}_{44221} + \mathfrak{s}^{(6)}_{43222}.$$

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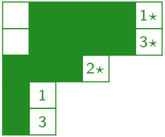

inside $= 44311$

spin $= 1 + 0 + 1 = 2$

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Gromov-Witten invariants

- A word is *cyclically increasing* if some rotation of it is increasing.
- $\mathrm{Inv}_i(w) = \big|\{j > i : w_i > w_j\}\big|$.
- $\theta : \mathcal{S}_{k+1} \to k$-bounded partitions

Corollary

Let $u, v, w \in \mathcal{S}_{k+1}$ and $\mathbf{d} \in \mathbb{Z}^k_{\geq 0}$. Suppose u has only one descent at position j and $v_{m+1} \cdots v_{k+1}$ is cyclically increasing, where m is the maximum index such that $\mathrm{Inv}_1(u) = \cdots = \mathrm{Inv}_m(u)$. Then the Gromov-Witten invariant is given by

$$\langle u, v, w_\circ w \rangle_{\mathbf{d}} = \sum_{T \in \mathrm{SSYT}_{R/\theta(u)}(r)} \sum_{\substack{S \in \mathrm{SMT}^k(\mathrm{colword}(T)\,;R\theta(v)) \\ \mathrm{inside}(S) = \lambda}} 1\,,$$

where $r = k + 1 - j - \mathrm{Inv}_1(u)$, $R = (k - r + 1)^r$, and λ is determined from $\theta(w)$, $\mathbf{d}$, and the descent sets of v, w.

Gromov-Witten invariants

Let $u, v, w \in \mathcal{S}_{k+1}$ and $\mathbf{d} \in \mathbb{Z}^k_{\geq 0}$. Suppose u has only one descent at position j and $v_{m+1} \cdots v_{k+1}$ is cyclically increasing, where m is the maximum index such that $\mathrm{Inv}_1(u) = \cdots = \mathrm{Inv}_m(u)$.

Example. $k = 6$, $u = 1246357$, $v = 1734562$.

The only descent of u is at position $j = 4$, $\mathrm{Inv}_1(u) = \mathrm{Inv}_2(u) = 0$, and $v_3 \cdots v_7 = 34562$ is cyclically increasing.

$\theta(u) = 432$ and $\theta(v) = 211111$.

$$\mathbf{B}_{432}\, \mathfrak{s}^{(6)}_{21^5} = \mathfrak{s}^{(6)}_{43221^5} + t^2 \mathfrak{s}^{(6)}_{44221^4} + t^2 \mathfrak{s}^{(6)}_{43321^4} + t\, \mathfrak{s}^{(6)}_{4421^6} + t\, \mathfrak{s}^{(6)}_{4331^6} + t^3 \mathfrak{s}^{(6)}_{4431^5}$$

$$\sigma_u * \sigma_v = \sigma_{1746352} + \sigma_{2745361} + \sigma_{2736451} + q_2 q_3 q_4 q_5 q_6 \left(\sigma_{1245367} + \sigma_{1236457} + \sigma_{2135467}\right)$$

where $\sigma_u * \sigma_v = \sum_{w \in \mathcal{S}_{k+1}} \sum_{d \in \mathbb{N}^k} \langle u, v, w_\circ w \rangle_{\mathbf{d}}\, \sigma_w$

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Thank you!

Happy birthday Sergey!