The Rule of Three for commutation relations

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joint work with Sergey Fomin (University of Michigan)

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Symmetric functions

The ring of symmetric functions $\Lambda(\mathbf{x}) = \Lambda(x_1, \ldots, x_N) \subset \mathbb{Q}[x_1, \ldots, x_N]$ consists of those polynomials which are fixed by the action of the symmetric group $\mathcal{S}_N$ permuting the indices.

Basic families of symmetric functions:

- elementary symmetric functions

$$e_k(\mathbf{x}) = e_k(x_1, \ldots, x_N) = \sum_{N \geq i_1 > \cdots > i_k \geq 1} x_{i_1} \cdots x_{i_k}$$

- Schur functions $s_\lambda(\mathbf{x}) = s_\lambda(x_1, \ldots, x_N)$

Fact

The $e_k(\mathbf{x})$ are algebraically independent and generate $\Lambda(\mathbf{x})$. Hence $\Lambda(\mathbf{x}) \cong \mathbb{Q}[e_1(\mathbf{x}), \ldots, e_N(\mathbf{x})]$.

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Noncommutative elementary symmetric functions

$\mathcal{U} = \mathbb{Q}\langle u_1, \ldots, u_N \rangle$ = free associative ring generated by $\mathbf{u} = (u_1, \ldots, u_N)$

We identify the monomials in $\mathcal{U}$ with words in the alphabet $[N]$ and frequently write $312, \text{cab}$, etc. for words/monomials.

Definition (Noncommutative elementary symmetric functions)

$$e_k(\mathbf{u}) = \sum_{N \geq i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1} u_{i_2} \cdots u_{i_k}$$

Equivalently, $\displaystyle\sum_{k=0}^{N} x^k e_k(\mathbf{u}) = (1 + xu_N) \cdots (1 + xu_1) \in \mathcal{U}[x]$.

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The plactic algebra

The *plactic algebra* is the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned} \mathrm{bca} &= \mathrm{bac}, \quad \mathrm{cab} = \mathrm{acb} \qquad \text{for } a < b < c, \\ \mathrm{bba} &= \mathrm{bab}, \quad \mathrm{baa} = \mathrm{aba} \qquad \text{for } a < b. \end{aligned}$$

Theorem (Lascoux-Schützenberger)

The $e_k(\mathbf{u})$ pairwise commute in the plactic algebra.

Example ($e_3(u_1, u_2, u_3)$ commutes with $e_1(u_1, u_2, u_3)$)

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Generalizations to other algebras

Let R be the quotient of $\mathcal{U}$ by the relations

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Theorem (Fomin-Greene)

The $e_k(\mathbf{u})$ pairwise commute in R.

Other variations

- For LLT polynomials (Lam 2005),
- For k-Schur functions (Lam 2006),
- The forgotten monoid (Novelli-Schilling 2008),
- For Schubert times Schur (Benedetti-Bergeron 2014),
- The switchboard ideal (B.–Fomin 2015, A. N. Kirillov 2015).

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For $S \subset \{1, \ldots, N\}$, denote

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Theorem (A. N. Kirillov, B.–Fomin)

For a quotient R of $\mathcal{U}$, the following are equivalent:

- $e_k(\mathbf{u}_S)\, e_\ell(\mathbf{u}_S) = e_\ell(\mathbf{u}_S)\, e_k(\mathbf{u}_S)$ *in R, for all k, ℓ, and S of size ≤ 3;*
- $e_k(\mathbf{u}_S)\, e_\ell(\mathbf{u}_S) = e_\ell(\mathbf{u}_S)\, e_k(\mathbf{u}_S)$ *in R, for all k, ℓ, and S.*

Equivalent formulation

The elements $e_k(\mathbf{u}_S)\, e_\ell(\mathbf{u}_S) - e_\ell(\mathbf{u}_S)\, e_k(\mathbf{u}_S)$ for $|S| \geq 4$ lie in the two-sided ideal generated by $\left\{e_k(\mathbf{u}_S)\, e_\ell(\mathbf{u}_S) - e_\ell(\mathbf{u}_S)\, e_k(\mathbf{u}_S)\right\}_{k,\ell,|S|\leq 3}$

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Example

The relation $e_1(\mathbf{u}_S)\, e_2(\mathbf{u}_S) = e_2(\mathbf{u}_S)\, e_1(\mathbf{u}_S)$ for $|S| = 4$ follows from the relations of this type for $|S| \le 3$.

Example

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\begin{aligned}
& e_1(a,b,c,d)\, e_2(a,b,c,d) - e_2(a,b,c,d)\, e_1(a,b,c,d) \\
&= [a+b+c+d,\ ba+ca+cb+da+db+dc] \\
&= [e_1(a,b,c), e_2(a,b,c)] + [e_1(a,b,d), e_2(a,b,d)] \\
&+ [e_1(a,c,d), e_2(a,c,d)] + [e_1(b,c,d), e_2(b,c,d)] \\
&- [e_1(a,b), e_2(a,b)] - [e_1(a,c), e_2(a,c)] - [e_1(a,d), e_2(a,d)] \\
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Original motivation: positivity

Positivity problems in algebraic combinatorics ask to find positive combinatorial formulae for nonnegative quantities arising in geometry and representation theory.

Key example of a positive combinatorial formula

The Littlewood-Richardson Rule is a positive combinatorial formula for Littlewood-Richardson coefficients, the decomposition multiplicities of a tensor product of irreducible representations of GL_n.

Two important unsolved positivity problems

Find a positive combinatorial formula for

- Kronecker coefficients: decomposition multiplicities of a tensor product of irreducible representations of the symmetric group.
- Kostka-Macdonald coefficients: coefficients in the Schur expansion of a transformed Macdonald polynomial.

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Noncommutative Schur functions are a powerful tool for solving positivity problems. They have led to positive combinatorial formulae for

- the Schur expansion of Stanley symmetric functions and stable Grothendieck polynomials.
- the Schur expansion of LLT polynomials indexed by 3-tuples of skew shapes and transformed Macdonald polynomials indexed by shapes with 3 columns.
- Kronecker coefficients for one hook shape and two arbitrary shapes.

The first step to apply this tool is to identify a quotient of $\mathcal{U}$ in which the $e_k(\mathbf{u})$ commute.

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The Rule of Three for two sets of variables

For $S \subset [N]$, let $e_k(\mathbf{v}_S)$ denote the elementary symmetric function in the noncommuting variables $\{v_i \mid i \in S\}$.

Theorem (B.–Fomin)

For a quotient R of $\mathbb{Q}\langle u_1, \ldots, u_N, v_1, \ldots, v_N \rangle$, the following are equivalent:

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Reformulation using generating functions

Let $g_i = 1 + xu_i$ and $h_i = 1 + yv_i$ for $i = 1, \ldots, N$.

For $S \subset [N]$, let g_S denote the descending product of g_i, $i \in S$. Define h_S similarly. For example, $g_{[N]} = g_N g_{N-1} \cdots g_1$.

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Rule of Three for power series

Conjecture (B.–Fomin)

Let R be a quotient ring of $\mathbb{Q}\langle u_1, \ldots, u_N, v_1, \ldots, v_N\rangle$. Let $g_1, \ldots, g_N$, $h_1, \ldots, h_N \in R[[x, y]]$ be power series of the form $g_i = \sum_k \alpha_{ik}(u_i x)^k$, $h_i = \sum_k \beta_{ik}(v_i y)^k$, satisfying $\alpha_{i0} = \beta_{i0} = 1$ and $\alpha_{i1}\beta_{i1} \neq 0$. Then the following are equivalent:

- $g_S h_S = h_S g_S$ *for all S of size ≤ 3;*
- $g_S h_S = h_S g_S$ *for all S.*

Note: the previous theorem is the special case $g_i = 1 + xu_i$, $h_i = 1 + yv_i$.

Rule of Three for power series

Conjecture (B.–Fomin)

Let R be a quotient ring of $\mathbb{Q}\langle u_1, \dots, u_N, v_1, \dots, v_N \rangle$. Let $g_1, \dots, g_N$, $h_1, \dots, h_N \in R[[x, y]]$ be power series of the form $g_i = \sum_k \alpha_{ik}(u_i x)^k$, $h_i = \sum_k \beta_{ik}(v_i y)^k$, satisfying $\alpha_{i0} = \beta_{i0} = 1$ and $\alpha_{i1}\beta_{i1} \neq 0$. Then the following are equivalent:

- $g_S h_S = h_S g_S$ *for all* S *of size* ≤ 3*;*
- $g_S h_S = h_S g_S$ *for all* S*.*

Note: the previous theorem is the special case $g_i = 1 + xu_i$, $h_i = 1 + yv_i$.

van Kampen diagrams

Given a presentation of a group G, a *van Kampen diagram* is a planar finite cell complex, connected and simply connected, with a specific embedding in $\mathbb{R}^2$, and

- 1-cells labeled by generators,
- 2-cells labeled by relations.

Let G be the group with generators a, b, c and relations:

- $aba^{-1}b^{-1} = id$
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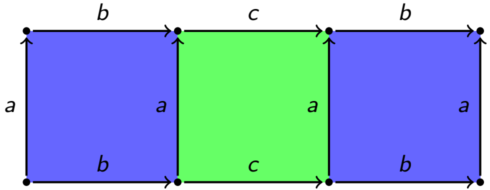

van Kampen's Lemma

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A word w in the generators and their inverses is equal to the identity if and only if there exists a van Kampen diagram whose boundary is labeled by w.

The given relations imply the boundary relation $abcba^{-1}b^{-1}c^{-1}b^{-1} = id$. The boundary relation can also be read as $abcb = bcba$.

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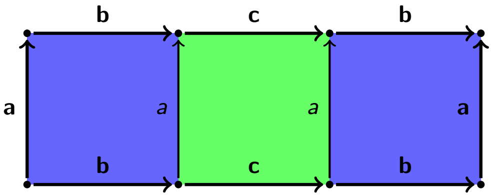

Simplified purely group-theoretic statement

Theorem

Let $g_1, \ldots, g_N, h_1, \ldots, h_N$ be elements of a group G satisfying the following relations:

$$\begin{aligned} g_b h_b &= h_b g_b && 1 < b < N, \\ g_{a+1} g_a h_{a+1} h_a &= h_{a+1} h_a g_{a+1} g_a && 1 \leq a < N, \\ g_a h_c &= h_c g_a && 1 \leq a, c \leq N \text{ with } |a - c| \geq 2. \end{aligned}$$

Then the following relation also holds in G:

$$g_N g_{N-1} \cdots g_1 h_N h_{N-1} \cdots h_1 = h_N h_{N-1} \cdots h_1 g_N g_{N-1} \cdots g_1.$$

Corollary

The $e_k(\mathbf{u})$ pairwise commute in the nilCoxeter algebra.

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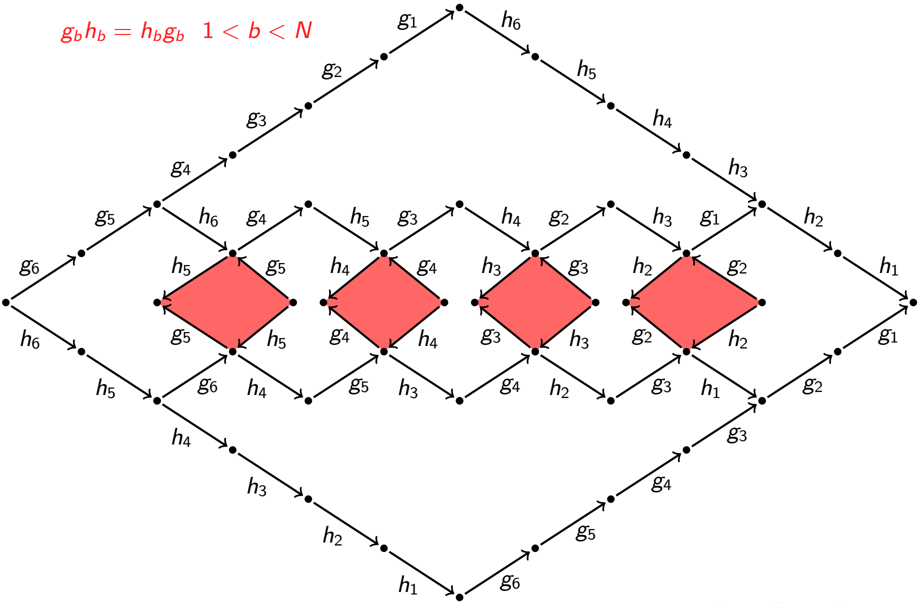
The van Kampen diagram proving the Theorem

$g_b h_b = h_b g_b \quad 1 < b < N$



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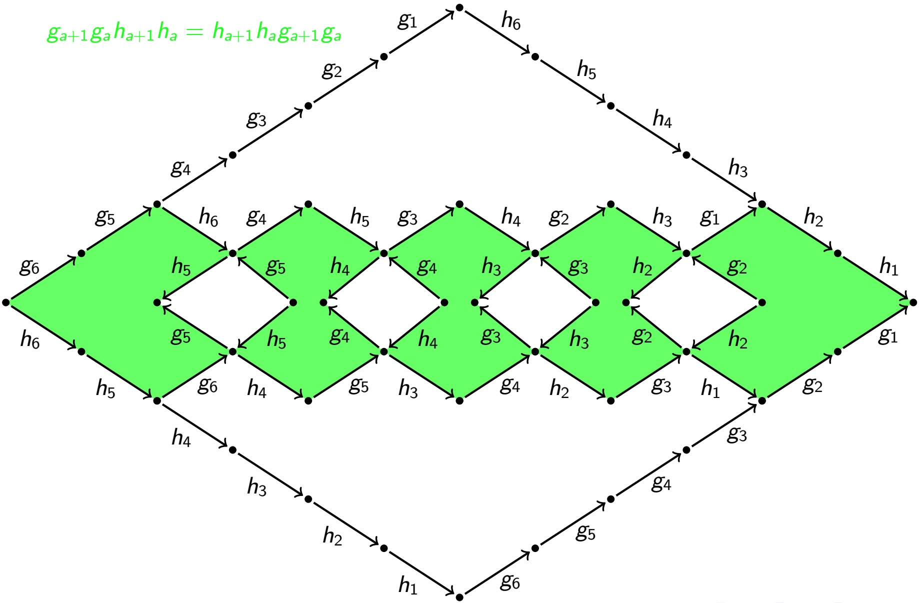
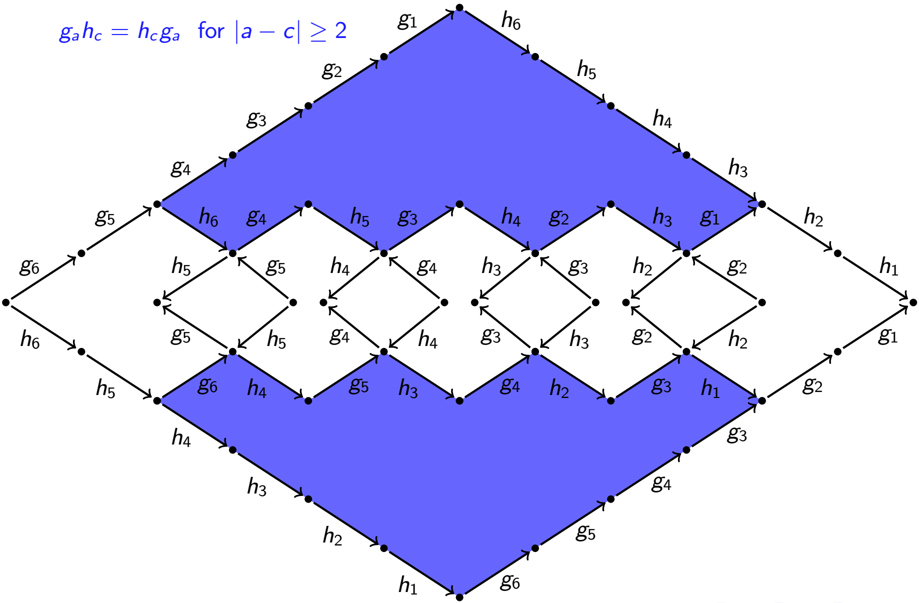
$$g_{a+1}g_a h_{a+1}h_a = h_{a+1}h_a g_{a+1}g_a$$



The van Kampen diagram proving the Theorem

$g_a h_c = h_c g_a$ for $|a - c| \geq 2$



The Rule of Three for power series

Special case of the main conjecture for one set of variables:

Theorem (B.–Fomin)

Let R be a quotient of $\mathcal{U}$. Let $g_1, \ldots, g_N, h_1, \ldots, h_N \in \mathcal{U}[[x,y]]$ be power series of the form $g_i = \sum_k \alpha_{ik}(u_i x)^k$, $h_i = \sum_k \beta_{ik}(u_i y)^k$, satisfying $\alpha_{i0} = \beta_{i0} = 1$ and $\alpha_{i1}\beta_{i1} \neq 0$. The following are equivalent:

- $g_S h_S = h_S g_S$ *in* $R[[x,y]]$ *for all* S *of size* ≤ 3;
- $g_S h_S = h_S g_S$ *in* $R[[x,y]]$ *for all* S.

Note: $R[[x,y]]$ is the ring of formal power series with coefficients in R.
g_S denotes the descending product of the g_i, $i \in S$.

Proof (Step 1)

The proof uses a lemma and theorem which are entirely group-theoretic, and a Lagrange inversion argument.

Lemma

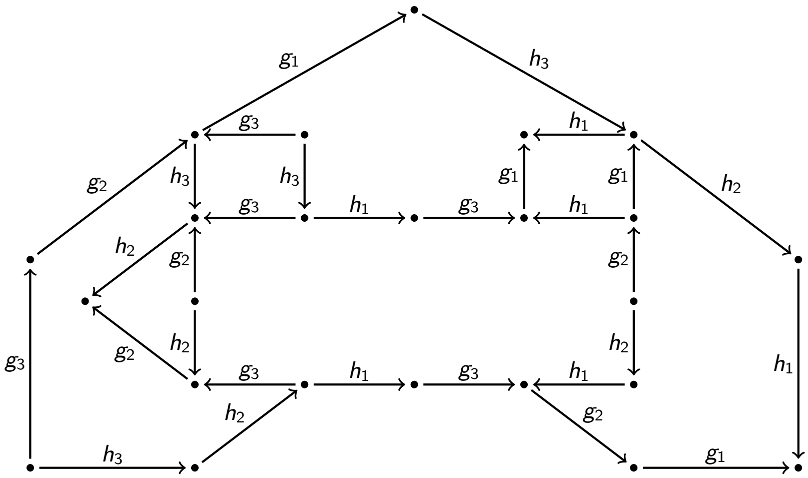
Let $g_1, g_2, g_3, h_1, h_2, h_3$ be elements of a group G satisfying the following relations:

$$\begin{aligned} g_a h_a &= h_a g_a && \text{for all } a, \\ g_b g_a h_b h_a &= h_b h_a g_b g_a && \text{for all } a < b. \end{aligned}$$

Then the following are equivalent:

- $h_2^{-1} g_2 (g_3^{-1} h_1 g_3 h_1^{-1}) = (g_3^{-1} h_1 g_3 h_1^{-1}) h_2^{-1} g_2$,
- $g_3 g_2 g_1 h_3 h_2 h_1 = h_3 h_2 h_1 g_3 g_2 g_1$.

The van Kampen diagram proving the lemma



Proof (Step 2)

Proposition

Let R be a quotient of $\mathcal{U}$. Suppose $g_i = \sum_k \alpha_{ik}(u_i x)^k$, $h_i = \sum_k \beta_{ik}(u_i y)^k \in R[[x, y]]$, satisfying $\alpha_{i0} = \beta_{i0} = 1$ and $\alpha_{i1}\beta_{i1} \neq 0$. Then if $g_i h_i$ commutes with some $z \in R[[x, y]]$, then g_i and h_i also commute with z.

This follows from a Lagrange inversion argument.

Proof (Step 3)

Theorem

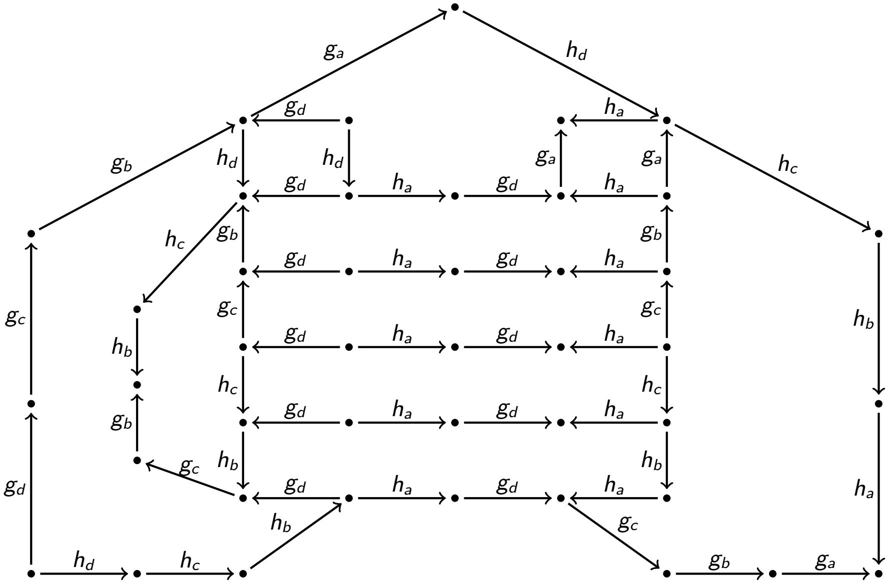
Let $g_1, \ldots, g_N, h_1, \ldots, h_N$ be elements of a group G satisfying the following relations:

$$\begin{aligned}
g_a h_a &= h_a g_a && \text{for all } a, \\
g_b g_a h_b h_a &= h_b h_a g_b g_a && \text{for all } a < b, \\
g_b (g_c^{-1} h_a g_c h_a^{-1}) &= (g_c^{-1} h_a g_c h_a^{-1}) g_b && \text{for all } a < b < c, \\
h_b^{-1} (g_c^{-1} h_a g_c h_a^{-1}) &= (g_c^{-1} h_a g_c h_a^{-1}) h_b^{-1} && \text{for all } a < b < c.
\end{aligned}$$

Then the following relation also holds in G:

$$g_N g_{N-1} \cdots g_1 h_N h_{N-1} \cdots h_1 = h_N h_{N-1} \cdots h_1 g_N g_{N-1} \cdots g_1.$$

The van Kampen diagram proving the theorem



Computations

Definition

Let $g_1, \ldots, g_N, h_1, \ldots, h_N$ be invertible elements of the ring $\mathbb{Q}\langle u_1, \ldots, u_M, v_1, \ldots, v_M\rangle[[x, y]]$. The *Rule of Three holds* (for $g_1, \ldots, g_N, h_1, \ldots, h_N$) if for any quotient R of $\mathbb{Q}\langle u_1, \ldots, u_M, v_1, \ldots, v_M\rangle$, the following are equivalent:

- $g_S h_S = h_S g_S$ in $R[[x, y]]$ for all S of size ≤ 3;
- $g_S h_S = h_S g_S$ in $R[[x, y]]$ for all S.

Problem

Determine natural conditions on the g_i, h_i which ensure that the Rule of Three holds.

Computations

Rule of Three fails:

$$\begin{aligned} g_4 &= (1 + xu_4) & h_4 &= (1 + xv_4) \\ g_3 &= (1 + xu_3) & h_3 &= (1 + xv_3) \\ g_2 &= (1 + xu_2) & h_2 &= (1 + xv_2) \\ g_1 &= (1 + xu_1) & h_1 &= (1 + xv_1) \end{aligned}$$

Corollary

The purely group-theoretic version of the Rule of Three is false: there exists a group G with elements $g_1, \ldots, g_4, h_1, \ldots, h_4$ such that $g_S h_S = h_S g_S$ for all S of size ≤ 3, but $g_S h_S \neq h_S g_S$ for $S = \{1, 2, 3, 4\}$.

Computations

Rule of Three fails:

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\end{aligned}
$$

Corollary

The purely group-theoretic version of the Rule of Three is false: there exists a group G with elements $g_1, \ldots, g_4, h_1, \ldots, h_4$ such that $g_S h_S = h_S g_S$ for all S of size ≤ 3, but $g_S h_S \neq h_S g_S$ for $S = \{1, 2, 3, 4\}$.

Computations

Rule of Three fails:

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\begin{aligned}
g_4 &= (1+xu_8)(1+xu_7) & h_4 &= (1+yu_8)(1+yu_7)\\
g_3 &= (1+xu_6)(1+xu_5) & h_3 &= (1+yu_6)(1+yu_5)\\
g_2 &= (1+xu_4)(1+xu_3) & h_2 &= (1+yu_4)(1+yu_3)\\
g_1 &= (1+xu_2)(1+xu_1) & h_1 &= (1+yu_2)(1+yu_1)
\end{aligned}
$$

Conjecturally, the Rule of Three holds:

$$
\begin{aligned}
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g_1 &= (1+xu_2) & h_1 &= (1+yu_2)
\end{aligned}
$$

Computations

Consider g_i, h_i of the following form:

$$g_i = (1 + z_i u_i),\ \ h_i = (1 + z_i' v_i),\ \text{for } z_i, z_i' \in \{x, y\}.$$

A *pattern* is a sequence $z_1 z_2 z_3 z_4\, z_1' z_2' z_3' z_4' \in \{x, y\}^8$.

Rule of Three fails for the following 34 patterns:

xxxx xxxx	*yyyy yyyy*		
xxyy xxyy	*yyxx yyxx*	*xyxy xyxy*	*yxyx yxyx*
xyyx xyyx	*yxxy xyyx*	*xyyx yxxy*	*yxxy yxxy*
xyyx yyyy	*yxxy xxxx*	*xxxx yxxy*	*yyyy xyyx*
xyyy xyyy	*yxxx yxxx*	*xxxy xxxy*	*yyyx yyyx*
xxyx xxyx	*yyxy yyxy*	*xxyx yxyy*	*yyxy xyxx*
xyxx yyxy	*yxyy xxyx*	*xyxx xyxx*	*yxyy yxyy*
xyyx yxyy	*yxxy xyxx*	*xyyx yyxy*	*yxxy xxyx*
xyxx yxxy	*yxyy xyyx*	*xxyx yxxy*	*yyxy xyyx*

Rule of Three holds for the other $2^8 - 34$ patterns.

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$xxyy\ xxyy$	$yyxx\ yyxx$	$xyxy\ xyxy$	$yxyx\ yxyx$
$xyyx\ xyyx$	$yxxy\ xyyx$	$xyyx\ yxxy$	$yxxy\ yxxy$
$xyyx\ yyyy$	$yxxy\ xxxx$	$xxxx\ yxxy$	$yyyy\ xyyx$
$xyyy\ xyyy$	$yxxx\ yxxx$	$xxxy\ xxxy$	$yyyx\ yyyx$
$xxyx\ xxyx$	$yyxy\ yyxy$	$xxyx\ yxyy$	$yyxy\ xyxx$
$xyxx\ yyxy$	$yxyy\ xxyx$	$xyxx\ xyxx$	$yxyy\ yxyy$
$xyyx\ yxyy$	$yxxy\ xyxx$	$xyyx\ yyxy$	$yxxy\ xxyx$
$xyxx\ yxxy$	$yxyy\ xyyx$	$xxyx\ yxxy$	$yyxy\ xyyx$

Rule of Three holds for the other $2^8 - 34$ patterns.

Further results

Theorem (B.–Fomin)

The commutation relations $e_k(\mathbf{u}_S)e_\ell(\mathbf{v}_S) = e_\ell(\mathbf{v}_S)e_k(\mathbf{u}_S)$ *hold for all* k, ℓ, *and* $S \subset [N]$ *if and only if the following relations hold:*

$$\begin{aligned}
e_1(\mathbf{u}_S)e_1(\mathbf{v}_S) &= e_1(\mathbf{v}_S)e_1(\mathbf{u}_S) && \text{for } 1 \le |S| \le 2,\\
e_2(\mathbf{u}_S)e_1(\mathbf{v}_S) &= e_1(\mathbf{v}_S)e_2(\mathbf{u}_S) && \text{for } 2 \le |S| \le 3,\\
e_1(\mathbf{u}_S)e_2(\mathbf{v}_S) &= e_2(\mathbf{v}_S)e_1(\mathbf{u}_S) && \text{for } 2 \le |S| \le 3,\\
e_3(\mathbf{u}_S)e_1(\mathbf{v}_S) &= e_1(\mathbf{v}_S)e_3(\mathbf{u}_S) && \text{for } |S| = 3,\\
e_1(\mathbf{u}_S)e_3(\mathbf{v}_S) &= e_3(\mathbf{v}_S)e_1(\mathbf{u}_S) && \text{for } |S| = 3.
\end{aligned}$$

Amazingly, the relations $e_2(\mathbf{u}_S)e_2(\mathbf{v}_S) = e_2(\mathbf{v}_S)e_2(\mathbf{u}_S)$ for S of size 2 and 3 and the relations $e_2(\mathbf{u}_S)e_3(\mathbf{v}_S) = e_3(\mathbf{v}_S)e_2(\mathbf{u}_S)$ for $|S| = 3$ follow from the above relations.

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Further results

Let $g_1, \ldots, g_N, h_1, \ldots, h_N \in \mathcal{U}[[x, y]]$ have the following form:

- $g_i = (1 + xu_i)$ and $h_i = (1 + yu_i)$, or
- $g_i = (1 - xu_i)^{-1}$ and $h_i = (1 - yu_i)^{-1}$.

Definition

Define the *noncommutative super elementary symmetric functions* $e_k(\overline{\mathbf{u}})$ by $g_N \cdots g_1 = \sum_{k=0}^{N} x^k e_k(\overline{\mathbf{u}})$.

By the main theorem, the Rule of Three holds for this choice of g_i, h_i. The following stronger result holds:

Theorem (B.–Fomin)

For a quotient R of $\mathcal{U}$, the following are equivalent:

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