Catalan functions and k-Schur positivity

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joint work with Jennifer Morse, Anna Pun, and Dan Summers

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Strengthened Macdonald positivity conjecture

Theorem (Haiman)

The modified Macdonald polynomials are Schur positive:

$$H_\mu(\mathbf{x}; q, t) = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda(\mathbf{x}) \qquad \text{for } K_{\lambda\mu}(q, t) \in \mathbb{N}[q, t].$$

Conjecture (Lapointe-Lascoux-Morse)

The atom k-Schur functions $\{A_\lambda(\mathbf{x}; t)\}_{\lambda_1 \leq k}$

- *form a basis for* $\Lambda^k = \mathrm{span}_{\mathbb{Q}(q,t)}\{H_\mu(\mathbf{x}; q, t)\}_{\mu_1 \leq k}$, *and*
- *are Schur positive;*
- *expansion of* $H_\mu(\mathbf{x}; q, t) \in \Lambda^k$ *in this basis has coefficients in* $\mathbb{N}[q, t]$.

Conjecture (Lapointe-Lascoux-Morse)

The $k+1$*-Schur expansion of a k-Schur function has coefficients in* $\mathbb{N}[t]$.

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Strengthened Macdonald positivity conjecture

Example. $k = 2$

$$H_{1^4} = t^4\left(s_{22} + ts_{31} + t^2 s_{4}\right) + (t^2 + t^3)\left(s_{211} + ts_{31}\right) + \left(s_{1111} + ts_{211} + t^2 s_{22}\right)$$

$$H_{211} = t\left(s_{22} + ts_{31} + t^2 s_{4}\right) + (1 + qt^2)\left(s_{211} + ts_{31}\right) + q\left(s_{1111} + ts_{211} + t^2 s_{22}\right)$$

$$H_{22} = \left(s_{22} + ts_{31} + t^2 s_{4}\right) + \underbrace{(tq + q)}_{\substack{\text{positive sum of} \\ q,\, t\text{-monomials}}} \underbrace{\left(s_{211} + ts_{31}\right)}_{\substack{t\text{-positive sum} \\ \text{of schur functions}}} + q^2\left(s_{1111} + ts_{211} + t^2 s_{22}\right)$$

Strengthened Macdonald positivity conjecture

Example. $k = 2$

$$H_{1^4} = t^4\left(s_{(2,2)} + ts_{(3,1)} + t^2 s_{(4)}\right) + \left(t^2 + t^3\right)\left(s_{(2,1,1)} + ts_{(3,1)}\right) + \left(s_{(1,1,1,1)} + ts_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{211} = t\left(s_{(2,2)} + ts_{(3,1)} + t^2 s_{(4)}\right) + \left(1 + qt^2\right)\left(s_{(2,1,1)} + ts_{(3,1)}\right) + q\left(s_{(1,1,1,1)} + ts_{(2,1,1)} + t^2 s_{(2,2)}\right)$$

$$H_{22} = \underbrace{\left(s_{(2,2)} + ts_{(3,1)} + t^2 s_{(4)}\right)}_{s^{(2)}_{(2,2)}} + \underbrace{(tq + q)}_{\text{positive sum of } q,\, t\text{-monomials}} \underbrace{\left(s_{(2,1,1)} + ts_{(3,1)}\right)}_{t\text{-positive sum of schur functions};\ s^{(2)}_{(2,1,1)}} + q^2 \underbrace{\left(s_{(1,1,1,1)} + ts_{(2,1,1)} + t^2 s_{(2,2)}\right)}_{s^{(2)}_{(1,1,1,1)}}$$

basis for restricted span Λ^k of Macdonald polynomials

Conjecturally equivalent definitions of k-Schurs

	basis	symmetric	Schur positive	branching
[1998:Lapointe,Lascoux,Morse] Tableaux and katabolism		✓	✓	
[2003:Lapointe,Morse] Jing vertex operators	✓	✓		
[2006:Lam,Lapointe,Morse,Shimozono] Bruhat order on type-A affine Weyl group / strong tableaux				
[2010:Chen,Haiman] $GL_\ell(\mathbb{C})$-equivariant Euler characteristics / Demazure operators		✓		
[2012:Assaf,Billey] Quasisymmetric functions				
[2015:Dalal,Morse] Inverting affine Kostka matrix	✓	✓		

Overview

The k-Schur functions appear in

- the study of Macdonald polynomials,
- the homology of the affine Grassmannian,
- graded representations of the symmetric group.

Prior work on the branching rule:

- Geometric proof at $t = 1$ (Lam 2011).
- Formula for branching at $t = 1$ as equivalence classes on the k-shape poset (Lam-Lapointe-Morse-Shimozono 2013).

Main results:

- Strong tableaux k-Schur functions form a Schur positive basis for Λ^k.
- (Branching rule) positive combinatorial formula for the $k + 1$-Schur expansion of k-Schur functions.
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[2015:Dalal,Morse] Inverting affine Kostka matrix	✓	✓		
[2018:B,Morse,Pun,Summers] Strong tableaux = Catalan functions	✓	✓	✓	✓

k-bounded partitions and $k+1$-cores

Def. A *k-bounded partition* is a partition with parts of size $\leq k$.

Def. A *$k+1$-core* is a partition whose diagram has no box with hook length $k+1$.

Proposition. There is a bijection $\kappa \mapsto \mathfrak{p}(\kappa)$ from $k+1$-cores to k-bounded partitions.

Example. $k = 4$.

κ $\qquad$ $\mathfrak{p}(\kappa)$

Def. The k-skew diagram of a $k+1$-core κ is the skew shape obtained by removing boxes of hook length $> k$.

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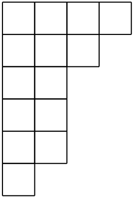
14	12	9	7	6	4	3	2	1
9	7	4	2	1				
6	4	1						
4	2							
3	1							
1								

κ



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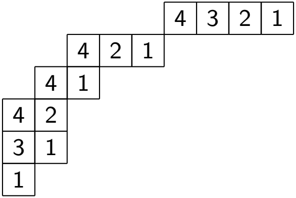
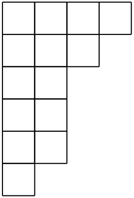


k-skew(κ)



$\mathfrak{p}(\kappa)$

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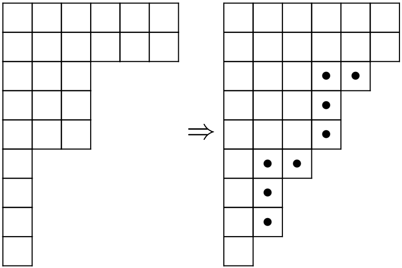
Strong covers

Def. An inclusion $\tau \subset \kappa$ of $k+1$-cores is a *strong cover*, denoted $\tau \Rightarrow \kappa$, if $|\mathfrak{p}(\tau)| + 1 = |\mathfrak{p}(\kappa)|$.

Example.

Strong cover with $k = 4$:



corresponding k-skew diagrams:

$\mathfrak{p}(\tau) = 332221111$ $\quad \mathfrak{p}(\kappa) = 222222221$

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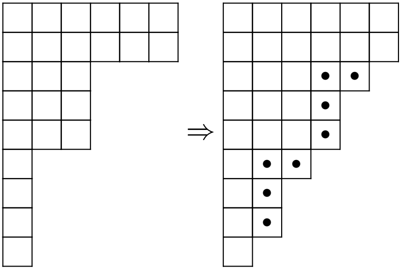

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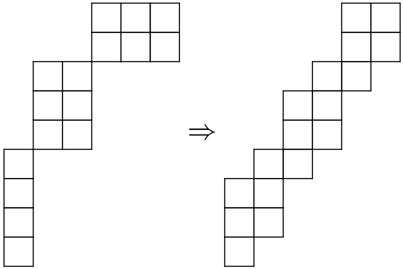
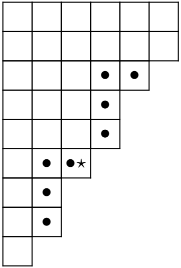


$\mathfrak{p}(\tau) = 332221111$ $\mathfrak{p}(\kappa) = 222222221$

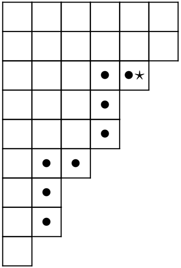
Strong marked covers

Def. A *strong marked cover* $\tau \overset{r}{\Longrightarrow} \kappa$ is a strong cover $\tau \Rightarrow \kappa$ together with a positive integer r which is allowed to be the smallest row index of any connected component of the skew shape κ/τ.

Example. The two possible markings of the previous strong cover:



$\tau \overset{6}{\Longrightarrow} \kappa$



$\tau \overset{3}{\Longrightarrow} \kappa$

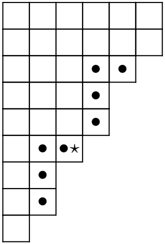
Spin

Def.

$$\textit{spin}(\tau \overset{r}{\Longrightarrow} \kappa) = c \cdot (h-1) + N, \quad \text{where}$$

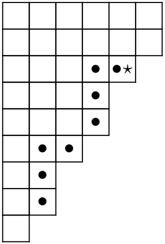
- c = number of connected components of κ/τ,
- h = height (number of rows) of each component,
- N = number of components below the marked one.

Example.



$\tau \overset{6}{\Longrightarrow} \kappa$

spin $= 4$



$\tau \overset{3}{\Longrightarrow} \kappa$

spin $= 5$

spin $= c \cdot (h-1) + N = 2 \cdot (3-1) + 0 = 4$ spin $= 2 \cdot (3-1) + 1 = 5$

Vertical strong marked tableaux

Def. A *vertical strong marked tableau* T of *weight* $\eta = (\eta_1, \eta_2, \dots)$ is a sequence

$$\kappa^{(0)} \xrightarrow{r_1} \kappa^{(1)} \xrightarrow{r_2} \cdots \xrightarrow{r_m} \kappa^{(m)}$$

such that $r_{v_i+1} < r_{v_i+2} < \cdots < r_{v_i+\eta_i}$ for all i, where $v_i := \eta_1 + \cdots + \eta_{i-1}$.

- $\text{inside}(T) := \mathfrak{p}(\kappa^{(0)})$
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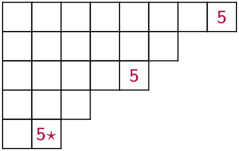
Example. For $k = 4$, a vertical strong marked tableau of weight (5):



$$\kappa^{(4)} \xrightarrow{5} \kappa^{(5)}$$

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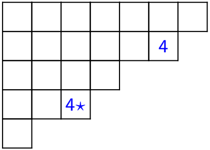

$$\kappa^{(3)} \xRightarrow{4} \kappa^{(4)}$$

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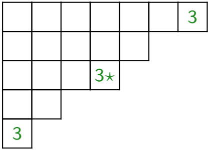

$$\kappa^{(2)} \xRightarrow{3} \kappa^{(3)}$$

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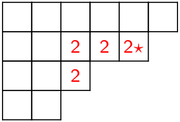

$$\kappa^{(1)} \xRightarrow{2} \kappa^{(2)}$$

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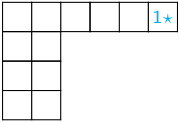

$$\kappa^{(0)} \xRightarrow{1} \kappa^{(1)}$$

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					$1\star$	3	5
		2	2	$2\star$	4		
		2	$3\star$	5			
		$4\star$					
3	$5\star$						

Spin k-Schur functions

- We work in the ring of symmetric functions in infinitely many variables $\mathbf{x} = (x_1, x_2, \ldots)$.
- $\mathrm{SMT}^k_\eta(\mu)$ = set of strong marked tableaux T of weight η with outside$(T) = \mu$.
- spin(T) = sum of the spins of the strong marked covers comprising T.

Def. For a k-bounded partition μ, let

$$s^{(k)}_\mu(\mathbf{x}; t) = \sum_{\eta \in \mathbb{Z}^\infty_{\geq 0},\, |\eta| = |\mu|} \sum_{T \in \mathrm{SMT}^k_\eta(\mu)} t^{\mathrm{spin}(T)} \mathbf{x}^\eta.$$

Their $t = 1$ specializations

- agree with another combinatorial definition using weak tableaux (Lam-Lapointe-Morse-Shimozono 2010),
- are Schubert classes in the homology of the affine Grassmannian $\mathrm{Gr}_{SL_{k+1}}$ of SL_{k+1} (Lam 2008).

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Properties of k-Schur functions

Theorem (B.-Morse-Pun-Summers)

The k-Schur functions $\{s_\mu^{(k)} \mid \mu$ *is k-bounded of length* $\leq \ell\}$ *satisfy*

(vertical dual Pieri rule) $$e_d^\perp s_\mu^{(k)} = \sum_{T \in \mathrm{VSMT}^k_{(d)}(\mu)} t^{\mathrm{spin}(T)} s^{(k)}_{\mathrm{inside}(T)},$$

(shift invariance) $$s_\mu^{(k)} = e_\ell^\perp s^{(k+1)}_{\mu+1^\ell},$$

(Schur function stability) *if* $k \geq |\mu|$, *then* $s_\mu^{(k)} = s_\mu$.

- $e_d^\perp \in \mathrm{End}(\Lambda)$ is defined by $\langle e_d^\perp(g), h\rangle = \langle g, e_d h\rangle$ for all $g, h \in \Lambda$.
- $\mathrm{VSMT}^k_\eta(\mu)$ = set of vertical strong marked tableaux T of weight η with outside$(T) = \mu$.

k-Schur branching rule

Theorem (B.-Morse-Pun-Summers)

For μ a k-bounded partition of length $\leq \ell$, the expansion of the k-Schur function $s_\mu^{(k)}$ into $k+1$-Schur functions is given by

$$s_\mu^{(k)} = \sum_{T \in \mathrm{VSMT}_{(\ell)}^{k+1}(\mu+1^\ell)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}^{(k+1)}.$$

Proof.

The shift invariance property followed by the vertical dual Pieri rule yields

$$s_\mu^{(k)} = e_\ell^\perp \, s_{\mu+1^\ell}^{(k+1)} = \sum_{T \in \mathrm{VSMT}_{(\ell)}^{k+1}(\mu+1^\ell)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}^{(k+1)}. \qquad \square$$

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k-Schur branching rule

$$s^{(3)}_{22221} = t^2 s^{(4)}_{3222} + t^2 s^{(4)}_{3321} + t^2 s^{(4)}_{33111} + s^{(4)}_{22221}$$

					1⋆	3	5
		2	2	2⋆	4		
		2	3⋆	5			
		4⋆					
3	5⋆						

					1⋆	3	5
				2⋆	4		
		1	3⋆	5			
	2	4⋆					
3	5⋆						

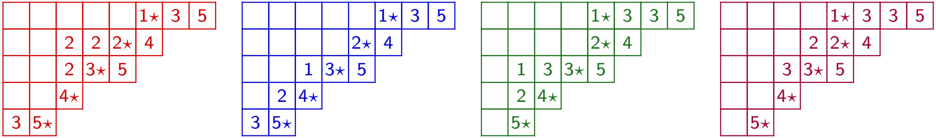
				1⋆	3	3	5
				2⋆	4		
	1	3	3⋆	5			
	2	4⋆					
	5⋆						

				1⋆	3	3	5
			2	2⋆	4		
		3	3⋆	5			
		4⋆					
	5⋆						

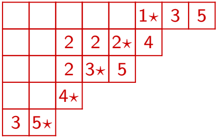
$$\mathrm{VSMT}^4_{(5)}(33332)$$

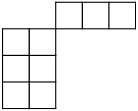
k-Schur branching rule

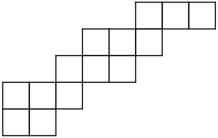
$$s^{(3)}_{22221} = t^2 s^{(4)}_{3222} + t^2 s^{(4)}_{3321} + t^2 s^{(4)}_{33111} + s^{(4)}_{22221}$$



$$\mathrm{VSMT}^4_{(5)}(33332)$$

$T =$ 





$\mathrm{spin}(T) = 0 + 1 + 1 + 0 + 0 = 2$ $\quad$ $\mathrm{inside}(T) = 3222$ $\quad$ $\mathrm{outside}(T) = 33332$

Root ideals

- Set of positive roots $\Delta^+ := \{(i,j) \mid 1 \leq i < j \leq \ell\}$.
- $\Psi \subseteq \Delta^+$ is an upper order ideal of positive roots.

Example. $\Psi = \{(1,3),(1,4),(1,5),(1,6),(2,5),(2,6),(3,6)\}$

		$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
				$(2,5)$	$(2,6)$
					$(3,6)$

Catalan functions

Def. (Panyushev, Chen-Haiman)

- $\Psi \subseteq \Delta^+$ is an upper order ideal of positive roots,
- $\gamma \in \mathbb{Z}^\ell$.

The *Catalan function* indexed by Ψ and γ:

$$H_\gamma^\Psi(\mathbf{x};t) := \prod_{(i,j)\in\Psi} (1 - tR_{ij})^{-1} s_\gamma(\mathbf{x})$$

where the raising operator R_{ij} acts by $R_{ij}(s_\gamma(\mathbf{x})) = s_{\gamma+\epsilon_i-\epsilon_j}(\mathbf{x})$.

Example. Let $\mu = (\mu_1, \ldots, \mu_\ell)$ be a partition.

- Empty root set: $H_\mu^\varnothing(\mathbf{x};t) = s_\mu(\mathbf{x})$.
- Full root set: $H_\mu^{\Delta^+}(\mathbf{x};t) = H_\mu(\mathbf{x};t)$, the modified Hall-Littlewood polynomial.

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k-Schur Catalan functions

Def. For μ a k-bounded partition of length $\leq \ell$, define the root ideal

$$\Delta^k(\mu) = \{(i,j) \in \Delta^+ \mid k - \mu_i + i < j\},$$

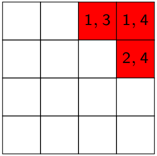
and the Catalan function

$$\mathfrak{s}_\mu^{(k)}(\mathbf{x};t) := H_\mu^{\Delta^k(\mu)} = \prod_{i=1}^{\ell} \prod_{j=k+1-\mu_i+i}^{\ell} \left(1 - tR_{ij}\right)^{-1} s_\mu(\mathbf{x})\,.$$

" # nonroots in row $i \;=\; k - \mu_i$ "

Examples of Catalan functions

Example. $k = 4$, $\mu = 3321$.

$\Delta^k(\mu) =$ 

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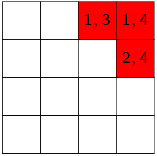
$\Delta^k(\mu) =$ 

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$$= (1 - tR_{13})^{-1}(1 - tR_{24})^{-1}(1 - tR_{14})^{-1} s_{3321}(\mathbf{x})$$

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$\Delta^k(\mu) =$

		1, 3	1, 4
			2, 4

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$$= s_{3321} + t(s_{3420} + s_{4311} + s_{4320}) + t^2(s_{4410} + s_{5301} + s_{5310})$$

$$+ t^3(s_{63-11} + s_{5400} + s_{6300}) + t^4(s_{64-10} + s_{73-10})$$

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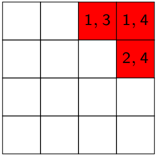
$\Delta^k(\mu) =$ 

$$\begin{aligned}
\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) &= \prod_{(i,j)\in\Delta^k(\mu)} (1 - tR_{ij})^{-1} s_\mu(\mathbf{x}) \\
&= (1 - tR_{13})^{-1}(1 - tR_{24})^{-1}(1 - tR_{14})^{-1} s_{3321}(\mathbf{x}) \\
&= s_{3321} + t(s_{3420} + s_{4311} + s_{4320}) + t^2(s_{4410} + s_{5301} + s_{5310}) \\
&\quad + t^3(s_{63-11} + s_{5400} + s_{6300}) + t^4(s_{64-10} + s_{73-10}) \\
&= s_{3321} + t(s_{4320} + s_{4311}) + t^2(s_{4410} + s_{5310}) + t^3 s_{5400}.
\end{aligned}$$

Chen-Haiman conjecture

Theorem (B.-Morse-Pun-Summers)

For any k-bounded partition μ, the k-Schur function $s_\mu^{(k)}(\mathbf{x};t)$ is the Catalan function $\mathfrak{s}_\mu^{(k)}(\mathbf{x};t)$.

k-Schur into Schur

Theorem (B.-Morse-Pun-Summers)

Let μ be a k-bounded partition of length $\leq \ell$ and set $m = \max(|\mu| - k, 0)$. The Schur expansion the k-Schur function $s_\mu^{(k)}$ is given by

$$s_\mu^{(k)} = \sum_{T \in \mathrm{VSMT}^{k+m}_{(\ell^m)}(\mu + m^\ell)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}.$$

Proof.

Applying the shift invariance property m times followed by the vertical dual Pieri rule, we obtain

$$s_\mu^{(k)} = (e_\ell^\perp)^m \, s_{\mu+m^\ell}^{(k+m)} = \sum_{T \in \mathrm{VSMT}^{k+m}_{(\ell^m)}(\mu + m^\ell)} t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}.$$

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Schur expansion of $s^{(1)}_{111} = H_{111}$

			$1\star$	2	4	$4\star$	5	6	
1	$2\star$	4	4	$5\star$	6				$t^3\, s_3$
$3\star$	5	$6\star$							

		1	$1\star$	2	4	$4\star$	5	6	
	$2\star$	4	4	$5\star$	6				$t^2\, s_{21}$
$3\star$	5	$6\star$							

		$1\star$	3	$4\star$	5	5	5	6	
	$2\star$	5	5	$5\star$	6				$t\, s_{21}$
1	$3\star$	$6\star$							

	1	$1\star$	3	$4\star$	5	5	5	6	
	$2\star$	5	5	$5\star$	6				s_{111}
	$3\star$	$6\star$							

$$s^{(1)}_{111} = t^3 s_3 + t^2 s_{21} + t s_{21} + s_{111}$$

The Schur expansion of the 1-Schur function $s^{(1)}_{111}$ is obtained by summing $t^{\mathrm{spin}(T)} s_{\mathrm{inside}(T)}$ over the set $\mathrm{VSMT}^3_{(3,3)}(3,3,3)$ of vertical strong marked tableaux T given above.

Schur function straightening

Schur functions may be defined for any $\gamma \in \mathbb{Z}^\ell$. The Schur function $s_\gamma(x_1, x_2, \ldots, x_\ell) = s_\gamma(\mathbf{x})$ is straightened as follows:

$$s_\gamma(\mathbf{x}) = \begin{cases} \operatorname{sgn}(\gamma+\rho) s_{\mathrm{sort}(\gamma+\rho)-\rho}(\mathbf{x}) & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\mathrm{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\mathrm{sort}(\beta)$.

Example. $\ell = 4$, $\gamma = 3125$.

$\gamma + \rho = (3,1,2,5) + (3,2,1,0) = (6,3,3,5)$ has a repeated part.

Hence $s_{3125}(\mathbf{x}) = 0$.

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Example. $\ell = 4$, $\gamma = 4716$.

$$\gamma + \rho = (4,7,1,6) + (3,2,1,0) = (7,9,2,6)$$

$$\mathrm{sort}(\gamma+\rho) = (9,7,6,2)$$

$$\mathrm{sort}(\gamma+\rho) - \rho = (6,5,5,2)$$

Hence $s_{4716}(\mathbf{x}) = s_{6552}(\mathbf{x})$.