Kronecker coefficients and noncommutative super Schur functions

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joint work with Ricky Liu (North Carolina State University)

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The Kronecker problem

M_λ denotes the irreducible representation of the symmetric group corresponding to the partition λ.

The *Kronecker coefficient* $g_{\lambda\mu\nu}$ is the multiplicity of M_ν in the tensor product $M_\lambda \otimes M_\mu$, where λ, μ, ν are partitions of n.

Kronecker problem

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- Formula for Kronecker coefficients when one of the shapes is a hook
- Noncommutative Schur functions and switchboards
- Noncommutative super Schur functions for the Kronecker problem
- Lascoux's heuristic for Kronecker coefficients

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Colored Yamanouchi words

- $\{1, 2, \ldots, N\}$ = alphabet of *unbarred letters*
- $\{\bar{1}, \bar{2}, \ldots, \overline{N}\}$ = alphabet of *barred letters*
- $\mathcal{A}$ = alphabet of barred and unbarred letters ordered by $1 < \bar{1} < 2 < \bar{2} < \cdots < N < \overline{N}$
- A *colored word* is a word in the alphabet $\mathcal{A}$.

$\mathrm{w}^{\mathrm{brgt}}$:= the ordinary word formed from w by shuffling the barred letters to the right, reversing this subword of barred letters and removing their bars.

w is *Yamanouchi* of content λ if $\mathrm{w}^{\mathrm{brgt}}$ is Yamanouchi of content λ.

Example

$$\mathrm{w} = 2\bar{1}\bar{2}1\bar{3}\bar{1}21$$

$$\mathrm{w}^{\mathrm{brgt}} = 21211321$$

These words are Yamanouchi of content $(4, 3, 1)$.

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Colored tableaux

A *colored tableau* is a tableau with entries in $\mathcal{A}$ such that

- each row and column is weakly increasing,
- the unbarred letters in each column are strictly increasing,
- the barred letters in each row are strictly increasing.

For a colored tableau T, the word $\mathrm{barread}(T)$ is obtained by reading the diagonals of T one by one, starting from the southwest corner. In each diagonal, read the unbarred entries in the direction $\nwarrow$, followed by the barred entries in the direction $\searrow$.

Example

$$\mathrm{barread}\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & \bar{3} & \bar{4} & 6 \\ \hline \bar{2} & 3 & 4 & \bar{4} \\ \cline{1-4} 3 & \bar{3} & \bar{4} & 5 \\ \cline{1-4} \end{array}\right) = 3\,\bar{2}\,\bar{3}\,3\,1\,\bar{4}\,5\,4\,1\,\bar{3}\,\bar{4}\,\bar{4}\,6.$$

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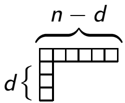
Kronecker coefficients for one hook shape

$\mu(d)$ denotes the hook partition $(n-d, 1^d) =$



Theorem (R. Liu)

For any partitions λ, ν of n and $d \leq n-1$, the Kronecker coefficient $g_{\lambda\,\mu(d)\,\nu}$ is the number of colored tableaux T such that

- $\mathrm{barread}(T)$ *is Yamanouchi of content λ,*
- *T has exactly d barred letters,*
- *T has shape ν,*
- *the northeast corner of T is unbarred.*

1	1	1	3
$\bar{1}$	2	$\bar{2}$	4
$\bar{1}$	3		
$\bar{1}$	4		

$\lambda = (6, 2, 2, 2)$
$\mu(d) = (8, 1, 1, 1, 1)$
$\nu = (4, 4, 2, 2)$

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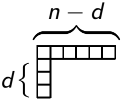
$\mu(d)$ denotes the hook partition $(n-d, 1^d) =$ 

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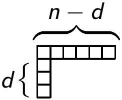
$\mu(d)$ denotes the hook partition $(n-d, 1^d) =$ 

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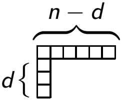

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Kronecker coefficients for one hook shape

The colored tableaux T and $\mathrm{barread}(T)$ such that

- $\mathrm{barread}(T)$ is Yamanouchi of content $(6,2,2,2)$,
- T has exactly 4 barred letters,
- T has shape $(4,4,2,2)$,
- the northeast corner of T is unbarred.

1	1	1	1
$\bar{1}$	$\bar{2}$	3	3
$\bar{1}$	$\bar{2}$		
4	4		

$44\bar{1}\bar{1}\bar{2}1\bar{2}31311$

1	1	1	3
$\bar{1}$	2	$\bar{2}$	4
$\bar{1}$	3		
$\bar{1}$	4		

$\bar{1}4\bar{1}3\bar{1}211\bar{2}413$

1	1	1	1
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$\bar{1}$	3		
4	4		

$44\bar{1}3\bar{1}211\bar{2}1\bar{3}1$

$$g_{(6,2,2,2)\,(8,1,1,1)\,(4,4,2,2)} = 3$$

Noncommutative Schur functions

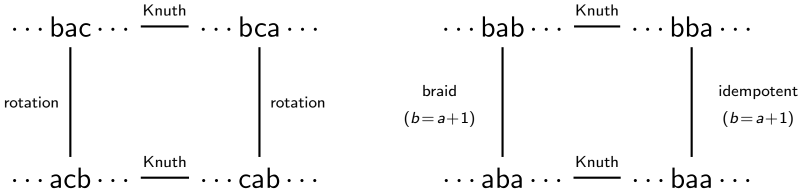
Noncommutative Schur functions are a powerful tool for solving positivity problems. They have led to positive combinatorial formulae for

- the Schur expansion of Stanley symmetric functions and stable Grothendieck polynomials.
- the Schur expansion of LLT polynomials indexed by 3-tuples of skew shapes and transformed Macdonald polynomials indexed by shapes with 3 columns.
- Kronecker coefficients for one hook shape and two arbitrary shapes.

Switchboards

Two words of the same length are related by a *switch* in position i if

- their entries match to the left of position $i-1$ and to the right of $i+1$;
- their entries in positions $i-1, i, i+1$ are related in one of these ways:



A *switchboard* is an edge-labeled graph on a vertex set of words such that

- each edge labeled i corresponds to a switch in position i;
- if a word w appearing in the switchboard can in principle be switched in position i, then w is incident to exactly one edge labeled i.

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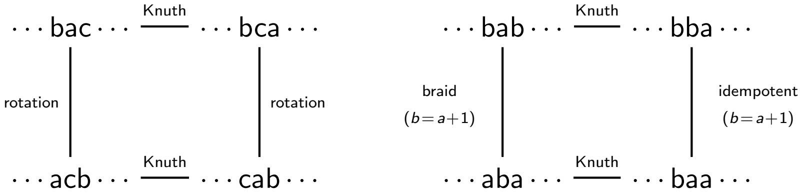

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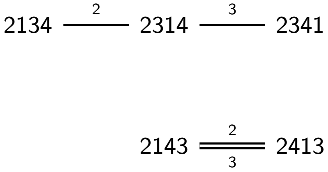


Figure : A switchboard whose 3-edges are Knuth switches.

Switchboards

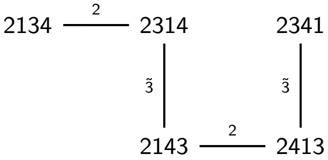
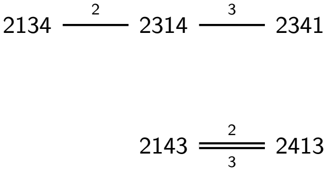


Figure : A switchboard whose 3-edges are rotation switches.

Switchboards



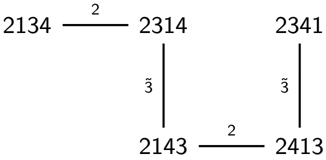
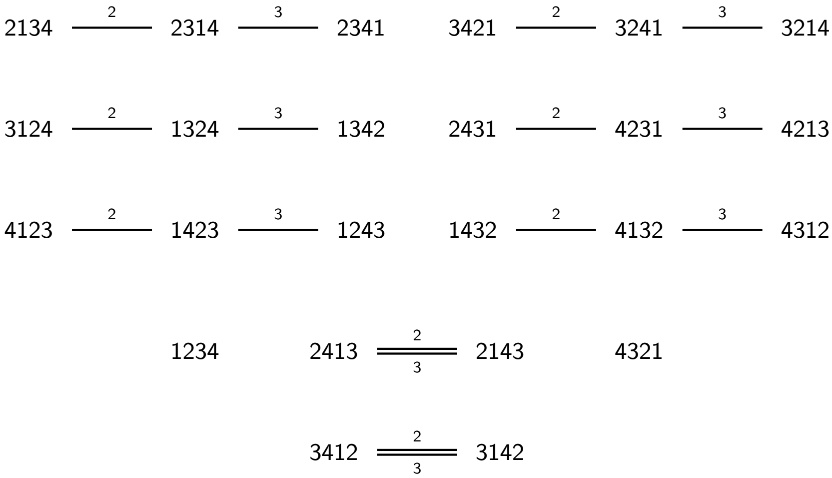


Figure : Different switchboards on the same set of vertices.

The switchboard on $\mathcal{S}_4$ with only Knuth switches



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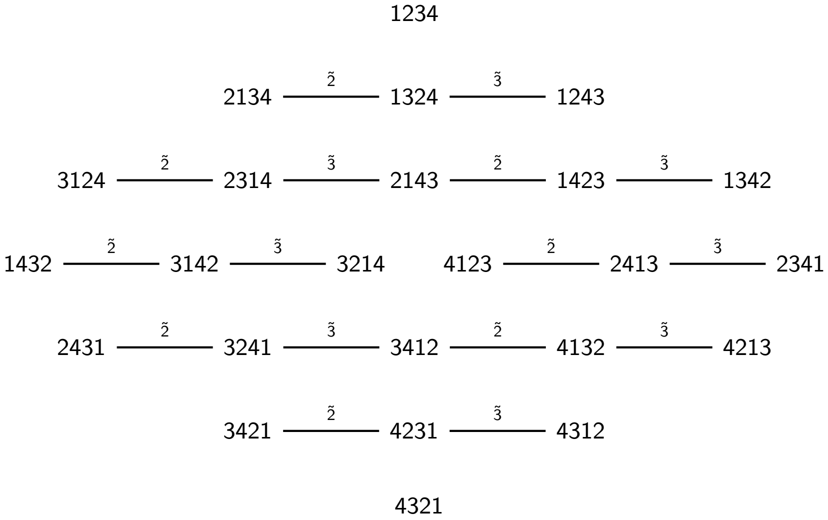

The symmetric function associated to a switchboard

- Let $\mathbf{x} = (x_1, x_2, \dots)$ be commuting variables.
- $\mathrm{Des}(\mathrm{w}) = \{i \in \{1, \dots, n-1\} \mid \mathrm{w}_i > \mathrm{w}_{i+1}\}$ denotes the *descent set* of a word $\mathrm{w} = \mathrm{w}_1 \cdots \mathrm{w}_n$.
- Gessel's *fundamental quasisymmetric function* is

$$Q_{\mathrm{Des}(\mathrm{w})}(\mathbf{x}) = \sum_{\substack{1 \le i_1 \le \cdots \le i_n \\ j \in \mathrm{Des}(\mathrm{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

- For a switchboard Γ, define

$$F_\Gamma(\mathbf{x}) = \sum_{\mathrm{w} \in \mathrm{Vert}(\Gamma)} Q_{\mathrm{Des}(\mathrm{w})}(\mathbf{x}).$$

Theorem (B.-Fomin)

For any switchboard Γ, the function $F_\Gamma(\mathbf{x})$ is symmetric in $x_1, x_2, \dots$.

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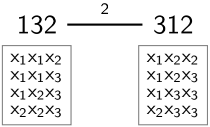
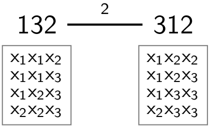


Figure : A switchboard Γ with $F_\Gamma = s_{21}$. The sum of the outlined monomials is $s_{21}(x_1, x_2, x_3)$.

Figure : A switchboard Γ' with $F_{\Gamma'} = s_{31} + s_{22}$. The sum of the outlined monomials is $s_{31}(x_1, x_2, x_3) + s_{22}(x_1, x_2, x_3)$.

The symmetric function associated to a switchboard



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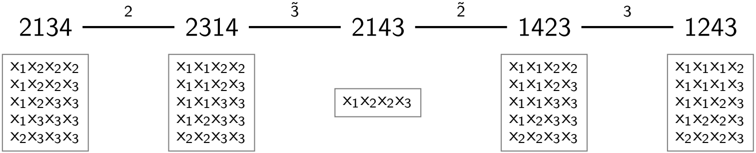

Figure : A switchboard Γ' with $F_{\Gamma'} = s_{31} + s_{22}$. The sum of the outlined monomials is $s_{31}(x_1, x_2, x_3) + s_{22}(x_1, x_2, x_3)$.

Noncommutative Schur functions

Let $\mathcal{U} = \mathbb{Z}\langle u_1, u_2, \ldots, u_N \rangle$. We identify the monomials in $\mathcal{U}$ with words in the alphabet $[N]$ and write $312, \mathrm{cab}$, etc. for words/monomials.

Noncommutative elementary symmetric functions:

$$e_k(\mathbf{u}) = \sum_{N \geq i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1} u_{i_2} \cdots u_{i_k}$$

Noncommutative Schur functions $\mathfrak{J}_\lambda(\mathbf{u})$:

- $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition.
- λ' is the conjugate partition.
- $t = \lambda_1$ is the number of parts of λ'.

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{\pi \in \mathcal{S}_t} \mathrm{sgn}(\pi)\, e_{\lambda'_1+\pi(1)-1}(\mathbf{u})\, e_{\lambda'_2+\pi(2)-2}(\mathbf{u}) \cdots e_{\lambda'_t+\pi(t)-t}(\mathbf{u}).$$

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Schur expansion of $F_\gamma(\mathbf{x})$

- For $\gamma = \sum_{\text{words w}} \gamma_{\text{w}} \text{w} \in \mathcal{U}$, define $F_\gamma(\mathbf{x}) = \sum_{\text{words w}} \gamma_{\text{w}}\, Q_{\mathrm{Des(w)}}(\mathbf{x})$.
- We consider $\mathcal{U}$ to be endowed with the symmetric bilinear form for which the monomials/words form an orthonormal basis.
- Note that any element of $\mathcal{U}/I$ has a well-defined pairing with any element of $I^\perp$, for any ideal I of $\mathcal{U}$.

Theorem (Fomin-Greene)

Let I be an ideal of $\mathcal{U}$ such that the $e_k(\mathbf{u})$ commute pairwise in $\mathcal{U}/I$. Then for any $\gamma \in I^\perp$,

$$F_\gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x}) \langle \mathfrak{J}_\lambda(\mathbf{u}), \gamma \rangle.$$

The switchboard ideal

Let $\mathcal{U}/I_{\mathrm{S}}$ be the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned}
&\mathrm{b(ca-ac)=(ca-ac)b} && \text{for } a<b<c,\\
&\mathrm{caa=aca}, \quad \mathrm{cca=cac} && \text{for } c-a>1,\\
&\mathrm{bab+baa=bba+aba} && \text{for } b=a+1.
\end{aligned}$$

Theorem (A. N. Kirillov, B.-Fomin)

The $e_k(\mathbf{u})$ commute pairwise in $\mathcal{U}/I_{\mathrm{S}}$.

Proposition

For a set of words W of the same length, the following are equivalent:

- $\sum_{\mathrm{w}\in W} \mathrm{w} \in I_{\mathrm{S}}^{\perp}$;
- *W is the vertex set of a switchboard.*

Corollary

For any switchboard Γ, $F_\Gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x})\langle \mathfrak{J}_\lambda(\mathbf{u}), \sum_{\mathrm{w}\in\mathrm{Vert}(\Gamma)} \mathrm{w}\rangle$.

Recipe for positive combinatorial formulae

We obtain the following recipe for producing positive combinatorial formulae for the coefficients in the Schur expansions of many classes of symmetric functions:

(1) Find a suitable ideal $I \supset I_{\mathrm{S}}$ such that symmetric functions of interest are of the form $F_\Gamma(\mathbf{x})$ with $\gamma = \sum_{\mathrm{w} \in \mathrm{Vert}(\Gamma)} \mathrm{w} \in I^\perp$.

(2) Find a monomial positive expression for $\mathfrak{J}_\lambda(\mathbf{u})$ in $\mathcal{U}/I$.

(3) Apply $(*)$ to obtain a positive formula for the Schur expansion of F_Γ.

$$F_\Gamma(\mathbf{x}) = F_\gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x}) \langle \mathfrak{J}_\lambda(\mathbf{u}), \gamma \rangle \qquad (*)$$

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Noncommutative super Schur functions

Ordinary	Super
alphabet $\{1, 2, \ldots, N\}$	$\mathcal{A} = \{1, 2, \ldots, N\} \sqcup \{\bar{1}, \bar{2}, \ldots, \overline{N}\}$
$\mathcal{U} = \mathbb{Z}\langle u_1, \ldots, u_N \rangle$	$\overline{\mathcal{U}} = \mathbb{Z}\langle u_1, u_{\bar{1}}, \ldots, u_N, u_{\overline{N}} \rangle$
$e_k(\mathbf{u}) = \sum_{N \geq i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1} \cdots u_{i_k}$	$\bar{e}_k(\mathbf{u}) = \sum_{\substack{N \geq i_1 \geq i_2 \geq \cdots \geq i_k \geq 1 \\ i_j \text{ unbarred} \Rightarrow i_j > i_{j+1}}} u_{i_1} \cdots u_{i_k}$
$\mathfrak{J}_\lambda(\mathbf{u}) = \det(e_{\lambda'_i + j - i}(\mathbf{u}))$	$\bar{\mathfrak{J}}_\lambda(\mathbf{u}) = \det(\bar{e}_{\lambda'_i + j - i}(\mathbf{u}))$
$\mathrm{Des}(\mathrm{w}) = \{i \mid \mathrm{w}_i > \mathrm{w}_{i+1}\}$	$\mathrm{Des}(\mathrm{w}) = \left\{ i \mid \substack{\mathrm{w}_i > \mathrm{w}_{i+1} \text{ or} \\ \mathrm{w}_i = \mathrm{w}_{i+1} \text{ are barred}} \right\}$

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$e_k(\mathbf{u})=\sum_{N\geq i_1>i_2>\cdots>i_k\geq 1} u_{i_1}\cdots u_{i_k}$	$\bar{e}_k(\mathbf{u})=\sum_{\substack{N\geq i_1\geq i_2\geq\cdots\geq i_k\geq 1\\ i_j \text{ unbarred}\Rightarrow i_j>i_{j+1}}} u_{i_1}\cdots u_{i_k}$
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Recipe for positivity with $I = I_{\mathrm{Kron}}$, $\gamma = \sum_{\mathsf{w} \in \mathrm{CYW}_{\lambda,d}} \mathsf{w}$

$\mathrm{CYW}_{\lambda,d}$ = set of colored Yamanouchi words of content λ having exactly d barred letters

Proposition

For any partitions λ, ν of n and $d \leq n$,

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = \Big(\text{the coefficient of } s_\nu(\mathbf{x}) \text{ in } F_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x})\Big).$$

Fact: $\sum_{\mathsf{w} \in \mathrm{CYW}_{\lambda,d}} \mathsf{w} \in (I_{\mathrm{Kron}})^\perp$

where I_{Kron} is the ideal of $\overline{\mathcal{U}}$ corresponding to the relations

$$
\begin{aligned}
&(\mathsf{xz} - \mathsf{zx})\mathsf{y} = \mathsf{y}(\mathsf{xz} - \mathsf{zx}) && \text{for } x, y, z \in \mathcal{A},\\
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&\mathsf{yyx} = \mathsf{yxy}, \quad \mathsf{zyy} = \mathsf{yzy} && \text{for } x, z \in \mathcal{A},\ y \text{ unbarred},\ x < y < z,\\
&\mathsf{xyy} = \mathsf{yxy}, \quad \mathsf{yyz} = \mathsf{yzy} && \text{for } x, z \in \mathcal{A},\ y \text{ barred},\ x < y < z,
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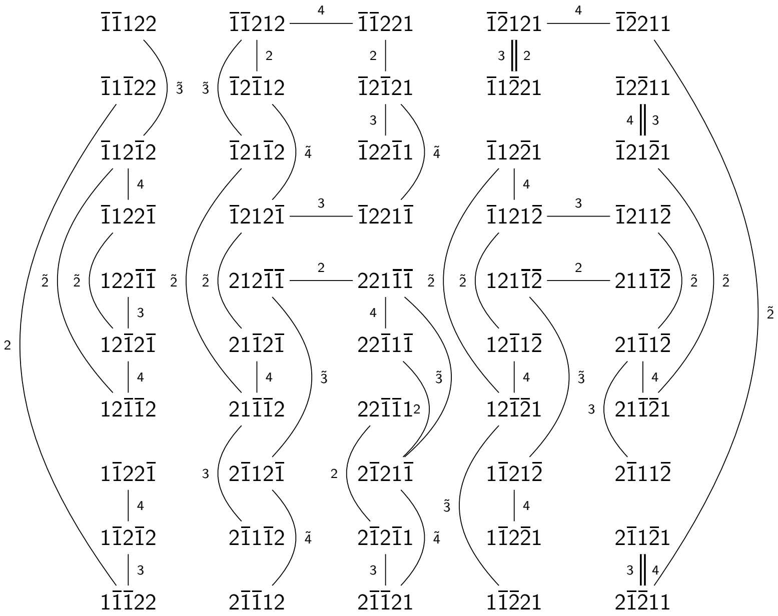
The set $\mathrm{CYW}_{(3,2),2}$

$\bar{1}\bar{1}122$	$\bar{1}\bar{1}212$	$\bar{1}\bar{1}221$	$\bar{1}\bar{2}121$	$\bar{1}\bar{2}211$
$\bar{1}1\bar{1}22$	$\bar{1}2\bar{1}12$	$\bar{1}2\bar{1}21$	$\bar{1}1\bar{2}21$	$\bar{1}2\bar{2}11$
$\bar{1}12\bar{1}2$	$\bar{1}21\bar{1}2$	$\bar{1}22\bar{1}1$	$\bar{1}12\bar{2}1$	$\bar{1}21\bar{2}1$
$\bar{1}122\bar{1}$	$\bar{1}212\bar{1}$	$\bar{1}221\bar{1}$	$\bar{1}121\bar{2}$	$\bar{1}211\bar{2}$
$122\bar{1}\bar{1}$	$212\bar{1}\bar{1}$	$221\bar{1}\bar{1}$	$121\bar{1}\bar{2}$	$211\bar{1}\bar{2}$
$12\bar{1}2\bar{1}$	$21\bar{1}2\bar{1}$	$22\bar{1}1\bar{1}$	$12\bar{1}1\bar{2}$	$21\bar{1}1\bar{2}$
$12\bar{1}\bar{1}2$	$21\bar{1}\bar{1}2$	$22\bar{1}\bar{1}1$	$12\bar{1}\bar{2}1$	$21\bar{1}\bar{2}1$
$1\bar{1}22\bar{1}$	$2\bar{1}12\bar{1}$	$2\bar{1}21\bar{1}$	$1\bar{1}21\bar{2}$	$2\bar{1}11\bar{2}$
$1\bar{1}2\bar{1}2$	$2\bar{1}1\bar{1}2$	$2\bar{1}2\bar{1}1$	$1\bar{1}2\bar{2}1$	$2\bar{1}1\bar{2}1$
$1\bar{1}\bar{1}22$	$2\bar{1}\bar{1}12$	$2\bar{1}\bar{1}21$	$1\bar{1}\bar{2}21$	$2\bar{1}\bar{2}11$



A super switchboard on the vertex set $\mathrm{CYW}_{(3,2),2}$

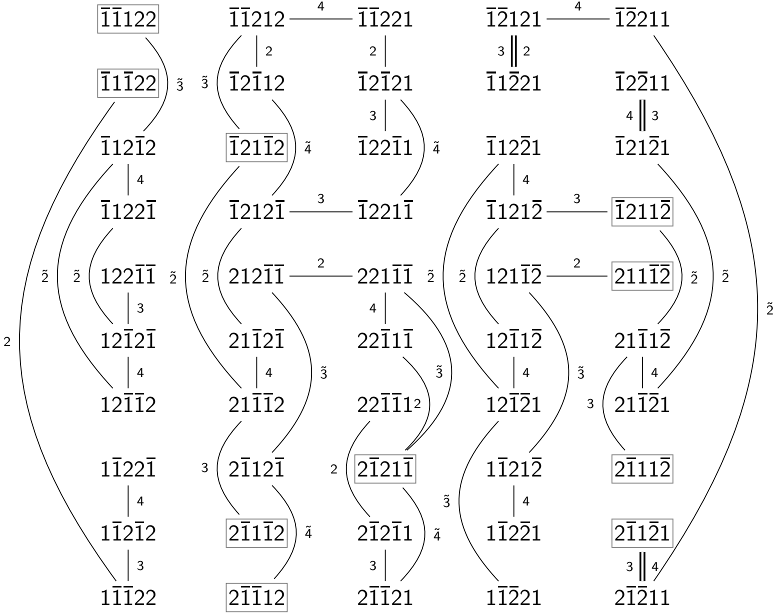
Main theorem: monomial positivity of $\bar{\mathfrak{J}}_\nu(\mathbf{u})$

Theorem (B.-Liu)

In the algebra $\overline{\mathcal{U}}/I_{\mathrm{Kron}}$, the noncommutative super Schur functions have the following monomial positive expression:

$$\bar{\mathfrak{J}}_\nu(\mathbf{u}) = \sum_{T \in \mathrm{CT}(\nu)} \mathrm{barread}(T) \qquad \textit{in } \overline{\mathcal{U}}/I_{\mathrm{Kron}}.$$

$\mathrm{CT}(\nu)$ denotes the set of colored tableaux of shape ν.



The words of the form $\mathrm{barread}(T)$ are outlined

Colored tableaux T such that $\mathrm{barread}(T) \in \mathrm{CYW}_{(3,2),2}$

$$\begin{array}{cccc} 1 & 1 & \bar{1} & \bar{2} \\ 2 & & & \end{array} \qquad \begin{array}{cccc} 1 & \bar{1} & 2 & 2 \\ \bar{1} & & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & \bar{2} \\ \bar{1} & 2 & \end{array} \qquad \begin{array}{ccc} 1 & \bar{1} & 2 \\ \bar{1} & 2 & \end{array}$$

$$2 1 1 \bar{1} \bar{2} \qquad \bar{1} 1 \bar{1} 2 2 \qquad \bar{1} 2 1 1 \bar{2} \qquad \bar{1} 2 1 \bar{1} 2$$

$$\begin{array}{ccc} 1 & \bar{1} & 2 \\ \bar{1} & & \\ 2 & & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 2 \\ \bar{1} & & \\ \bar{1} & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & \bar{2} \\ \bar{1} & & \\ 2 & & \end{array} \qquad \begin{array}{cc} 1 & \bar{1} \\ \bar{1} & 2 \\ 2 & \end{array} \qquad \begin{array}{cc} 1 & 1 \\ \bar{1} & \bar{2} \\ 2 & \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \bar{1} & \\ \bar{1} & \\ 2 & \end{array}$$

$$2 \bar{1} 1 \bar{1} 2 \qquad \bar{1} \bar{1} 1 2 2 \qquad 2 \bar{1} 1 1 \bar{2} \qquad 2 \bar{1} 2 1 \bar{1} \qquad 2 \bar{1} 1 \bar{2} 1 \qquad 2 \bar{1} \bar{1} 1 2$$

Colored tableaux T such that $\mathrm{barread}(T) \in \mathrm{CYW}_{(3,2),2}$

$\begin{array}{|c|c|c|c|}\hline 1 & 1 & \bar{1} & \bar{2} \\ \hline 2 \\ \cline{1-1}\end{array}$ $2 1 1 \bar{1} \bar{2}$

$\begin{array}{|c|c|c|c|}\hline 1 & \bar{1} & 2 & 2 \\ \hline \bar{1} \\ \cline{1-1}\end{array}$ $\bar{1} 1 \bar{1} 2 2$

$\begin{array}{|c|c|c|}\hline 1 & 1 & \bar{2} \\ \hline \bar{1} & 2 \\ \cline{1-2}\end{array}$ $\bar{1} 2 1 1 \bar{2}$

$\begin{array}{|c|c|c|}\hline 1 & \bar{1} & 2 \\ \hline \bar{1} & 2 \\ \cline{1-2}\end{array}$ $\bar{1} 2 1 \bar{1} 2$

$\begin{array}{|c|c|c|}\hline 1 & \bar{1} & 2 \\ \hline \bar{1} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ $2 \bar{1} 1 \bar{1} 2$

$\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \bar{1} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array}$ $\bar{1} \bar{1} 1 2 2$

$\begin{array}{|c|c|c|}\hline 1 & 1 & \bar{2} \\ \hline \bar{1} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ $2 \bar{1} 1 1 \bar{2}$

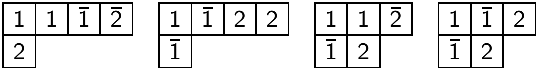
$\begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline \bar{1} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ $2 \bar{1} 2 1 \bar{1}$

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{1} & \bar{2} \\ \hline 2 \\ \cline{1-1}\end{array}$ $2 \bar{1} 1 \bar{2} 1$

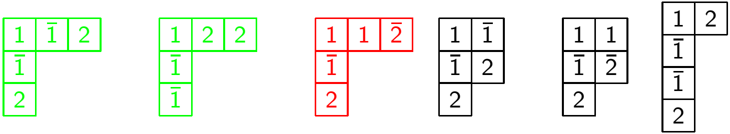
$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{1} \\ \cline{1-1} \bar{1} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$ $2 \bar{1} \bar{1} 1 2$

$$g_{(3,2)\,(3,1,1)\,(3,1,1)} + g_{(3,2)\,(4,1)\,(3,1,1)} = 3$$

Colored tableaux T such that $\mathrm{barread}(T) \in \mathrm{CYW}_{(3,2),2}$



$211\bar{1}\bar{2}$ $\bar{1}1\bar{1}22$ $\bar{1}211\bar{2}$ $\bar{1}21\bar{1}2$



$2\bar{1}1\bar{1}2$ $\bar{1}\bar{1}122$ $2\bar{1}11\bar{2}$ $2\bar{1}21\bar{1}$ $2\bar{1}1\bar{2}1$ $2\bar{1}\bar{1}12$

$g_{(3,2)\,(3,1,1)\,(3,1,1)} = 2$ $g_{(3,2)\,(4,1)\,(3,1,1)} = 1$

Corollary for Kronecker coefficients

Theorem (B.-Liu)

For any $\gamma \in (I_{\mathrm{Kron}})^{\perp}$, the function $F_\gamma(\mathbf{x})$ is symmetric and

$$F_\gamma(\mathbf{x}) = \sum_{\nu} s_\nu(\mathbf{x}) \langle \bar{\mathfrak{J}}_\nu(\mathbf{u}), \gamma \rangle.$$

Corollary

For any set of colored words W such that $\sum_{\mathsf{w} \in W} \mathsf{w} \in (I_{\mathrm{Kron}})^{\perp}$,

$$\begin{aligned} &\textit{the coefficient of } s_\nu(\mathbf{x}) \textit{ in } F_W(\mathbf{x}) \\ &= \left\langle \bar{\mathfrak{J}}_\nu(\mathbf{u}), \textstyle\sum_{\mathsf{w} \in W} \mathsf{w} \right\rangle \\ &= \left| \left\{ T \in \mathrm{CT}(\nu) : \mathrm{barread}(T) \in W \right\} \right|. \end{aligned}$$

Corollary (R. Liu)

For any partitions λ, ν of n and $d \leq n$,

$$g_{\lambda\, \mu(d)\, \nu} + g_{\lambda\, \mu(d-1)\, \nu} = \left| \left\{ T \in \mathrm{CT}(\nu) : \mathrm{barread}(T) \in \mathrm{CYW}_{\lambda, d} \right\} \right|.$$

Lascoux's heuristic for Kronecker coefficients

Γ_λ = the set of permutations obtained by standardizing the Yamanouchi words of content λ.

For partitions λ and μ, define the multiset of permutations

$$\Gamma_\lambda \circ \Gamma_\mu := \{ u \circ v \mid u \in \Gamma_\lambda, v \in \Gamma_\mu \},$$

where $\circ$ denotes ordinary composition of permutations.

$\Gamma_\lambda \circ \Gamma_\mu$ for $\lambda = (3,1)$, $\mu = (2,1,1)$

$\circ$ (Γ_λ \ Γ_μ)	4312	4132	1432
4123	3241	3421	4321
1423	3214	3124	1324
1243	3412	3142	1342

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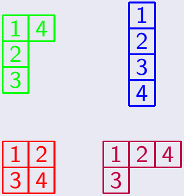
	$\circ$	4312 (Γ_μ)	4132 (Γ_μ)	1432 (Γ_μ)
Γ_λ	4123	3241	3421	4321
Γ_λ	1423	3214	3124	1324
Γ_λ	1243	3412	3142	1342



Lascoux's heuristic for Kronecker coefficients

Theorem (Lascoux)

If λ and μ are hook shapes, then $\Gamma_\lambda \circ \Gamma_\mu$ is a union of Knuth equivalence classes, and the Kronecker coefficient $g_{\lambda\mu\nu}$ is the number of these classes with insertion tableau of shape ν.

Though this rule no longer holds outside the hook-hook case, it seems to approximate Kronecker coefficients amazingly well for any three partitions and therefore gives a useful heuristic.

Lascoux's heuristic for Kronecker coefficients

$\lambda \backslash \mu$	6	51	42	411	33	321	3111	222	2211	21^4	1^6
6	0	0	0	0	0	1	0	0	0	0	0
51	0	0	1	1	1	2	1	1	1	0	0
42	0	1	2	2	1	3	2	1	2	1	0
411	0	1	2	2	1	4	2	1	2	1	0
33	0	1	1	1	0	2	1	0	1	1	0
321	1	2	3	4	2	5	4	2	3	2	1
3111	0	1	2	2	1	4	2	1	2	1	0
222	0	1	1	1	0	2	1	0	1	1	0
2211	0	1	2	2	1	3	2	1	2	1	0
21^4	0	0	1	1	1	2	1	1	1	0	0
1^6	0	0	0	0	0	1	0	0	0	0	0

$g_{\lambda\,\mu\,(3,2,1)}$

Lascoux's heuristic for Kronecker coefficients

$\lambda \backslash \mu$	6	51	42	411	33	321	3111	222	2211	21^4	1^6
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51	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0
411	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	−1	1	0	0	0	0	0
321	0	0	0	0	1	1	0	0	0	0	0
3111	0	0	0	0	0	0	0	0	0	0	0
222	0	0	0	0	0	0	0	0	0	0	0
2211	0	0	0	0	0	0	0	0	0	0	0
21^4	0	0	0	0	0	0	0	0	0	0	0
1^6	0	0	0	0	0	0	0	0	0	0	0

$$g_{\lambda\,\mu\,(3,2,1)} - \left|\left\{ \mathsf{w} \in \Gamma_\lambda \circ \Gamma_\mu : Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \right\}\right|$$

Lascoux's heuristic for Kronecker coefficients

$\lambda \backslash \mu$	7	61	52	511	43	421	4111	331	322	3211	31^4	2221	221^3	21^5	1^7
7	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
61	0	0	0	0	1	1	0	1	1	1	0	0	0	0	0
52	0	0	1	1	1	2	1	2	1	2	0	1	1	0	0
511	0	0	1	0	1	3	1	1	2	2	1	1	0	0	0
43	0	1	1	1	1	2	1	1	1	2	1	1	1	0	0
421	0	1	2	3	2	5	3	3	3	5	2	2	2	1	0
4111	0	0	1	1	1	3	2	2	2	3	1	1	1	0	0
331	1	1	2	1	1	3	2	1	2	3	2	1	1	1	0
322	0	1	1	2	1	3	2	2	1	3	1	1	2	1	1
3211	0	1	2	2	2	5	3	3	3	5	3	2	2	1	0
31^4	0	0	0	1	1	2	1	2	1	3	0	1	1	0	0
2221	0	0	1	1	1	2	1	1	1	2	1	1	1	1	0
221^3	0	0	1	0	1	2	1	1	2	2	1	1	1	0	0
21^5	0	0	0	0	0	1	0	1	1	1	0	1	0	0	0
1^7	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

$g_{\lambda\,\mu\,(3,3,1)}$

Lascoux's heuristic for Kronecker coefficients

$\lambda \backslash \mu$	7	61	52	511	43	421	4111	331	322	3211	31^4	2221	221^3	21^5	1^7
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52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
511	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0
421	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
4111	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
331	0	0	0	0	0	0	0	−1	1	0	0	0	0	0	0
322	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
3211	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
31^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2221	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0
221^3	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
21^4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1^7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$g_{\lambda\,\mu\,(3,3,1)} - \left| \left\{ w \in \Gamma_\lambda \circ \Gamma_\mu : Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & & \\ \hline \end{array} \right\} \right|$$

Lascoux's heuristic for Kronecker coefficients

When only μ is a hook, $\Gamma_\lambda \circ \Gamma_\mu$ is not in general a union of Knuth equivalence classes, but

Proposition

If μ is a hook, the quasisymmetric function $F_{\Gamma_\lambda \circ \Gamma_\mu}(\mathbf{x}) := \sum_{w \in \Gamma_\lambda \circ \Gamma_\mu} Q_{\mathrm{Des(w)}}(\mathbf{x})$ is equal to $\sum_\nu g_{\lambda\mu\nu} s_\nu(\mathbf{x})$.

$$\mathrm{CYW}_{\lambda,d} \xrightarrow{\text{standardize}} (\Gamma_\lambda \circ \Gamma_{\mu(d)}) \sqcup (\Gamma_\lambda \circ \Gamma_{\mu(d-1)}).$$

The theory of noncommutative super Schur functions can then be used to go from this quasisymmetric function expansion to the Schur function expansion, thereby obtaining a formula for $g_{\lambda\mu\nu}$.

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