Noncommutative Schur functions

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Based on joint work with Sergey Fomin (University of Michigan) and Ricky Liu (North Carolina State University).

October 2015

Positivity problems in algebraic combinatorics

Positivity problems in algebraic combinatorics ask to find positive combinatorial formulae for nonnegative quantities arising in geometry and representation theory.

Key example of a positive combinatorial formula

The Littlewood-Richardson rule is a positive combinatorial formula for Littlewood-Richardson coefficients, the decomposition multiplicities of a tensor product of irreducible representations of GL_n.

Two important unsolved positivity problems

Find a positive combinatorial formula for

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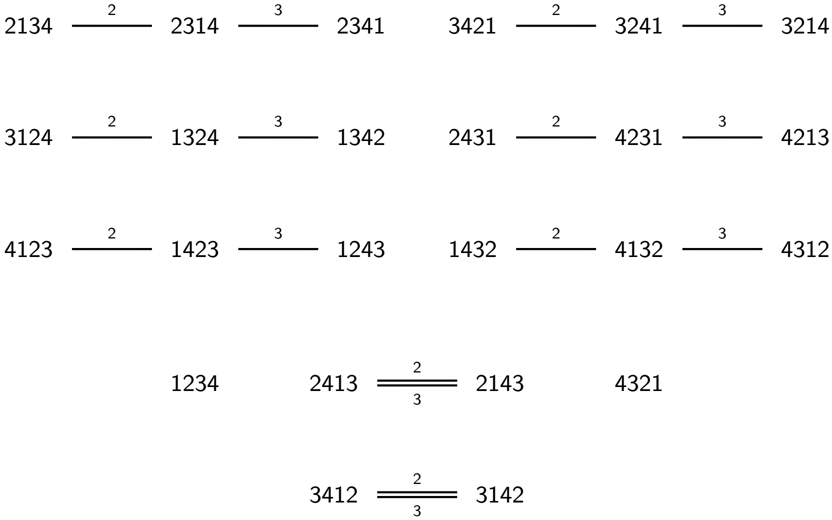
Noncommutative Schur functions

Noncommutative Schur functions are a powerful tool for solving positivity problems. They have led to positive combinatorial formulae for

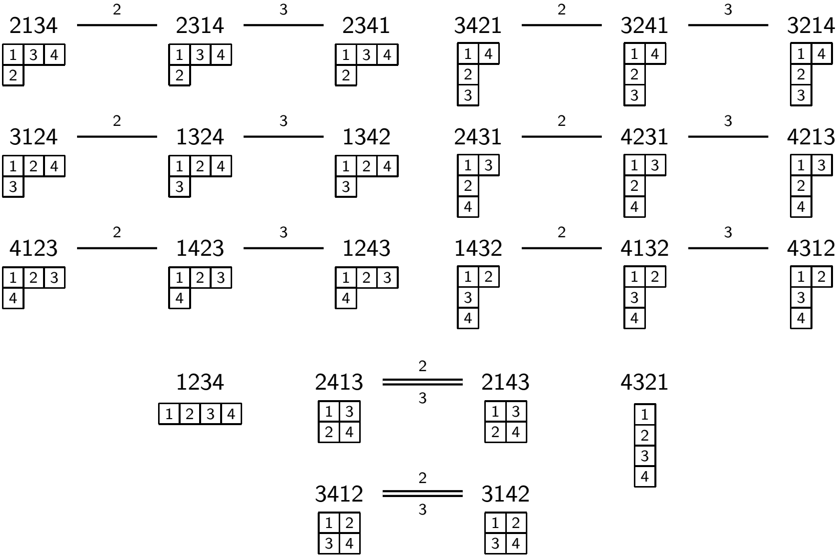
- the Schur expansion of Stanley symmetric functions and stable Grothendieck polynomials.
- the Schur expansion of LLT polynomials indexed by 3-tuples of skew shapes and transformed Macdonald polynomials indexed by shapes with 3 columns.
- Kronecker coefficients for one hook shape and two arbitrary shapes.

The Knuth equivalence graph on $\mathcal{S}_4$

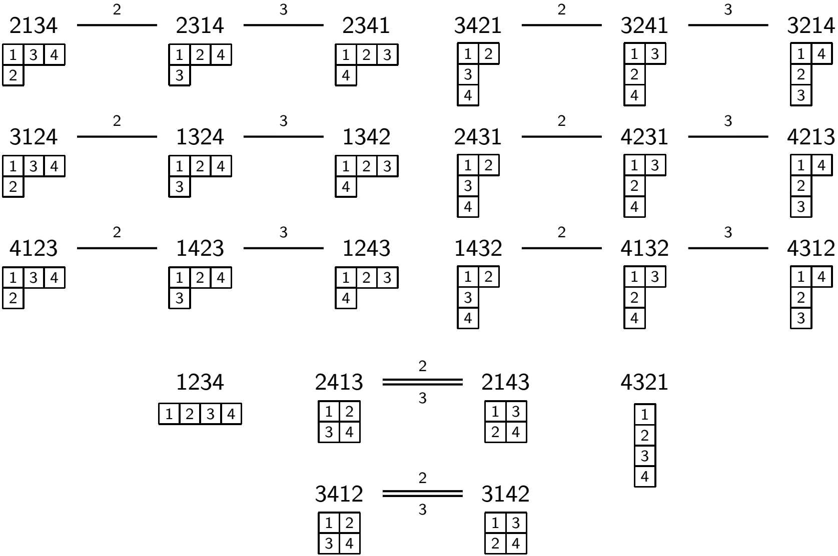


Edges are Knuth transformations $\cdots \mathsf{bac} \cdots \leftrightarrow \cdots \mathsf{bca} \cdots$, $\cdots \mathsf{acb} \cdots \leftrightarrow \cdots \mathsf{cab} \cdots$.

Knuth equivalence graph on $\mathcal{S}_4$ with insertion tableaux



Knuth equivalence graph on $\mathcal{S}_4$ with recording tableaux



Switchboards

Definition

Words $w = w_1 \cdots w_n$ and $w' = w'_1 \cdots w'_n$ are related by a *switch* in position i if $w_j = w'_j$ for $j \notin \{i-1, i, i+1\}$, while the unordered pair $\{w_{i-1}w_iw_{i+1}, w'_{i-1}w'_iw'_{i+1}\}$ fits one of the following patterns:

- $\{bac, bca\}$ or $\{acb, cab\}$, $a < b < c$ (a *Knuth switch*);
- $\{bac, acb\}$ or $\{bca, cab\}$, $a < b < c$ (a *rotation switch*);
- $\{bab, bba\}$ or $\{aba, baa\}$, $a < b$ (a *Knuth switch*);
- $\{bab, aba\}$ or $\{bba, baa\}$, $b = a + 1$ (a *braid/idempotent switch*).

Definition

A *switchboard* is an edge-labeled graph on a vertex set of words of fixed length in the alphabet $\{1, 2, \ldots, N\}$ such that

- each i-edge corresponds to a switch in position i;
- each vertex which has exactly one descent in positions $i-1$ and i belongs to exactly one i-edge.

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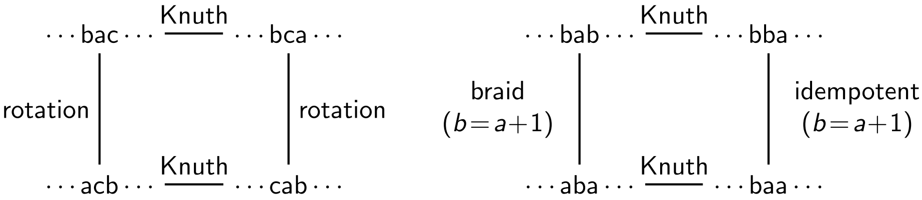
Switches of different types



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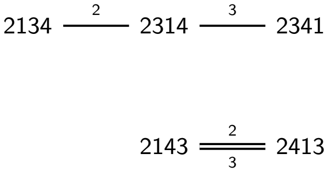


Figure : A switchboard whose 3-edges are Knuth switches.

Switchboards

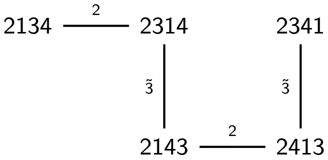


Figure : A switchboard whose 3-edges are rotation switches.

Switchboards

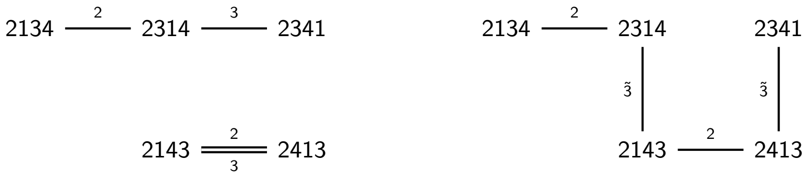
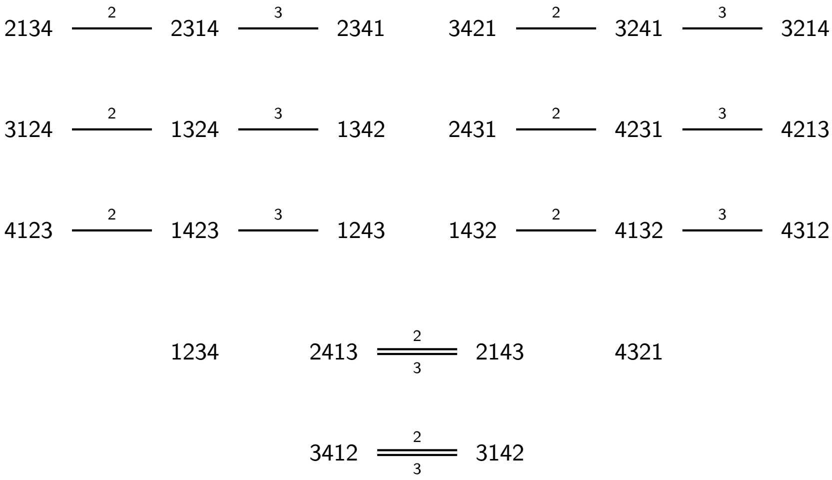
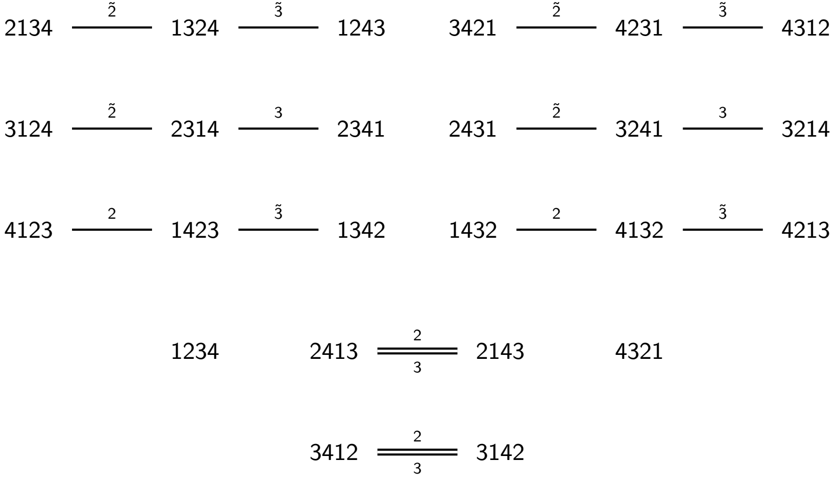


Figure : Different switchboards on the same set of vertices.

The switchboard on $\mathcal{S}_4$ with only Knuth switches



The switchboard on $\mathcal{S}_4$ with rotation switches for $c - a \leq 2$ and Knuth switches otherwise.



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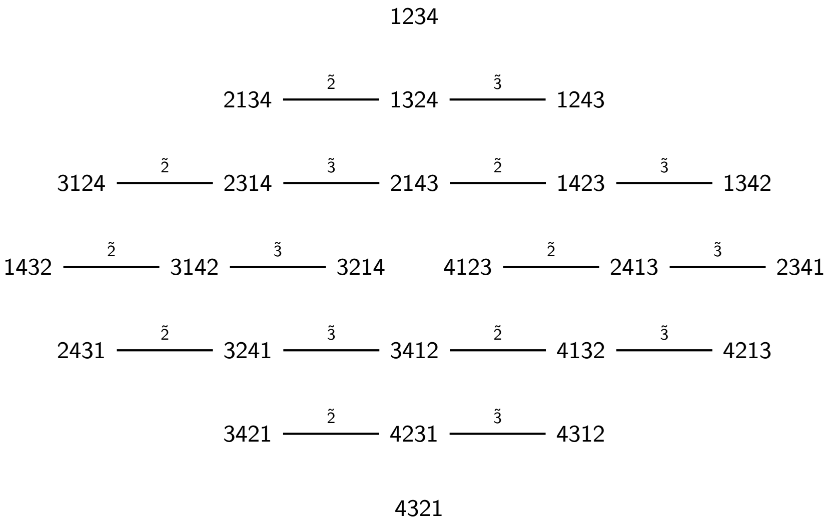

The symmetric function associated to a switchboard

- Let $\mathbf{x} = (x_1, x_2, \dots)$ be commuting variables.
- $\mathrm{Des}(\mathrm{w}) = \{i \in \{1, \dots, n-1\} \mid \mathrm{w}_i > \mathrm{w}_{i+1}\}$ denotes the *descent set* of a word $\mathrm{w} = \mathrm{w}_1 \cdots \mathrm{w}_n$.
- Gessel's *fundamental quasisymmetric function* is

$$Q_{\mathrm{Des}(\mathrm{w})}(\mathbf{x}) = \sum_{\substack{1 \le i_1 \le \cdots \le i_n \\ j \in \mathrm{Des}(\mathrm{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

- For a switchboard Γ, define

$$F_\Gamma(\mathbf{x}) = \sum_{\mathrm{w} \in \mathrm{Vert}(\Gamma)} Q_{\mathrm{Des}(\mathrm{w})}(\mathbf{x}).$$

Theorem (B.-Fomin)

For any switchboard Γ, the function $F_\Gamma(\mathbf{x})$ is symmetric in $x_1, x_2, \dots$.

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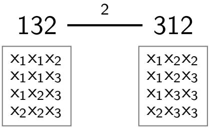

Figure : A switchboard Γ with $F_\Gamma = s_{21}$. The sum of the outlined monomials is $s_{21}(x_1, x_2, x_3)$.

Figure : A switchboard Γ' with $F_{\Gamma'} = s_{31} + s_{22}$. The sum of the outlined monomials is $s_{31}(x_1, x_2, x_3) + s_{22}(x_1, x_2, x_3)$.

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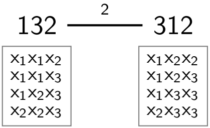

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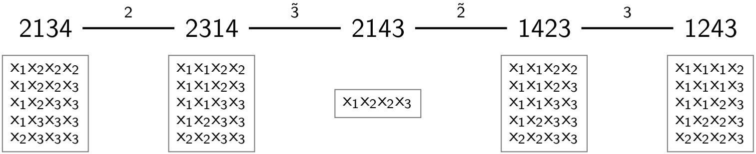

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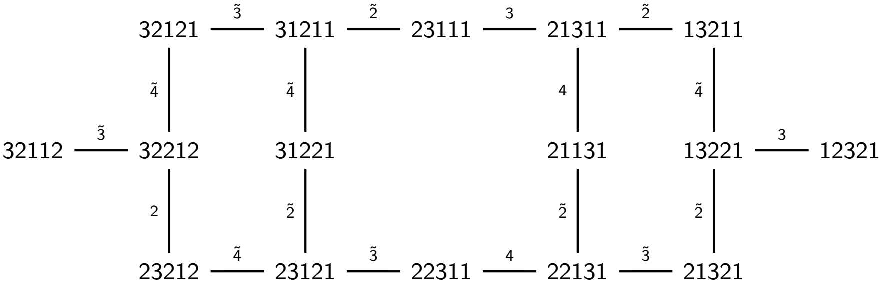

Figure : A switchboard Γ with $F_\Gamma = s_{32} + s_{311} + s_{221}$.

Switchboards and Schur positivity

Big question

For which switchboards Γ is F_Γ Schur positive?

Switchboards and Schur positivity

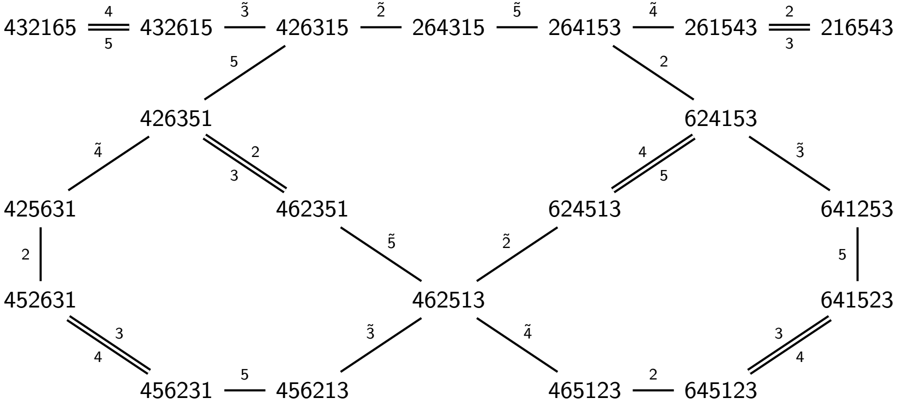

Figure : A switchboard Γ with $F_\Gamma = s_{321} + s_{2211} - s_{222}$.

Noncommutative Schur functions

Let $\mathcal{U} = \mathbb{Z}\langle u_1, u_2, \ldots, u_N \rangle$. We identify the monomials in $\mathcal{U}$ with words in the alphabet $[N]$ and write $312, \text{cab}$, etc. for words/monomials.

Definition (Noncommutative elementary symmetric functions)

$$e_k(\mathbf{u}) = \sum_{N \geq i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1} u_{i_2} \cdots u_{i_k}$$

for any positive integer k; set $e_0(\mathbf{u}) = 1$ and $e_k(\mathbf{u}) = 0$ for $k < 0$.

Definition (Noncommutative Schur functions $\mathfrak{J}_\lambda(\mathbf{u})$)

- $\lambda = (\lambda_1, \lambda_2, \ldots)$ is an integer partition.
- λ' is the conjugate partition.
- $t = \lambda_1$ is the number of parts of λ'.

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{\pi \in \mathcal{S}_t} \mathrm{sgn}(\pi)\, e_{\lambda'_1+\pi(1)-1}(\mathbf{u})\, e_{\lambda'_2+\pi(2)-2}(\mathbf{u}) \cdots e_{\lambda'_t+\pi(t)-t}(\mathbf{u}).$$

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Schur expansion of $F_\gamma(\mathbf{x})$

- For $\gamma = \sum_{\text{words w}} \gamma_{\text{w}} \text{w} \in \mathcal{U}$, define $F_\gamma(\mathbf{x}) = \sum_{\text{words w}} \gamma_{\text{w}} \, Q_{\mathrm{Des}(\text{w})}(\mathbf{x})$.
- We consider $\mathcal{U}$ to be endowed with the symmetric bilinear form for which the monomials/words form an orthonormal basis.
- Note that any element of $\mathcal{U}/I$ has a well-defined pairing with any element of $I^\perp$, for any ideal I of $\mathcal{U}$.

Theorem (Fomin-Greene)

Let I be an ideal of $\mathcal{U}$ such that the $e_k(\mathbf{u})$ commute pairwise in $\mathcal{U}/I$. Then for any $\gamma \in I^\perp$,

$$F_\gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x}) \langle \mathfrak{J}_\lambda(\mathbf{u}), \gamma \rangle.$$

The switchboard ideal

Let $\mathcal{U}/I_{\mathrm{S}}$ be the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned} &\mathrm{b}(\mathrm{ca}-\mathrm{ac}) = (\mathrm{ca}-\mathrm{ac})\mathrm{b} && \text{for } a<b<c,\\ &\mathrm{caa}=\mathrm{aca}, \quad \mathrm{cca}=\mathrm{cac} && \text{for } c-a>1,\\ &\mathrm{bab}+\mathrm{baa}=\mathrm{bba}+\mathrm{aba} && \text{for } b=a+1. \end{aligned}$$

Theorem (A. N. Kirillov, B.-Fomin)

The $e_k(\mathbf{u})$ commute pairwise in $\mathcal{U}/I_{\mathrm{S}}$.

Proposition

For a set of words W of the same length, the following are equivalent:

- $\sum_{\mathrm{w}\in W} \mathrm{w} \in I_{\mathrm{S}}^{\perp}$;
- *W is the vertex set of a switchboard.*

Corollary

For any switchboard Γ, $F_\Gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x})\langle \mathfrak{J}_\lambda(\mathbf{u}), \sum_{\mathrm{w}\in\mathrm{Vert}(\Gamma)} \mathrm{w}\rangle$.

Recipe for positive combinatorial formulae

We obtain the following recipe for producing positive combinatorial formulae for the coefficients in the Schur expansions of many classes of symmetric functions:

(1) Find a suitable ideal $I \supset I_{\mathrm{S}}$ such that symmetric functions of interest are of the form $F_\Gamma(\mathbf{x})$ for $\gamma = \sum_{\mathrm{w} \in \mathrm{Vert}(\Gamma)} \mathrm{w} \in I^\perp$.

(2) Find a monomial positive expression for $\mathfrak{J}_\lambda(\mathbf{u})$ in $\mathcal{U}/I$.

(3) Apply $(*)$ to obtain a positive formula for the Schur expansion of F_Γ.

$$F_\Gamma(\mathbf{x}) = F_\gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x}) \langle \mathfrak{J}_\lambda(\mathbf{u}), \gamma \rangle \qquad (*)$$

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Monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

Big question

- For which ideals $I \supset I_{\mathrm{S}}$ of $\mathcal{U}$ does $\mathfrak{J}_\lambda(\mathbf{u})$ have a monomial positive expression in $\mathcal{U}/I$ for all λ?

Monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$ (first example)

Let $\mathcal{U}/I$ be the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned} &\mathsf{ab} = \mathsf{ba} && \text{for all } a, b \in [N], \\ &\mathsf{v} && \text{for words } \mathsf{v} \text{ with a repeated letter.} \end{aligned}$$

Example ($\mathfrak{J}_{(2,2)}(\mathbf{u})$ in $\mathcal{U}/I$)

Computing modulo I,

$$\begin{aligned} \mathfrak{J}_{(2,2)}(\mathbf{u}) &= e_2(\mathbf{u})e_2(\mathbf{u}) - e_3(\mathbf{u})e_1(\mathbf{u}) \\ &\equiv 2143 + 3142 + 3241 + 4132 + 4231 + 4321 \\ &- 3214 - 4213 - 4312 - 4321 \\ &\equiv 1234 + 1234. \end{aligned}$$

Monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$ (second example)

Let $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{plac}}$ be the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned} &\mathrm{acb} = \mathrm{cab}, \quad \mathrm{bac} = \mathrm{bca} && \text{for } a < b < c, \\ &\mathrm{v} && \text{for words v with a repeated letter.} \end{aligned}$$

Example ($\mathfrak{J}_{(2,2)}(\mathbf{u})$ in $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{plac}}$)

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Monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$ (third example)

Let $\mathcal{U}/I_{\mathrm{S}}^{\mathrm{st}}$ be the quotient of $\mathcal{U}$ by the relations

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Example ($\mathfrak{J}_{(2,2)}(\mathbf{u})$ in $\mathcal{U}/I_{\mathrm{S}}^{\mathrm{st}}$)

Computing modulo $I_{\mathrm{S}}^{\mathrm{st}}$,

$$\begin{aligned}\mathfrak{J}_{(2,2)}(\mathbf{u}) &= 2143+3142+3241+4132+4231+4321\\ &\quad -3214-4213-4312-4321\\ &= 2143+(3412+3[1,4]2)+32[4,1]+4[1,3]2+42[3,1]\\ &\equiv 2143+3412+32[1,4]+32[4,1]+42[1,3]+42[3,1]\\ &= 2143+3412.\end{aligned}$$

Fact

This is the only monomial positive expression for $\mathfrak{J}_{(2,2)}(\mathbf{u})$ in $\mathcal{U}/I_{\mathrm{S}}^{\mathrm{st}}$.

Monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$ (third example)

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$$\begin{aligned} &\mathsf{b}(\mathsf{ca}-\mathsf{ac}) = (\mathsf{ca}-\mathsf{ac})\mathsf{b} && \text{for } a<b<c, \\ &\mathsf{v} && \text{for words v with a repeated letter.} \end{aligned}$$

Example ($\mathfrak{J}_{(2,2)}(\mathbf{u})$ in $\mathcal{U}/I_{\mathrm{S}}^{\mathrm{st}}$)

Computing modulo $I_{\mathrm{S}}^{\mathrm{st}}$,

$$\begin{aligned} \mathfrak{J}_{(2,2)}(\mathbf{u}) &= 2143 + 3142 + 3241 + 4132 + 4231 + 4321 \\ &\quad - 3214 - 4213 - 4312 - 4321 \\ &= 2143 + (3412 + 3[1,4]2) + 32[4,1] + 4[1,3]2 + 42[3,1] \\ &\equiv 2143 + 3412 + 32[1,4] + 32[4,1] + 42[1,3] + 42[3,1] \\ &= 2143 + 3412. \end{aligned}$$

Fact

This is the only monomial positive expression for $\mathfrak{J}_{(2,2)}(\mathbf{u})$ in $\mathcal{U}/I_{\mathrm{S}}^{\mathrm{st}}$.

First theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

The *column reading word* $\mathrm{colword}(T)$ of a tableau T is the word obtained by concatenating the columns of T (reading each column bottom to top), starting with the leftmost column. For example,

$$\mathrm{colword}\left(\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}\right) = 5216217322.$$

First theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

Let $\mathcal{U}/I_{\oslash}$ be the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned} &\mathrm{acb} = \mathrm{cab}, \quad \mathrm{bac} = \mathrm{bca} && \text{for } a < b < c, \\ &\mathrm{caa} = \mathrm{aca}, \quad \mathrm{cca} = \mathrm{cac} && \text{for } c - a > 1, \\ &\mathrm{bab} + \mathrm{baa} = \mathrm{bba} + \mathrm{aba} && \text{for } b = a + 1. \end{aligned}$$

Theorem (Fomin-Greene)

The noncommutative Schur functions have the following monomial positive expression in $\mathcal{U}/I_{\oslash}$:

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathrm{colword}(T) \quad \text{in } \mathcal{U}/I_{\oslash}.$$

Second theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

For a partition λ, let $\mathrm{RSST}(\lambda)$ denote the set of semistandard Young tableaux T of shape λ satisfying:

- the entries strictly increase across the rows and down the columns;
- the entries increase in increments of at least 3 along diagonals.

For $T \in \mathrm{RSST}(\lambda)$, we define the word $\mathrm{sqread}(T)$ by reading the entries of T as follows: first circle each entry c of T such that the entry immediately west of c is $c - 1$. Then read the diagonals of T, starting from the southwest corner and finishing at the northeast corner. In each diagonal, first read the circled entries in the direction $\nwarrow$, then the uncircled entries in the direction $\searrow$.

Example

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 4 & 5 & 7 \\ \hline 5 & 6 & 8 & 9 \\ \hline \end{array}$$

Second theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

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Example

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1	②	4	6
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5	⑥	8	⑨

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$$\mathrm{sqread}(T) = 563418952476$$

Second theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

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Second theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

Definition (Lam's algebra of ribbon Schur operators)

Define $\mathcal{U}_{\mathbb{Q}(q)}/I_{\mathrm{L},k}$ to be the quotient of $\mathcal{U}_{\mathbb{Q}(q)} := \mathbb{Q}(q) \otimes_{\mathbb{Z}} \mathcal{U}$ by the relations

$$\begin{aligned}
\mathrm{ac} &= \mathrm{ca} && \text{for } c - a > k,\\
\mathrm{ba} &= q\,\mathrm{ab} && \text{for } 0 < b - a < k,\\
\mathrm{cac} &= \mathrm{aca} = \mathrm{aa} = 0 && \text{for } c = a + k.
\end{aligned}$$

Theorem (Lam)

LLT polynomials are Schur positive if and only if the noncommutative Schur functions $\mathfrak{J}_\lambda(\mathbf{u})$ are monomial positive in $\mathcal{U}_{\mathbb{Q}(q)}/I_{\mathrm{L},k}$ for all k.

Theorem (B.)

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{T \in \mathrm{RSST}(\lambda)} \mathrm{sqread}(T) \quad \textit{in } \mathcal{U}_{\mathbb{Q}(q)}/I_{\mathrm{L},3}.$$

Second theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

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Corollary for LLT polynomials

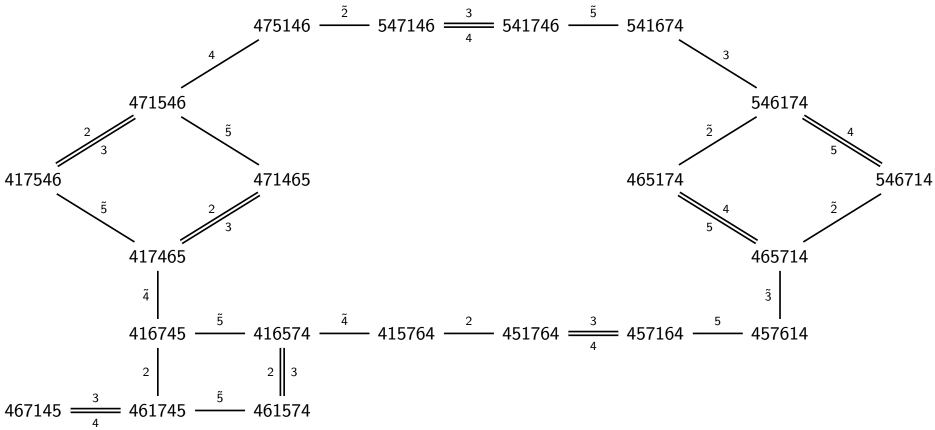

Figure : A switchboard Γ with $F_\Gamma = s_{33} + s_{321}$. This is the coefficient of q^3 in the LLT polynomial indexed by the 3-tuple $(3/2, 33/11, 2/1)$.

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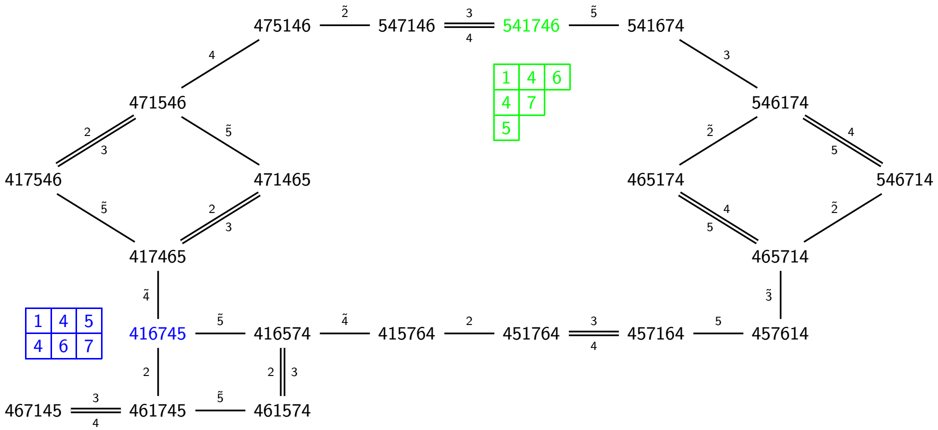

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Third theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

- Let $\mathcal{A}_\varnothing = \{1, 2, \ldots, N\}$ denote the alphabet of unbarred letters and $\mathcal{A}_- = \{\overline{1}, \overline{2}, \ldots, \overline{N}\}$ the alphabet of barred letters.
- We will work with the alphabet $\mathcal{A} = \mathcal{A}_\varnothing \sqcup \mathcal{A}_-$ ordered by $1 < \overline{1} < 2 < \overline{2} < \cdots < N < \overline{N}$.
- A *colored tableau* is a tableau with entries in $\mathcal{A}$ such that each row and column is weakly increasing with respect to the order $<$, while the unbarred letters in each column and the barred letters in each row are strictly increasing.

Third theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

For a colored tableau T, the word $\mathrm{barread}(T)$ is obtained by reading the diagonals of T one by one, starting from the southwest corner and finishing at the northeast corner. In each diagonal, we first read the unbarred entries in the direction $\nwarrow$, followed by the barred entries in the direction $\searrow$.

Example

$$\mathrm{barread}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \overline{3} & \overline{4} & 6 \\ \hline \overline{2} & 3 & 4 & \overline{4} & \\ \cline{1-4} 3 & \overline{3} & \overline{4} & 5 & \\ \cline{1-4} \end{array}\right) = 3\,\overline{2}\,\overline{3}\,3\,1\,\overline{4}\,5\,4\,1\,\overline{3}\,\overline{4}\,\overline{4}\,6.$$

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For a colored tableau T, the word $\mathrm{barread}(T)$ is obtained by reading the diagonals of T one by one, starting from the southwest corner and finishing at the northeast corner. In each diagonal, we first read the unbarred entries in the direction $\nwarrow$, followed by the barred entries in the direction $\searrow$.

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$$\mathrm{barread}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \bar{3} & \bar{4} & 6 \\ \hline \bar{2} & 3 & 4 & \bar{4} & \\ \cline{1-4} 3 & \bar{3} & \bar{4} & 5 & \\ \cline{1-4}\end{array}\right) = 3\,\bar{2}\,\bar{3}\,3\,1\,\bar{4}\,5\,4\,1\,\bar{3}\,\bar{4}\,\bar{4}\,6.$$

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$$\mathrm{barread}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \overline{3} & \color{gray}{\overline{4}} & \color{gray}{6} \\ \hline \overline{2} & 3 & 4 & \overline{4} \\ \cline{1-4} 3 & \overline{3} & \overline{4} & 5 \\ \cline{1-4} \end{array}\right) = 3\,\overline{2}\,\overline{3}\,3\,1\,\overline{4}\,5\,4\,1\,\overline{3}\,\overline{4}\,\color{gray}{\overline{4}\,6}.$$

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Third theorem on the monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

Let $\overline{\mathcal{U}}/I_{\mathrm{Kron}}$ be the quotient of $\overline{\mathcal{U}}$ by the relations

$$
\begin{array}{lll}
(\mathsf{xz}-\mathsf{zx})\mathsf{y}=\mathsf{y}(\mathsf{xz}-\mathsf{zx}) & & \text{for } x,y,z\in\mathcal{A},\\
\mathsf{xz}=\mathsf{zx} & & \text{for } x,z\in\mathcal{A},\ x<z-1,\\
\mathsf{yyx}=\mathsf{yxy}, & \mathsf{zyy}=\mathsf{yzy} & \text{for } x,z\in\mathcal{A},\ y\in\mathcal{A}_\varnothing,\ x<y<z,\\
\mathsf{xyy}=\mathsf{yxy}, & \mathsf{yyz}=\mathsf{yzy} & \text{for } x,z\in\mathcal{A},\ y\in\mathcal{A}_-,\ x<y<z,
\end{array}
$$

where for a barred letter $\overline{a}$, we define $\overline{a}-1$ to be the barred letter $\overline{a-1}$.

Theorem (B.-Liu)

In the algebra $\overline{\mathcal{U}}/I_{\mathrm{Kron}}$, the noncommutative super Schur functions have the following monomial positive expression:

$$\mathfrak{J}_\nu(\overline{\mathbf{u}})=\sum_{T\in\mathrm{CT}(\nu)}\mathrm{barread}(T)\qquad \textit{in } \overline{\mathcal{U}}/I_{\mathrm{Kron}}.$$

Corollaries on positive combinatorial formulae

Recall: for any switchboard Γ, $F_\Gamma(\mathbf{x}) = \sum_\lambda s_\lambda(\mathbf{x})\langle \mathfrak{J}_\lambda(\mathbf{u}), \sum_{\mathrm{w}\in \mathrm{Vert}(\Gamma)} \mathrm{w}\rangle$.

The previous theorems yield positive combinatorial formulae for

- the Schur expansion of Stanley symmetric functions and stable Grothendieck polynomials.
- the Schur expansion of LLT polynomials indexed by 3-tuples of skew shapes and transformed Macdonald polynomials indexed by shapes with 3 columns.
- Kronecker coefficients for one hook shape and two arbitrary shapes.

Recent Papers

- J. Blasiak and R. Liu. *Kronecker coefficients and noncommutative super Schur functions*. October 2015, `arXiv:1510.00644`.
- J. Blasiak and S. Fomin. *Noncommutative Schur functions, switchboards, and Schur positivity*. October 2015, `arXiv:1510.00657`.
- J. Blasiak. *Haglund's conjecture on 3-column Macdonald polynomials*. November 2014, `arXiv:1411.3646`.
- J. Blasiak. *What makes a D_0 graph Schur positive?* November 2014, `arXiv:1411.3624`.

Schur positivity of $F_\gamma(\mathbf{x})$ vs. monomial positivity of $\mathfrak{J}_\lambda(\mathbf{u})$

Let $\mathcal{U}_{\geq 0} \subset \mathcal{U}$ denote the set of all nonnegative integer linear combinations of monomials.

Theorem (B.-Fomin)

For a homogeneous ideal $I \subset \mathcal{U}$ such that the $e_k(\mathbf{u})$ commute pairwise in $\mathcal{U}/I$, the following are equivalent:

- *all symmetric functions $F_\gamma(\mathbf{x})$, for $\gamma \in \mathcal{U}_{\geq 0} \cap I^\perp$, are Schur positive;*
- *all noncommutative Schur functions $\mathfrak{J}_\lambda(\mathbf{u})$ have a monomial positive expression in $\mathbb{Q} \otimes \mathcal{U}/I$.*