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Interaction Manifolds for Reaction Diffusion Equations in Two Dimensions*

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Abstract. We consider a general planar reaction diffusion equation which we hypothesize has a localized traveling wave solution. Under assumptions which are no stronger than those needed to prove the stability of a single pulse, we prove that the PDE has solutions which are roughly the linear superposition of two pulses, so long as they move along trajectories which are not parallel. In particular, we prove that if the initial data for the equation is close to the sum of two separated pulses, then the solution converges exponentially fast to such a superposition so long as the distance between the two pulses remains sufficiently large.

Key words. reaction diffusion equations, multiuse solutions, traveling wave interactions, nonautonomous differential equations

AMS subject classifications. 35K57, 35K40, 35K55, 37L15

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1. Introduction.

1.1. The system and hypotheses.

Given the existence and stability of localized traveling wave solutions (a.k.a. pulses) to a reaction diffusion equation, one expects that there are solutions (called *multipulse* solutions) which are nearly the linear superposition of two or more such pulses, at least until such time as those pulses come close to one another. There are rigorous justifications for this conjecture in a number of settings, most of which are posed for one spatial variable. [2, 8, 9, 13, 19] treat existence and stability of multipulse standing solutions and use primarily dynamical systems and geometric techniques. [5, 10, 11, 12, 15, 23] handle counterpropagating fronts and pulses and give very complete descriptions of interactions by means of the maximum principle. In [7, 24], the authors handle long-distance weak interactions between standing pulses—in particular, they compute the induced motion between the pulses. [24] is particularly notable here in that it covers any number of pulses in any spatial dimension. Our work is most closely related to that of [4, 20, 22], which deal with counterpropagating fronts or pulses in reaction diffusion systems. The chief difficulty in this case is the fact that the problem cannot be made autonomous by transforming it into a moving reference frame.

For planar reaction diffusion systems, a host of numerical simulations bear out the expectation (see [16, 18]) that there are solutions which are roughly the sum of pulses moving in different directions, and in this paper we prove the corresponding rigorous results. Here we

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consider a general reaction diffusion system:

$$u_t = \mathcal{L}u + F(u), \tag{1.1}$$

where

$$\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2, \quad u(\mathbf{x}, t) \in \mathbf{R}^N$$

and

$$F \in C^2 \text{ with } F(0) = 0, \quad \text{and} \quad \mathcal{L} \text{ is a constant coefficient second order elliptic operator.}$$

We further demand that $\mathcal{L}$ be invariant under rotations. Equations of this type model excitable media in cardiac tissue, numerous chemical reactions, cell growth, blood clotting, etc. See [1] for a comprehensive overview. A particularly interesting example which is of the form (1.1) after a slight change of coordinates is the model for gas-discharge dynamics developed in [18] and studied in [6]:

$$\begin{aligned} \partial_t u_1 &= \Delta u_1 + u_1 - u_1^3 - \epsilon(\alpha u_2 + \beta u_3 + \gamma), \\ \tau \partial_t u_2 &= \epsilon^{-2} \Delta u_2 + u_1 - u_2, \\ \theta \partial_t u_3 &= D^2 \epsilon^{-2} \Delta u_3 + u_1 - u_3. \end{aligned}$$

The dynamics of pulse interactions in this system are quite complex, and a first step towards understanding them is to study long-distance interactions between pulses.

Our first result states that if the initial data for (1.1) is sufficiently close to the linear superposition of two pulses which, in the absence of any interaction, would not pass too close to one another, then the solution of (1.1) is asymptotically close to the linear superposition of two (possibly different) pulses. We refer to this situation as an *exit*. Our second main result concerns pulses which *shoot* in towards one another from spatial infinity; in particular, we prove that for a given configuration of the pulse trajectories there is a unique, stable, backward-in-time time solution of (1.1) which is asymptotically close to the linear superposition to pulses moving along that trajectory. (See Figure 1.)

We make precise our assumptions about the existence and stability of pulses.

Hypothesis 1.1. *There exist a C^3 function $Q(\mathbf{y})$ and $c > 0$ such that*

$$u(\mathbf{x}, t) = Q(\mathbf{x} - ct\mathbf{i})$$

solves (1.1). Here $\mathbf{i} = (1, 0)$. Furthermore, we assume there exists $\beta_1 > 0$ so that for all multi-indices $|\mathbf{n}| \leq 3$ and $b \in [0, \beta_1]$ we have

$$\cosh(b|\mathbf{y}|) D^{\mathbf{n}} Q(\mathbf{y}) \in L^\infty.$$

We assume without loss of generality that $\int \mathbf{y}Q(\mathbf{y})d\mathbf{y} = 0$, which is to say that the center of mass of the pulse (in a frame moving with the pulse) is located at the origin. Thus in the fixed frame, the pulse $Q(\mathbf{x} - ct\mathbf{i})$ will be "located" at $ct\mathbf{i}$.

Remark 1.2. *The fact that $c > 0$ is crucial in what follows. The method used here to prove the main results requires that the pulses move relative to one another. Since the equation*

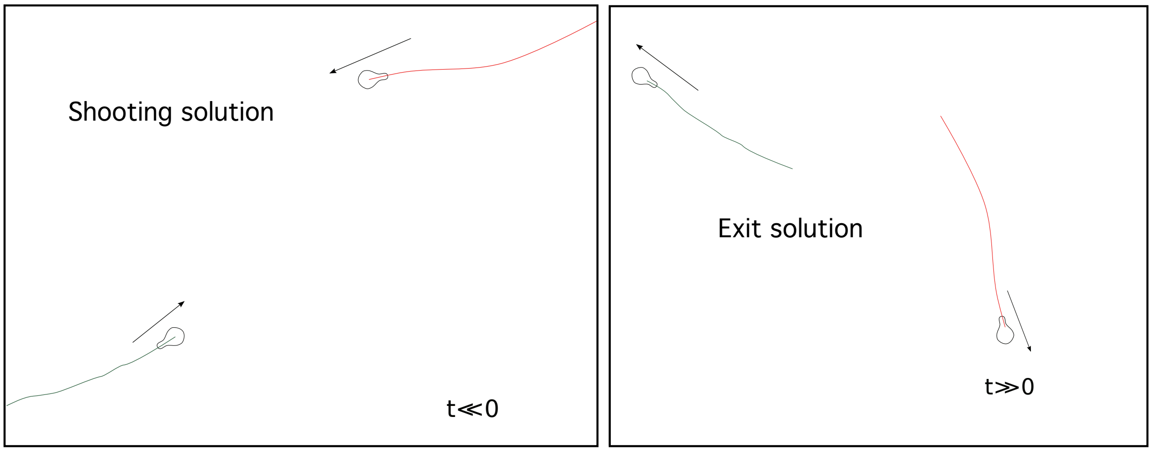

Figure 1. *Schematics of "shooting" and "exit" solutions.*

is invariant under rotations, the fact that $c > 0$ immediately implies that pulses are not rotationally symmetric.

Equation (1.1) is invariant under spatial translations and rotations. So if we let $\mathcal{R}[\theta](x_1, x_2) := (\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2)$ and if $\mathbf{v}(\theta) := \mathcal{R}[\theta]\mathbf{i}$, then

$$Q\left(\mathcal{R}[-\theta]\left(\mathbf{x} - ct\mathbf{v}(\theta) - \mathbf{r}_0\right)\right)$$

is another pulse solution which moves in the direction $\mathbf{v}(\theta)$, i.e., at angle θ.

If we insert

$$u(\mathbf{x}, t) = Q\left(\mathcal{R}[-\theta]\left(\mathbf{x} - ct\mathbf{v}(\theta) - \mathbf{r}_0\right)\right) + W\left(\mathcal{R}[-\theta]\left(\mathbf{x} - ct\mathbf{v}(\theta) - \mathbf{r}_0\right), t\right)$$

into (1.1), we find

$$W_t = AW + O(|W|^2),$$

where the linearization of (1.1) about the pulse is

$$A := c\mathbf{i} \cdot \nabla + \mathcal{L} + F'(Q(\mathbf{y})).$$

We use $\mathbf{y}$ to denote the independent spatial variable of W, which corresponds to the frame moving with the pulse. Note that the kernel of A contains Q_{y_1} and Q_{y_2} due to the translation invariance of (1.1), and also $\mathbf{y}^\perp \cdot \nabla Q$ because of its rotational invariance. (Here $(y_1, y_2)^\perp := (-y_2, y_1)$.) We assume that these three eigenvalues fully characterize the kernel and that the rest of the spectrum is stable. To wit, we have the following.

Hypothesis 1.3. *The spectrum of A viewed as an unbounded operator on L^2 consists of a triple eigenvalue at zero due to the translation and rotation invariance of the problem and the rest of the spectrum which lies in the set*

$$S := \left\{\lambda \mid \Re\lambda < -\alpha,\ |\arg(\lambda)| > \pi - \varphi_0\right\},$$

where $\alpha > 0$ and $\varphi_0 \in (0, \pi/2)$. The center eigenspace is

$$E^c := \operatorname{span}\left\{Q_{y_1}(\mathbf{y}), Q_{y_2}(\mathbf{y}), \mathbf{y}^\perp \cdot \nabla Q(\mathbf{y})\right\}.$$

Hypothesis 1.3 has several important consequences, which we summarize in the following lemmas. First, since A is elliptic, this implies (see [3, 17]) the following.

Lemma 1.4. *The spectral properties of A when viewed as an operator on L^p spaces are identical for all $1 < p < \infty$. In particular, we will work with $p = 5$. (See section 2 below.) Moreover, the fact that the zero eigenfunctions of A are exponentially decaying and C^1 implies that the spectral projection onto E^c is given by*

$$\Pi^c f = \Big(\langle \psi_1^\dagger, f\rangle,\ \langle \psi_2^\dagger, f\rangle,\ \langle \psi_3^\dagger, f\rangle\Big) = \langle \psi^\dagger, f\rangle,$$

where the adjoint eigenfunctions $\psi^\dagger = (\psi_1^\dagger, \psi_2^\dagger, \psi_3^\dagger)$ are also C^1 and exponentially decaying. That is, $\cosh(b|\mathbf{x}|)\psi^\dagger(\mathbf{x}) \in W^{1,\infty}$ for $b \in [0, \beta_2]$, for some $\beta_2 > 0$. The strong stable eigenspace is denoted $E^s := \ker \Pi^c$ with projection $\Pi^s := 1 - \Pi^c$. Note that $\langle \cdot, \cdot\rangle$ denotes the L^2 inner product.

Similarly we have the following estimate.

Lemma 1.5. *There is a positive constant M such that the resolvent estimate*

$$(1 + |\lambda|)\left\|(\lambda - A)^{-1}\right\|_{L^5 \longrightarrow L^5} \le M$$

is true for all $\lambda \notin S$. Additionally A generates an analytic semigroup e^{At} on L^5.

Last, the conjugated operator $A_b := \cosh(b|\mathbf{x}|)A[\operatorname{sech}(b|\mathbf{x}|)\cdot]$ is a small bounded perturbation of A when b is not too large. The spectral properties of A_b as an unbounded operator on L^5 coincide with those of A on

$$L_b^5 := \left\{ f(\mathbf{y}) : \cosh(b|\mathbf{y}|) f(\mathbf{y}) \in L^5 \right\}.$$

Since the center eigenfunctions of A are, by hypothesis, in L_b^5, we know that the center eigenspace of A_b is the same as that of A. Thus we have the next lemma.

Lemma 1.6. *There exists $\beta_3 > 0$ so that Hypothesis 1.3 and Lemmas 1.4–1.5 remain true when L^5 is replaced by the exponentially weighted space L_b^5 for $b \in [0, \beta_3]$. All estimates are uniform in b.*

In what follows, let

$$\beta := \min\{\beta_1, \beta_2, \beta_3\} \quad \text{and} \quad b_0 = \beta/2.$$

Hypothesis 1.3 and Lemmas 1.5 and 1.6 can be used to show, for instance, that the pulse Q is stable with asymptotic phase for small initial perturbations in

$$W_b^{8/5,5} := \left\{ f(\mathbf{y}) : \cosh(b|\mathbf{y}|) f(\mathbf{y}) \in W^{8/5,5} \right\}$$

and $b \in [0, b_0]$ (see [14]).

We refer the reader to [7] for discussion of when one can expect Hypotheses 1.1 and 1.3 to be satisfied for a fairly general class of equations of type (1.1).

1.2. Description of main results. We are interested, however, in multipulse solutions. In the planar setting, pulses can approach or separate from one another obliquely, and this complicates our analysis in comparison with our work in one-dimensional problems in [20, 22]. For instance, two pulses may be separated by a great distance while at the same time be moving along nearly parallel trajectories; the effect of each pulse on the other is small but acts over very long times. Thus we must carefully quantify the interaction between pulses. To this end, we define for $j = 1, 2$

$$\mathbf{r}_j^\star(t) := \mathbf{r}_j^\star(t; \theta_{j0}, \mathbf{r}_{j0}) := \mathbf{r}_{j0} + ct\mathbf{v}(\theta_{j0}),$$

which gives the projected location of a pulse indexed by j if it travels along a straight line pointing in the direction θ_{j0} and is initially located at $\mathbf{r}_{j0}$. Let $\Delta\mathbf{r}_0 := \mathbf{r}_{20} - \mathbf{r}_{10}$, $\Delta\mathbf{v}_0 := \mathbf{v}(\theta_{20}) - \mathbf{v}(\theta_{10})$,

$$\mu^\star(t) := \mu^\star(t; \theta_{10}, \theta_{20}, \mathbf{r}_{10}, \mathbf{r}_{20}) = \left|\mathbf{r}_2^\star(t) - \mathbf{r}_1^\star(t)\right|,$$

and

$$\mu_0^\star := \min_{t \geq 0} \mu^\star(t).$$

Elementary considerations show that this minimum is achieved at $T^\star := \max\{\frac{-\Delta\mathbf{r}_0 \cdot \Delta\mathbf{v}_0}{c|\Delta\mathbf{v}_0|^2}, 0\}$, and if $T^\star > 0$, then

$$\mu_0^\star = \left[|\Delta\mathbf{r}_0|^2 - \left(\frac{\Delta\mathbf{r}_0 \cdot \Delta\mathbf{v}_0}{|\Delta\mathbf{v}_0|}\right)^2\right]^{1/2}.$$

Otherwise $\mu_0^\star = |\Delta\mathbf{r}_0|$. Likewise, for all $t \geq 0$, we have

$$\mu^\star(t) \geq k^\star t,$$

where

$$k^\star := c|\Delta\mathbf{v}_0|\frac{\mu_0^\star}{|\Delta\mathbf{r}_0|}.$$

The constant $k^\star$ is optimal. (See Figure 2.)

We define the first set of function spaces on which we will work. For $0 < T \leq \infty$, $a \geq 0$, and $b \geq 0$,

$$X_{a,b}[T] := \left\{f(\mathbf{x}, t) : e^{at}\cosh(b\,|\mathbf{x}|)f(\mathbf{x}, t) \in L^5([0, T]\,; L^5(\mathbf{R}^2))\right\},$$
$$X'_{a,b}[T] := \left\{f(\mathbf{x}, t) : e^{at}\cosh(b\,|\mathbf{x}|)f(\mathbf{x}, t) \in L^5([0, T]; W^{2,5}(\mathbf{R}^2)) \cap W^{1,5}([0, T]; L^5(\mathbf{R}^2))\right\}.$$

See section 2 for further discussion of these particular choices.

We now precisely state our result concerning exit solutions, as follows.

Theorem 1.7. *Given Hypotheses 1.1 and 1.3, for all $\epsilon_0 > 0$ there exist positive constants δ_{exit}, M_0, and $\mathcal{K}_0$ so that*

$$\mu_0^\star \geq M_0, \quad |\Delta\mathbf{v}_0| \geq \epsilon_0,$$

and

$$\left\| u_0 - \sum_{j=1}^{2} Q\left(\mathcal{R}[-\theta_{j0}]\left(\cdot - \mathbf{r}_{j0}\right)\right)\right\|_{W^{8/5,5}} \leq \delta_{exit}$$

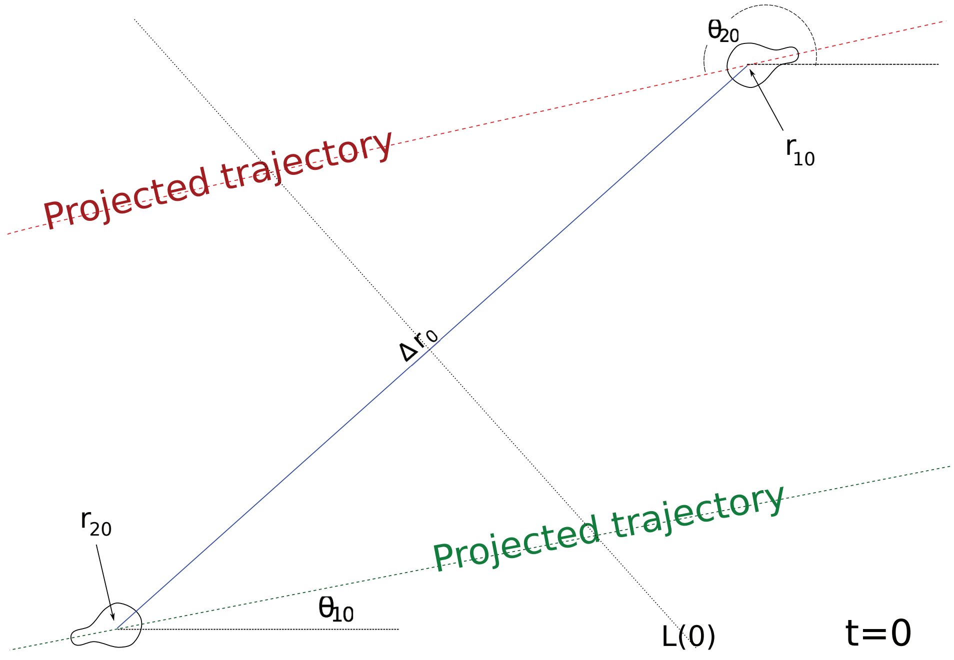

Figure 2. *Initial configuration for an exit. The dashed lines represent the projected trajectories* $\mathbf{r}_1^\star$ *and* $\mathbf{r}_2^\star$.

imply that $u(\mathbf{x}, t)$*, the solution of* (1.1) *with initial data* u_0*, satisfies*

$$\left\| u(\cdot, \diamond) - \sum_{j=1}^{2} Q\left(\mathcal{R}[-\theta_j(\diamond)]\left(\cdot - \mathbf{r}_j(\diamond)\right)\right)\right\|_{X'_{a,0}[\infty]} \leq \mathcal{K}_0,$$

where $a = \min\{\alpha/2, b_0 k^\star/8\}$*. The functions* $\mathbf{r}_j(t)$ *and* $\theta_j(t)$ $(j = 1, 2)$ *are defined by*

$$\theta_j(t) = \theta_{j0} + \phi_j(t)$$

and

$$\mathbf{r}_j(t) = \mathbf{r}_{j0} + \mathbf{p}_j(t) + \int_0^t \mathcal{R}[\theta_j(s)] c\mathbf{i} ds,$$

where $\mathbf{p}_j$ *and* θ_j *are* C^1 *functions*[1] *for which*

$$\|\mathbf{p}_j(\diamond)\|_{C^1(\mathbf{R}^+)} + \|\phi_j(\diamond)\|_{C^1(\mathbf{R}^+)} \leq \mathcal{K}_0.$$

Remark 1.8. *In the proof, we find that* $M_0 = -c_1 \ln(|\Delta \mathbf{v}_0|/2) + c_2$ *for constants* $c_1, c_2 > 0$.

When dealing with *shooting* solutions, we work with the function spaces

$$Z_{\eta,b} := \left\{ f(\mathbf{x}, t) : e^{\eta t} \cosh(b\,|\mathbf{x}|) f(\mathbf{x}, t) \in L^5([-\infty, 0]\,; L^5(\mathbf{R}^2)) \right\},$$
$$Z'_{\eta,b} := \left\{ f(\mathbf{x}, t) : e^{\eta t} \cosh(b\,|\mathbf{x}|) f(\mathbf{x}, t) \in L^5([-\infty, 0]; W^{2,5}(\mathbf{R}^2)) \cap W^{1,5}([-\infty, 0]; L^5(\mathbf{R}^2)) \right\}.$$

[1] Note that we use the supremum norm for $C^1[0, T]$.

Our next theorem states that for suitable choices of $\mathbf{r}_{10}$, $\mathbf{r}_{20}$, θ_{10}, and θ_{20} there is a solution, defined for all $t \leq 0$, which is exponentially close to the sum of two pulses moving along trajectories $\mathbf{r}_1^\star$ and $\mathbf{r}_2^\star$.

Theorem 1.9. *Given Hypotheses 1.1 and 1.3, for all $\epsilon_0 > 0$ there exist positive constants M_1 and $\mathcal{K}_1$ so that if*

$$|\Delta\mathbf{v}_0| \geq \epsilon_0, \quad \Delta\mathbf{r}_0 \cdot \Delta\mathbf{v}_0 \leq 0, \quad \text{and} \quad |\Delta\mathbf{r}_0| \geq M_1,$$

then there exists $\Phi(\mathbf{x},t) = \Phi(\mathbf{x};\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20})$ which solves (1.1) and satisfies

$$\left\| \Phi(\cdot,\diamond) - \sum_{j=1}^{2} Q\left(\mathcal{R}[-\theta_{j0}]\left(\cdot - \mathbf{r}_j^\star(\diamond)\right)\right) \right\|_{Z'_{\eta,0}} \leq \mathcal{K}_1.$$

Here $\eta = -b_0 c|\Delta\mathbf{v}_0|/8$.

Moreover, Φ has a decomposition into spatially localized pieces. That is, there are $W_1^*, W_2^* \in Z'_{\eta,b_0}$ (with norms less than $\mathcal{K}_1$) so that

$$\Phi(\mathbf{x},t) = \sum_{j=1}^{2}\left(Q\left(\mathcal{R}[-\theta_{j0}]\left(\mathbf{x} - \mathbf{r}_j^\star(t)\right)\right) + W_j^*\left(\mathcal{R}[-\theta_{j0}]\left(\mathbf{x} - \mathbf{r}_j^\star(t)\right), t\right)\right).$$

W_1^* and W_2^* (and therefore Φ) are differentiable in their dependence on the parameters $\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20}$.

Remark 1.10. *The condition $\Delta\mathbf{r}_0 \cdot \Delta\mathbf{v}_0 \leq 0$ just implies that for $t \leq 0$ the pulses are closest at $t = 0$, not before.*

Our final result concerns the stability of the shooting solutions.

Theorem 1.11. *There exist positive constants δ_{shoot}, T_3, and $\mathcal{K}_2$ so that if $T \geq T_3$ and*

$$\|u_0 - \Phi(\cdot,-T;\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20})\|_{W^{8/5,5}} = \delta \leq \delta_{shoot},$$

then $u(\mathbf{x},t)$, the solution of (1.1) with initial data u_0, satisfies

$$\left\| u(\cdot,\diamond) - \Phi(\cdot,\diamond - T;\tilde{\mathbf{r}}_{10},\tilde{\mathbf{r}}_{20},\tilde{\theta}_{10},\tilde{\theta}_{20}) \right\|_{X'_{\alpha/2,0}[T-T_3]} \leq \mathcal{K}_2\delta,$$

where

$$\sum_j |\mathbf{r}_{j0} - \tilde{\mathbf{r}}_{j0}| + \left|\theta_{j0} - \tilde{\theta}_{j0}\right| \leq \mathcal{K}_2\delta.$$

Note that this theorem implies the following.

Corollary 1.12. *The function $\Phi(\mathbf{x},t;\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20})$ is the unique function with properties described in Theorem 1.9.*

1.3. Interaction manifolds. Given the freedom in choosing $\mathbf{r}_{10}$, $\mathbf{r}_{20}$, θ_{10}, and θ_{20} in our main results, we define the following *interaction manifolds* (defined for fixed ϵ_0):

$$\mathcal{M}_{exit} := \left\{Q(\mathcal{R}[\theta_{10}](\cdot - \mathbf{r}_{10})) + Q(\mathcal{R}[\theta_{20}](\cdot - \mathbf{r}_{20})) : \Delta\mathbf{v}_0 \geq \epsilon_0 \quad \text{and} \quad \mu_0^\star \geq M_0\right\},$$

$$\mathcal{M}_{shoot} := \left\{\Phi(\cdot,0;\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20}) : \Delta\mathbf{v}_0 \geq \epsilon_0,\ \Delta\mathbf{r}_0 \cdot \Delta\mathbf{v}_0 \leq 0, \quad \text{and} \quad |\Delta\mathbf{r}_0| \geq M_1\right\}.$$

(Note that, in the definition of $\mathcal{M}_{shoot}$, Φ is the solution of (1.1) described in Theorem 1.9.) Since Q and Φ are in $W^{8/5,5}$ and depend differentiably on $\mathbf{r}_{10}$, $\mathbf{r}_{20}$, θ_{10}, and θ_{20}, these are six-dimensional C^1 manifolds in $W^{8/5,5}$. Each has a boundary. Note that $\mathcal{M}_{shoot}$ is an invariant manifold for (1.1) since Φ solves this equation exactly, whereas $\mathcal{M}_{exit}$ is not. Nevertheless, our main results indicate that both are attracting sets for (1.1). That is, we have the following corollaries of our main theorems (and also (2.2)).

Corollary 1.13. *Suppose that*

$$dist_{W^{8/5,5}}(u(x,0),\mathcal{M}_{exit}) \le \delta_{exit}.$$

Then there exists $a > 0$ so that

$$dist_{W^{8/5,5}}(u(x,t),\mathcal{M}_{exit}) \le \mathcal{K}_0 e^{-at}$$

for all $t \ge 0$.

Corollary 1.14. *Suppose that*

$$dist_{W^{8/5,5}}(u(x,0),\mathcal{M}_{shoot}) = \delta \le \delta_{shoot}.$$

Then

$$dist_{W^{8/5,5}}(u(x,t),\mathcal{M}_{shoot}) \le \mathcal{K}_2 \delta e^{-\frac{b_0}{8}c|\Delta\mathbf{v}_0|t}$$

for all $0 \le t \le T$. Here $T > 0$ is a constant which depends only upon the parameters $\mathbf{r}_{10}$, $\mathbf{r}_{20}$, θ_{10}, and θ_{20} which minimize $dist_{W^{8/5,5}}(u(x,0),\mathcal{M}_{shoot})$.

We make two final remarks, the first of which is paraphrased from [20].

Remark 1.15. *Corollary 1.14 implies that collisions between two pulses are well-defined scattering problems. By this, we mean that strong interactions which take place during collisions will not be affected by specific choices for the initial data which lead to this collision. Moreover, Corollary 1.13 tells us that if there is an interaction which results after a finite amount of time in two pulses which are moving away from one another, then this state will persist for all time. That is to say, these results indicate that to study strong collisions one needs only finite time results.*

Remark 1.16. *Theorem 1.9 implies that if we take $\Delta\mathbf{r}_0$ to infinity while keeping $\Delta\mathbf{v}_0$ fixed, then*

$$\|\Phi(\cdot,0;\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20}) - Q(\mathcal{R}[\theta_{10}](\cdot-\mathbf{r}_{10})) - Q(\mathcal{R}[\theta_{20}](\cdot-\mathbf{r}_{20}))\|_{W^{8/5,5}} \to 0,$$

which is to say that

$$dist(\mathcal{M}_{shoot},\mathcal{M}_{exit}) = 0.$$

This implies that there are choices for $\mathbf{r}_{10}$, $\mathbf{r}_{20}$, θ_{10}, and θ_{20} so that $\Phi(\cdot,t;\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20})$ is in the attracting region for $\mathcal{M}_{exit}$ for some $t \le 0$. Thus there are global-in-time solutions of (1.1) which converge exponentially quickly for $|t| \to \infty$ to the linear superposition of two pulses.

1.4. The embedding and general strategy. Our strategy is to embed (1.1) into a larger system within which the multipulse problem can be viewed, in some sense, as a small perturbation of single pulse problem. To wit, we consider

$$U_t = \mathcal{L}U + F(U) + G_1(U,V),\\ V_t = \mathcal{L}V + F(V) + G_2(U,V), \tag{1.2}$$

where we choose G_1 and G_2 so that

$$F(U) + F(V) + G_1(U,V) + G_2(U,V) = F(U+V).$$

With this, if U and V solve (1.2), then

$$u = U + V$$

solves (1.1). Note that this idea is very similar to the "freezing" method used in [4]. The idea is then to show that there are solutions of (1.2) roughly of the form

$$U = Q\left(\mathcal{R}[-\theta_{10}]\left(\mathbf{x} - \mathbf{r}_1^\star(t)\right)\right) + \text{“small”},\\ V = Q\left(\mathcal{R}[-\theta_{20}]\left(\mathbf{x} - \mathbf{r}_2^\star(t)\right)\right) + \text{“small”},$$

where the parameters θ_{j0} and $\mathbf{r}_{j0}$ are taken so that $\mu_0^\star$ is large.

We select G_1 and G_2 as follows. Suppose that at time t our pulses are located at $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$. Let $L(t)$ be the perpendicular bisector of the segment connecting these two points. (See Figures 2 and 3.) The line $L(t)$ divides $\mathbf{R}^2$ into a disjoint union of two sets, $\Sigma_1(t)$ (which contains the pulse labeled "1") and $\Sigma_2(t)$ (which contains the other). For concreteness, we assume $L(t) \subset \Sigma_1(t)$.

Let $\chi_j(\mathbf{x}; \mathbf{r}_1(t), \mathbf{r}_2(t))$, $j = 1, 2$, be a C^∞ partition of unity subordinate to the sets $\Sigma_j(t)$ and which have derivatives whose support lies in $\{\mathbf{x} : \text{dist}(\mathbf{x}, L(t)) \le 1\}$. More compactly we will write these as $\chi_1(\mathbf{x}, t)$ and $\chi_2(\mathbf{x}, t)$. We set

$$G(U,V) := (F(U+V) - F(U) - F(V))$$

and

$$G_j(U,V) = \chi_j(\mathbf{x}, t) G(U,V).$$

We are thinking of U as being the pulse which lies in Σ_1, and V as being the one in Σ_2.

Our choice for G is motivated by the following heuristic. A straightforward application of Taylor's theorem, together with the fact that $F(0) = 0$, implies

$$|G(U,V)| \le C\,|U|\,|V|\,.$$

Now suppose that $U = Q_1 + W_1$ and $V = Q_1 + W_2$, where Q_1 and Q_2 are pulses and W_1 and W_2 are error functions. Thus

$$|G_1(U,V)| \le C\chi_1\,|Q_1|\,|Q_2| + C\chi_1\,|Q_1|\,|W_2| + C\chi_1\,|Q_2|\,|W_1| + C\chi_1\,|W_1|\,|W_2|\,.$$

Of the four terms here, three are "small."

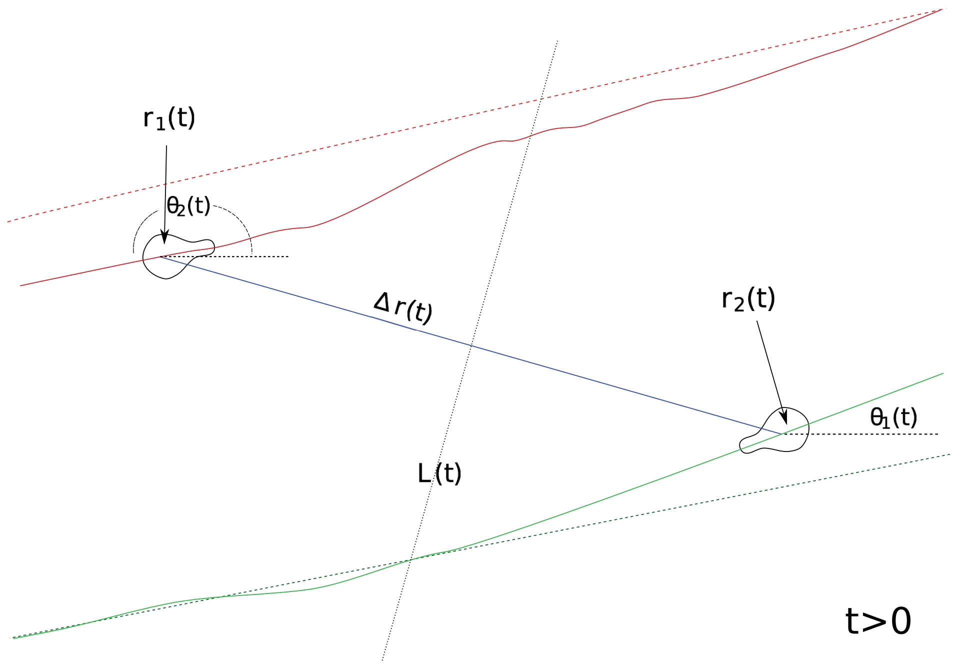

Figure 3. *Schematic of exit solution at $t > 0$. The solid red and green curves represent the actual pulse trajectories $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$.*

- Q_1 and Q_2 are exponentially localized, and so $|Q_1|\,|Q_2|$ is exponentially small in the separation distance.
- The term $|W_1|\,|W_2|$ is $O(|W_1|^2 + |W_2|^2)$ and thus very small if the error functions are small.
- The term $\chi_1\,|Q_2|$ is clearly bounded by the maximum value of Q_2 evaluated on the region Σ_1, which is to say that Q_2 is evaluated far from its center. Thus $\chi_1\,|Q_2|\,|W_1|$ is small as well.

The remaining term $\chi_1\,|Q_1|\,|W_2|$ is not small, at least not for general functions W_2; it is an $O(1)$ nonautonomous linear term and thus problematic. However, if W_2 is localized near the pulse Q_2, then we can conclude that this term is exponentially small in the pulse separation distance. Thus if we work with spaces of localized functions, we can handle this term. Nevertheless, we wish to allow arbitrary perturbations of the pulses, so this restriction is unsatisfying. We are able to circumvent this difficulty by making the observation that even if W_2 is not localized, $\chi_1\,|Q_1|\,|W_2|$ is. It turns out that this allows us to treat the term as if it were small.

The following toy model linear algebra problem demonstrates the core idea of why this is so. Let $\mathcal{A}$ be an invertible n by n matrix (which is our stand-in here for the linearization A). Suppose we are trying to solve the equation

$$(\mathcal{A} + \mathcal{B})w = j, \tag{1.3}$$

where $\mathcal{B}$ is a matrix that has the following properties:

1. $\mathcal{B} : \mathbf{R}^n \longrightarrow X_b$, where X_b is a subspace of $\mathbf{R}^n$.

2. $\|\mathcal{B}\|_{X_b \to X_b} = \epsilon \ll 1$.

(The matrix $\mathcal{B}$ is playing the role of the perturbation to A made by $\chi_1 |Q_1| |W_2|$, and X_b corresponds to the space of "exponentially decaying functions.") If in addition we know that $\mathcal{A}^{-1} : X_b \longrightarrow X_b$, then the fact that $\mathcal{B}$ is small on X_b means that $\mathcal{A} + \mathcal{B}$ is invertible on X_b. We denote this inverse by $(\mathcal{A} + \mathcal{B})^{-1}_{X_b}$. Now consider the augmented system

$$\begin{aligned} \mathcal{A} w^i &= j, \\ (\mathcal{A} + \mathcal{B}) w^l &= -\mathcal{B} w^i. \end{aligned} \tag{1.4}$$

The first equation can solved for any j in $\mathbf{R}^n$. Since $\mathcal{B}$ maps everything into X_b, we can solve the second equation. Moreover, adding these two equations shows that $w = w^i + w^l$ solves (1.3). Specifically, the solution map is

$$\mathcal{G} := \mathcal{A}^{-1} - (\mathcal{A} + \mathcal{B})^{-1}_{X_b} \mathcal{B} \mathcal{A}^{-1}.$$

Thus $\mathcal{G}$ is $(\mathcal{A} + \mathcal{B})^{-1}$ on all of $\mathbf{R}^n$.

The remainder of this paper is organized as follows. In section 2 we discuss the functional setting for our results. We compute the "almost" linearization of (1.2) about a multipulse solution in section 3 and collect a number of useful estimates. Section 4 concerns exit solutions and in particular contains the proof of Theorem 1.7. Section 5 is about the shooting manifold and has the proofs of Theorems 1.9 and 1.11 and Corollary 1.12. In the appendix we prove a local-in-time existence theorem for solutions of a system needed in section 4.

2. Functional setting and maximal regularity.

In this section we discuss important properties of A and solutions of equations of the form

$$W_t = AW + J(t) \quad \text{and} \quad W(t = 0) = W_0. \tag{2.1}$$

Specifically, we summarize the maximal regularity results for Sobolev spaces, found in section III.4 of [3]. The domain of A viewed as an operator on L^p is of course $W^{2,p}$, and so we set

$$Y := L^p([0,T]; L^p) \quad \text{and} \quad Y' := W^{1,p}([0,T]; L^p(\mathbf{R}^2)) \cap L^p([0,T]; W^{2,p}(\mathbf{R}^2)).$$

Hypothesis 1.3 and Theorem 4.10.7 in [3] allow us to conclude that there is a continuous solution map Γ (given explicitly by the Duhamel formula),

$$\Gamma : Y \times E_{1-1/p,p} \longrightarrow Y'.$$

That is, if $W = \Gamma(J, W_0)$, then W satisfies (2.1). The space

$$E_{1-1/p,p} = (L^p(\mathbf{R}^2), W^{2,p}(\mathbf{R}^2))_{1-1/p,p},$$

where $(\cdot,\cdot)_{\theta,q}$ is the real interpolation functor. We have (see, for instance, [21]) that

$$(L^p(\mathbf{R}^2), W^{2,p}(\mathbf{R}^2))_{1-1/p,p} = W^{2-2/p}(\mathbf{R}^2).$$

Also, Theorem 4.10.2 in [3] tells us that

$$Y' \subset\subset BUC([0,T]; W^{2-2/p}(\mathbf{R}^2)).$$

Our equation is nonlinear, and so, at the very least, we would like $W^{2-2/p} \subset\subset C(\mathbf{R}^2)$; this together with the above inclusion implies that Y' is an algebra. At several points, we require $W^{2-2/p} \subset\subset C^1(\mathbf{R}^2)$. From Sobolev embedding we get this inclusion when $1 - 2/p > 2/p$ or rather $p > 4$. We take $p = 5$.

Now we restate Theorem 4.10.7 and Remark 4.10.9(a) of [3] here, adapted to our problem.

Theorem 2.1. *There exists $C > 0$ so that for all $b \in [0, b_0]$ the following are true:*

1. *For all $a \in [0, \alpha/2]$ and $T \in (0, \infty]$, the map*

$$\Gamma_1(W^s, J) := e^{At}\Pi^s W^s + \int_0^t e^{A(t-\tau)}\Pi^s J(\tau)d\tau$$

is bounded from $(E^s \cap W_b^{8/5,5}) \times X_{a,b}[T]$ to $X'_{a,b}[T]$. Additionally, it satisfies

$$\|\Gamma_1(W^s, J)\|_{X'_{a,b}[T]} \leq C\left(\|W^s\|_{W_b^{8/5,5}} + \|J\|_{X_{a,b}[T]}\right),$$

and $W(\mathbf{y}, t) := \Gamma_1(W^s, J)$ solves (for $t \geq 0$ a.e.)

$$W_t = AW + \Pi^s J, \quad W(\mathbf{y}, 0) = W^s \in E^s.$$

2. *For all $\eta < 0$, the map*

$$\Gamma_2(J) := \int_{-\infty}^t e^{A(t-\tau)} J(\tau)d\tau$$

is bounded from $Z_{\eta,b}$ to $Z'_{\eta,b}$. Additionally, it satisfies

$$\|\Gamma_2(J)\|_{Z'_{\eta,b}} \leq C\,\|J\|_{Z_{\eta,b}},$$

and $W(\mathbf{y}, t) := \Gamma_2(J)$ solves (for $t \leq 0$ a.e.)

$$W_t = AW + J.$$

3. *For all $a \in [0, \alpha/2]$ and $T \in (0, \infty]$, the map*

$$\Gamma_3(W^s, J) := e^{At}\Pi^s W^s + \int_0^t e^{A(t-\tau)}\Pi^s J(\tau)d\tau - \int_t^T e^{A(t-\tau)}\Pi^c J(\tau)d\tau$$

is bounded from $(E^s \cap W_b^{8/5,5}) \times X_{a,b}[T]$ to $X'_{a,b}[T]$. Additionally, it satisfies

$$\|\Gamma(W^s, J)\|_{X'_{a,b}[T]} \leq C\left(\|W^s\|_{W_b^{8/5,5}} + \|J\|_{X_{a,b}[T]}\right),$$

and $W(\mathbf{y}, t) := \Gamma(W^s, J)$ solves (for $t \geq 0$ a.e.)

$$W_t = AW + \Pi^s J, \quad \Pi^s W(\mathbf{y}, 0) = W^s \in E^s, \quad \text{and} \quad \Pi^c W(\mathbf{y}, T) = 0.$$

Remark 2.2. *Point 1 will be used to prove the stability of exiting pulses, point 2 the existence of shooting solutions, and point 3 the stability of shooting solutions. For a further discussion of these results and how they follow from [3], see [20].*

Finally, we note that Theorem 4.10.2 gives

$$\begin{aligned} X'_{a,b}[T] &\subset\subset \left\{f(\mathbf{x}, t) : e^{at} f(\mathbf{x}, t) \in BUC([0, T]; W_b^{8/5,5})\right\}, \\ Z'_{\eta,b} &\subset\subset \left\{f(\mathbf{x}, t) : e^{\eta t} f(\mathbf{x}, t) \in BUC([-\infty, 0]; W_b^{8/5,5})\right\}. \end{aligned} \tag{2.2}$$

3. The "almost" linearization of (1.2) around a multipulse.

3.1. Reduction. We consider (1.2) with initial conditions

$$U(\mathbf{x},0) = Q\left(\mathcal{R}\left[-\theta_{10}\right]\left(\mathbf{x}-\mathbf{r}_{10}\right)\right) + W_{10}\left(\mathcal{R}\left[-\theta_{10}\right]\left(\mathbf{x}-\mathbf{r}_{10}\right)\right),$$

$$V(\mathbf{x},0) = Q\left(\mathcal{R}\left[-\theta_{20}\right]\left(\mathbf{x}-\mathbf{r}_{20}\right)\right) + W_{20}\left(\mathcal{R}\left[-\theta_{20}\right]\left(\mathbf{x}-\mathbf{r}_{20}\right)\right).$$

The following lemma tells us that we can without loss of generality assume that W_{10} and W_{20} lie in E^s.

Lemma 3.1. *There is a smooth map* $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)\colon W^{8/5,5} \longrightarrow \mathbf{R}^2 \times \mathbf{R} \times E^s$ *with the property that*

$$\mathcal{I}(0) = (0,0,0),$$

and if $\|W\|_{W^{8/5,5}}$ *is sufficiently small, then*

$$Q(\mathbf{x}) + W(\mathbf{x}) = Q(\mathcal{R}[\mathcal{I}_2(W)](\mathbf{x}-\mathcal{I}_1(W))) + \mathcal{I}_3(W)(\mathcal{R}[\mathcal{I}_2(W)](\mathbf{x}-\mathcal{I}_1(W))).$$

Proof. The proof follows immediately from applying the implicit function theorem to the map:

$$\Theta(\mathcal{I}, W) = Q(\mathbf{x}) + W(\mathbf{x}) - Q(\mathcal{R}[\mathcal{I}_2](\mathbf{x}-\mathcal{I}_1)) - \mathcal{I}_3(\mathcal{R}[\mathcal{I}_2](\mathbf{x}-\mathcal{I}_1)). \qquad \blacksquare$$

Now we let

$$\begin{aligned} U(\mathbf{x},t) &= Q\left(\mathbf{z}_1\right) + W_1\left(\mathbf{y}_1, t\right), \\ V(\mathbf{x},t) &= Q\left(\mathbf{z}_2\right) + W_2\left(\mathbf{y}_2, t\right), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \mathbf{y}_1 &= \mathcal{R}[-\theta_{10}]\left(\mathbf{x}-\mathbf{r}_1(t)\right) \quad \text{and} \quad \mathbf{z}_1 = \mathcal{R}[-\theta_1(t)]\left(\mathbf{x}-\mathbf{r}_1(t)\right), \\ \mathbf{y}_2 &= \mathcal{R}[-\theta_{20}]\left(\mathbf{x}-\mathbf{r}_2(t)\right) \quad \text{and} \quad \mathbf{z}_2 = \mathcal{R}[-\theta_2(t)]\left(\mathbf{x}-\mathbf{r}_2(t)\right). \end{aligned}$$

The collective coordinates $\mathbf{r}_j(t)$ and $\theta_j(t)$ are specified in sections 4 and 5. Note here that the "spatial coordinates" of the pulses Q and the error functions W_1 and W_2 are not all the same. The coordinates on the right-hand side for U are "centered" at the point $\mathbf{r}_1(t)$, and those for V at $\mathbf{r}_2(t)$. Moreover, the coordinates for the W's have their rotation matrices fixed at their initial values, $\theta_j(0) = \theta_{j0}$.

We use

$$\mathbf{m} := (\mathbf{p}_1, \phi_1, \mathbf{p}_2, \phi_2)$$

(with $\mathbf{p}_j$, ϕ_j defined as in Theorem 1.7) to refer to the collective coordinates in the aggregate, and define

$$\mu(t) := |\mathbf{r}_2(t) - \mathbf{r}_1(t)|$$

as the separation distance of the two pulses.

Inserting (3.1) into (1.2), we find

$$\begin{aligned} \partial_t W_1 &= AW_1 + B_1(\mathbf{m})W_2 \\ &\quad + \{\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}\} \cdot \nabla Q + \dot{\theta}_1 \mathbf{y}^{\perp} \cdot \nabla Q + J_1, \end{aligned} \tag{3.2}$$

$$\begin{aligned}(3.3)\qquad \partial_t W_2 &= AW_2 + B_2(\mathbf{m})W_1 \\ &\quad + \{\mathcal{R}[-\theta_2]\dot{\mathbf{r}}_2 - c\mathbf{i})\} \cdot \nabla Q + \dot{\theta}_2 \mathbf{y}^\perp \cdot \nabla Q + J_2,\end{aligned}$$

where

$$B_1(\mathbf{m})W_2 := \chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right] W_2(\mathbf{y}_2,t)$$

and

$$B_2(\mathbf{m})W_1 := \chi_2(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_1))\right] W_1(\mathbf{y}_1,t).$$

These terms correspond to the problematic term $\chi_1 |Q_1|$ from the introduction.

The J_1 term is given by

$$J_1 := K_1 + J_{1,mod} + J_{1,int} + J_{1,nl},$$

where

$$\begin{aligned}K_1 &:= \left\{(\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}) + \dot{\theta}_1 \mathcal{R}\left[-\theta_1+\theta_{10}\right]\mathbf{y}^\perp\right\} \cdot (\nabla Q(\mathbf{z}_1) - \nabla Q(\mathbf{y})) \\ &\quad + \dot{\theta}_1\left\{\mathcal{R}\left[-\theta_1+\theta_{10}\right]\mathbf{y}^\perp - \mathbf{y}^\perp\right\} \cdot \nabla Q(\mathbf{y}) \\ &\quad + \mathcal{R}[\theta_1-\theta_{10}]\left\{\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}\right\} \cdot \nabla W_1, \\ J_{1,mod} &:= (\mathcal{R}[\theta_1-\theta_{10}]c\mathbf{i} - c\mathbf{i}) \cdot \nabla W_1 \\ &\quad + \left\{F'(Q(\mathbf{z}_1)) - F'(Q(\mathbf{y}))\right\} W_1, \\ J_{1,int} &:= \chi_1(\mathbf{x},t)\left[F(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F(Q(\mathbf{z}_1)) - F(Q(\mathbf{z}_2))\right] \\ &\quad + \chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_1))W_1\right], \\ J_{1,nl} &:= F\left(Q\left(\mathbf{z}_1\right)+W_1\right) - \left(F(Q(\mathbf{z}_1)) + F'(Q(\mathbf{z}_1))W_1\right) \\ &\quad + \chi_1(\mathbf{x},t)N_{ct}(W_1, W_2(\mathbf{y}_2,t)),\end{aligned}$$

and

$$\begin{aligned}&N_{ct}(W_1,W_2) \\ &:= F(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)+W_1+W_2) - F(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2))(W_1+W_2) \\ &\quad - \left(F(Q(\mathbf{z}_1)+W_1) - F(Q(\mathbf{z}_1)) - F'(Q(\mathbf{z}_1))W_1\right) \\ &\quad - \left(F(Q(\mathbf{z}_2)+W_2) - F(Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))W_2\right).\end{aligned}$$

The terms in K_1 and $J_{1,mod}$ are small due to the choice of the collective coordinates, those in $J_{1,int}$ due to the types interactions described in the introduction, and those in $J_{1,nl}$ because they are nonlinear. We define J_2 analogously.

Remark 3.2. *In the definitions of B_j and J_j we use $\mathbf{y}$ as the independent spatial variable for both W_1 and W_2. If we do not specify the spatial coordinate of a function, it is assumed to be $\mathbf{y}$. When we have $\mathbf{y}_2$ (or $\mathbf{x}$, $\mathbf{z}_1$, $\mathbf{z}_2$) in the equation for W_1, it is implicit that we are viewing $\mathbf{y}_2$ as a function of $\mathbf{y}$ and t. For instance, when we write $W_2(\mathbf{y}_2,t)$ in $B_1(\mathbf{m})W_2$ we have*

$$\mathbf{y}_2 = \mathbf{y}_2(\mathbf{y},t) = \mathcal{R}[\theta_{10}-\theta_{20}]\mathbf{y} + \mathcal{R}[-\theta_{20}]\left(\mathbf{r}_2(t) - \mathbf{r}_1(t)\right).$$

In particular, we point out that the B_j are nonlocal operators.

3.2. Estimates on B_j and J_j. The following lemma, though easy to prove, is key to our strategy.

Lemma 3.3. *For $j = 1, 2$ and for $0 \le b' \le b \le b_0$ we have*

$$\|[B_j(\mathbf{m}(t))V](\cdot)\|_{L^5_b} \le Ce^{-\frac{b'}{2}\mu(t)} \|V(\cdot)\|_{L^5_{b'}} .$$

The constant $C > 0$ is independent of b' and of $\mathbf{m}$.

Proof. We carry out the details for $j = 1$. First notice that since F is C^2 and Q is exponentially localized we have

$$\begin{aligned} |B_1(\mathbf{m})| &= \chi_1(\mathbf{x}, t) \left|F'(Q(\mathbf{z}_1) + Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right| \\ &\le C\chi_1(\mathbf{x}, t) |Q(\mathbf{z}_1)| \\ &\le C\chi_1(\mathbf{x}, t)e^{-b_0|\mathbf{z}_1|} \\ &\le C\chi_1(\mathbf{x}, t)e^{-b_0|\mathcal{R}[-\theta_1]\mathbf{y}|} \\ &\le C\chi_1(\mathbf{x}, t)e^{-b_0|\mathbf{y}|}. \end{aligned}$$

Therefore

$$e^{b|\mathbf{y}|} |[B_jV](\mathbf{y})| \le C\chi_1(\mathbf{x}, t) |V(\mathbf{y}_2)| = C\chi_1(\mathbf{x}, t)e^{-b'|\mathbf{y}_2|}e^{b'|\mathbf{y}_2|} |V(\mathbf{y}_2)| . \tag{3.4}$$

Next,

$$\chi_1(\mathbf{x}, t)e^{-b'|\mathbf{y}_2|} \le \exp\left(-b' \inf_{\mathbf{x}\in\text{supp}\chi_1(t)} |\mathbf{y}_2(t)|\right) \le \exp\left(-b' \inf_{\mathbf{x}\in\text{supp}\chi_1(t)} |\mathbf{x} - \mathbf{r}_2(t)|\right).$$

By assumption, χ_1 is zero outside $\Sigma_1 \cup \{\mathbf{x} : \text{dist}(\mathbf{x}, L(t)) \le 1\}$, and Σ_1 is the set of points "on same side" of $L(t)$ as $\mathbf{r}_1$. Also recall that $L(t)$ is equidistant from $\mathbf{r}_1$ and $\mathbf{r}_2$. Thus we conclude that the minimum value of $|\mathbf{x} - \mathbf{r}_2|$ in Σ_1 occurs halfway between $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e., at $\mathbf{x}_0 = (\mathbf{r}_1(t) + \mathbf{r}_2(t))/2$. The minimum value of $|\mathbf{x} - \mathbf{r}_2|$ in the slightly larger set suppχ_1 must occur within a distance of 1 of $\mathbf{x}_0$, and so

$$\chi_1(\mathbf{x}, t)e^{-b'|\mathbf{y}_2|} \le Ce^{-\frac{b'}{2}\mu(t)}.$$

Taking the L^5 norm of (3.4) and using this last inequality finishes the proof. ∎

Notice two important features of this lemma. First, setting $b' = 0$ and $b = b_0$, we see that B_jV decays in space more rapidly than V does, although the norm of B_j as a map from L^5 to $L^5_{b_0}$ is $O(1)$. Second, if $b' = b = b_0$, then the norm of B_j as a map from $L^5_{b_0}$ to $L^5_{b_0}$ is exponentially small in the separation distance.

The next set of lemmas gives L^5 estimates for the terms which comprise J_j.

Lemma 3.4. *There exists a constant $C > 0$ so that for $j = 1, 2$, $t \ge 0$, and $|\phi_j(t)| \le \pi/2$,*

$$\|J_{j,mod}(\cdot, t)\|_{L^5} \le C |\mathbf{m}(t)| \|W_j(\cdot, t)\|_{W^{8/5,5}} .$$

Lemma 3.5. *We have a constant $C > 0$ so that for all $0 \le b \le b_0$, $j = 1, 2$, and all $t \ge 0$,*

$$\|J_{j,int}(\cdot, t)\|_{L^5_b} \le Ce^{-\frac{b_0}{2}\mu(t)} \left(1 + \|W_1(\cdot, t)\|_{L^5_b}\right).$$

Lemma 3.6. *We have a constant $C > 0$ so that for all $0 \le b \le b_0$, $j = 1, 2$, and all $t \ge 0$,*

$$\|J_{j,nl}(\cdot,t)\|_{L^5_b} \le C\left(\|W_1\|^2_{W_b^{8/5,5}} + \|W_2\|^2_{W_b^{8/5,5}}\right).$$

Lemma 3.7. *There exists $C > 0$ so that for $j = 1, 2$, $t \ge 0$, and $|\phi_j(t)| \le \pi/2$,*

$$\|K_j(\cdot,t)\|_{L^5} \le C\,|\dot{\mathbf{m}}(t)|\left(|\mathbf{m}(t)| + \|W_j(\cdot,t)\|_{W^{8/5,5}}\right).$$

We now prove these in order.

Proof of Lemma 3.4. We have $|\mathcal{R}[\theta_1 - \theta_{10}] - id| \le C|\phi_1|$, and so the first term in $J_{1,mod}$ is handled easily.

For the second term, we use that $F \in C^2$ to see

$$\left|F'(Q(\mathbf{z}_1)) - F'(Q(\mathbf{y}))\right| \le C\,|Q(\mathbf{z}_1) - Q(\mathbf{y})|.$$

By the mean value theorem there is a $\mathbf{y}_*$ on the line segment connecting $\mathbf{z}_1$ to $\mathbf{y}$ so that

$$|Q(\mathbf{z}_1) - Q(\mathbf{y})| = |\mathbf{z}_1 - \mathbf{y}|\,|\nabla Q(\mathbf{y}_*)| \le |(\mathcal{R}[\phi_1] - id)|\,|\mathbf{y}|\,|\nabla Q(\mathbf{y}_*)| \le C\,|\phi_1|\,|\mathbf{y}|\left|e^{-\beta|\mathbf{y}_*|}\right|.$$

Since $|\mathbf{z}_1| = |\mathbf{y}|$ and $\mathbf{y}_*$ is on the segment connecting them, we have $|\mathbf{y}_*| \ge |\mathbf{y}|\cos(\phi_1/2) \ge \sqrt{2}|\mathbf{y}|/2$. This implies $|\mathbf{y}|\,|e^{-\beta|\mathbf{y}_*|}| \le C$, and we are done. ∎

Proof of Lemma 3.5. The estimate for the first term goes as follows: We have $F \in C^2$ and $F(0) = 0$, and therefore there exists a constant $C > 0$ so that for all $Q_1, Q_2 \in \mathbf{R}^N$

$$|F(Q_1 + Q_2) - F(Q_1) - F(Q_2)| \le C|Q_1||Q_2|.$$

The proof can be obtained by the mean value theorem (see [22]). Thus

$$\begin{aligned}
&|\chi_1(\mathbf{x},t)\,[F(Q(\mathbf{z}_1) + Q(\mathbf{z}_2)) - F(Q(\mathbf{z}_1)) - F(Q(\mathbf{z}_2))]| \\
&\qquad \le C\chi_1(\mathbf{x},t)|Q(\mathbf{z}_1)||Q(\mathbf{z}_2)| \\
&\qquad \le C\left(\sup_{\mathbf{x}\in \mathrm{supp}\chi_1(t)} \exp(-b_0|\mathbf{z}_2|)\right) e^{-\beta|\mathbf{z}_1|} \le Ce^{-\frac{b_0}{2}\mu(t)}e^{-\beta|\mathbf{y}|}.
\end{aligned}$$

Note that we have used the same estimate as appeared in the proof of Lemma 3.3. Since $e^{-\beta|\mathbf{y}|}$ is in L^5_b we have completed the estimate for the first term.

The proof for the second term is nearly identical to the proof for Lemma 3.3, though it is easier. To wit, we have

$$\begin{aligned}
&\left|\chi_1(\mathbf{x},t)\left[\left(F'(Q(\mathbf{z}_1) + Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_1))\right) W_1(\mathbf{y})\right]\right| \\
&\qquad \le C\chi_1(\mathbf{x},t)\,|Q(\mathbf{z}_2)|\,|W_1(\mathbf{y})| \\
&\qquad \le C\left(\sup_{\mathbf{x}\in \mathrm{supp}\chi_1(t)} \exp(-b_0|\mathbf{z}_2|)\right)|W_1(\mathbf{y})| \le Ce^{-\frac{b_0}{2}\mu(t)}\,|W_1(\mathbf{y})|.
\end{aligned}$$ ∎

Proof of Lemma 3.6. This estimate follows from Taylor's theorem applied to F, combined with the fact that $W^{8/5,5}$ is an algebra. ■

Proof of Lemma 3.7. First,

$$\dot{\mathbf{p}}_1 = \dot{\mathbf{r}}_1 - \mathcal{R}[\theta_1(t)]c\mathbf{i} = \mathcal{R}[\theta_1(t)]\left(\mathcal{R}[-\theta_1(t)]\mathbf{r}_1(t) - c\mathbf{i}\right)$$

and

$$\dot{\phi}_1 = \dot{\theta}_1.$$

Then an argument nearly identical to that used in the proof of Lemma 3.4 shows that for $|\phi_1| \leq \pi/2$ we have

$$\left|(1 + \mathbf{y}^{\perp}\cdot)\left(\nabla Q(\mathbf{z}_1) - \nabla Q(\mathbf{y})\right)\right| \leq C|\phi_1(t)|(1 + |\mathbf{y}|)e^{-h|\mathbf{y}|}$$

for a constant $h > 0$. Since $|\mathbf{y}|e^{-h|\mathbf{y}|} \in L^5$ we can estimate the first line in K_1 by $C|\dot{\mathbf{m}}(t)||\mathbf{m}(t)|$.

The estimate for the second line follows from observing that $|\mathbf{y}||Q(\mathbf{y})| \in L^5$ and $|\mathcal{R}[\phi_1] - id| \leq C|\phi_1|$. Estimating the third line requires no intermediate steps. ■

4. The exit manifold.

4.1. Isolating the center directions. So long as U and V are not too far from the set of translations and rotations of Q in $W^{8/5,5}$, we can use Lemma 3.1 to select the collective coordinates $\mathbf{r}_j(t)$ and $\theta_j(t)$ so that

$$\Pi^c W_1(\cdot, t) = 0 \quad \text{and} \quad \Pi^c W_2(\cdot, t) = 0. \tag{4.1}$$

Applying Π^s to (3.2) and (3.3), we have, because of (4.1),

$$\begin{aligned} \partial_t W_1 &= AW_1 + \Pi^s B_1(\mathbf{m})W_2 + \Pi^s J_1, \\ \partial_t W_2 &= AW_2 + \Pi^s B_2(\mathbf{m})W_1 + \Pi^s J_2. \end{aligned} \tag{4.2}$$

We can isolate the equations of motion for $\mathbf{r}_j$ and θ_j by applying Π^c to (3.2) and (3.3). If we coordinatize E^c as

$$\Pi^c\left(\mathbf{a}\cdot\nabla Q(\mathbf{y}) + \phi\mathbf{y}^{\perp}\cdot\nabla Q(\mathbf{y})\right) = \begin{pmatrix} \mathbf{a} \\ \phi \end{pmatrix} \in \mathbf{R}^3,$$

applying the projections gives

$$\begin{aligned} \begin{pmatrix} \mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i} \\ \dot{\theta}_1 \end{pmatrix} &= -\Pi^c\left(B_1(\mathbf{m})W_2(\mathbf{y}_2, t) + K_1 + J_{1,mod} + J_{1,int} + J_{1,nl}\right), \\ \begin{pmatrix} \mathcal{R}[-\theta_2]\dot{\mathbf{r}}_2 - c\mathbf{i} \\ \dot{\theta}_2 \end{pmatrix} &= -\Pi^c\left(B_2(\mathbf{m})W_1(\mathbf{y}_1, t) + K_2 + J_{2,mod} + J_{2,int} + J_{2,nl}\right). \end{aligned} \tag{4.3}$$

Since K_1 and K_2 depend explicitly on $\dot{\mathbf{r}}_j(t)$ and $\dot{\theta}_j(t)$, (4.3) does not quite constitute the equations of motion. The following lemma helps us isolate $\dot{\mathbf{r}}_j(t)$ and $\dot{\theta}_j(t)$.

Lemma 4.1.

$$\Pi^c K_j = L_j(\mathbf{m}, \nabla W_j) \begin{pmatrix} \mathcal{R}[-\theta_j]\dot{\mathbf{r}}_j - c\mathbf{i} \\ \dot{\theta}_j \end{pmatrix}, \tag{4.4}$$

and if $|\phi_j(t)| \leq \pi/2$*, then*

$$\|L_j(\mathbf{m}(t), \nabla W_j(t))\| \leq C\,|\phi_j(t)| + C\,\|W_j(t)\|_{L^5}.$$

Proof. Notice that we can rewrite $\Pi^c K_1$ as

$$\begin{aligned}\Pi^c K_1 = \Pi^c\left((\nabla Q(\mathbf{z}_1) - \nabla Q(\mathbf{y})) + \mathcal{R}[-\phi_1]\nabla W_1\right)\cdot\{\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}\} \\ + \Pi^c\left(\mathcal{R}[-\phi_1]\mathbf{y}^\perp\cdot(\nabla Q(\mathbf{z}_1) - \nabla Q(\mathbf{y})) + \left(\mathcal{R}[-\phi_1]\mathbf{y}^\perp - \mathbf{y}^\perp\right)\cdot\nabla Q(\mathbf{y})\right)\dot{\theta}_1.\end{aligned}$$

Thus we have (4.4). Now we estimate the various terms in $\Pi^c K_1$. We use the estimates in Lemma 3.7 to observe that the second line above can be bounded above by

$$C\,|\phi_1(t)|\,\left|\dot{\theta}_1\right|.$$

Similarly, we saw in the proof of Lemma 3.7 that $\|\nabla Q(\mathbf{z}_1) - \nabla Q(\mathbf{y})\|_{L^2} \leq C\,|\phi_1(t)|$, and since we know from Hypothesis 1.3 that Π^c is computed by taking an L^2 inner product with the adjoint function, we have

$$|\Pi^c\left(\nabla Q(\mathbf{z}_1) - \nabla Q(\mathbf{y})\right)\cdot\{\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}\}| \leq C\,|\phi_1(t)|\,|\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}|.$$

Finally, we have by an integration by parts and Hölder's inequality

$$\begin{aligned}|\Pi^c(\mathcal{R}[-\phi_1]\nabla W_1)\cdot\{\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}\}| &= \left|\langle\psi^\dagger, [(\mathcal{R}[-\phi_1]\nabla W_1)\cdot\{\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}\}]\rangle\right| \\ &= \left|\langle[\mathcal{R}[\phi_1]\,(\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i})]\cdot\nabla\psi^\dagger, W_1\rangle\right| \\ &\leq |\mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i}|\,\left\|\nabla\psi^\dagger\right\|_{L^{5/4}}\|W_1\|_{L^5}.\end{aligned}$$

Recall that Hypothesis 1.3 guarantees that $\|\nabla\psi^\dagger\|_{L^{5/4}} < \infty$. This completes the proof. ∎

With this lemma, provided that $\mathbf{m}$ and W are small, we have no problem inverting $id + L_j$. Therefore

$$\begin{aligned}\begin{pmatrix} \mathcal{R}[-\theta_1]\dot{\mathbf{r}}_1 - c\mathbf{i} \\ \dot{\theta}_1 \end{pmatrix} &= -\Pi^c\left(B_1(\mathbf{m})W_2(\mathbf{y}_2, t) + J_1 - K_1 + \check{K}_1\right), \\ \begin{pmatrix} \mathcal{R}[-\theta_2]\dot{\mathbf{r}}_2 - c\mathbf{i} \\ \dot{\theta}_2 \end{pmatrix} &= -\Pi^c\left(B_2(\mathbf{m})W_1(\mathbf{y}_1, t) + J_2 - K_2 + \check{K}_2\right),\end{aligned}$$

where

$$\check{K}_1 = -\left((id + L_j)^{-1} - id\right)\Pi^c\left(B_1 W_2(\mathbf{y}_2, t) + J_1 - K_1\right)$$

and $\check{K}_2$ is defined analogously. Our previous estimates show that there is a constant $C > 0$ so that if $|\mathbf{m}|_{C^1[0,T_1]} \leq \pi/2$, then

$$\left|\check{K}_1(t)\right| \leq C\left(|\mathbf{m}(t)| + \|W_1\|_{L^5}\right)\left(\|W_2\|_{L^5} + \|J_1 - K_1\|_{L^5}\right). \tag{4.5}$$

Recalling their definitions, we see that the evolution equations for $\mathbf{p}_j$ and ϕ_j are

$$\begin{aligned}\begin{pmatrix} \dot{\mathbf{p}}_1 \\ \dot{\phi}_1 \end{pmatrix} &= -\tilde{\mathcal{R}}[\phi_1+\theta_{10}]\Pi^c\left(B_1W_2(\mathbf{y}_2,t)+J_1-K_1+\check{K}_1\right),\\ \begin{pmatrix} \dot{\mathbf{p}}_2 \\ \dot{\phi}_2 \end{pmatrix} &= -\tilde{\mathcal{R}}[\phi_2+\theta_{20}]\Pi^c\left(B_2W_1(\mathbf{y}_1,t)+J_2-K_2+\check{K}_2\right),\end{aligned} \tag{4.6}$$

with

$$\tilde{\mathcal{R}}[\theta] := \begin{bmatrix} \mathcal{R}[\theta] & 0 \\ 0 & 1 \end{bmatrix}.$$

Equations (4.2) and (4.6) are equivalent to (1.2), and our goal now is to solve this system. To compress notation, we let

$$\mathbf{W} = (W_1, W_2),$$

$$\mathbf{m} = (\mathbf{p}_1, \phi_1, \mathbf{p}_2, \phi_2),$$

$$\mathbf{J}_s(\mathbf{m},\mathbf{W}) = (J_1, J_2),$$

$$\mathbf{J}_{cc}(\mathbf{m},\mathbf{W}) = (J_1-K_1, J_2-K_2),$$

$$\mathbf{K}_{cc}(\mathbf{m},\mathbf{W}) = (\check{K}_1, \check{K}_2),$$

$$\mathbf{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

$$\mathbf{B}(\mathbf{m}) = \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix},$$

$$\mathbf{P}^s := \operatorname{diag}(\Pi^s, \Pi^s), \quad \mathbf{P}^c := \operatorname{diag}(\Pi^c, \Pi^c),$$

$$\mathbf{R}(\mathbf{m}) := \begin{bmatrix} \tilde{\mathcal{R}}[\phi_1+\theta_{10}] & 0 \\ 0 & \tilde{\mathcal{R}}[\phi_2+\theta_{20}] \end{bmatrix},$$

and then we see that (4.2) and (4.6) become

$$\begin{aligned}\mathbf{W}_t &= \mathbf{A}\mathbf{W} + \mathbf{P}^s\left(\mathbf{B}(\mathbf{m})\mathbf{W} + \mathbf{J}_s(\mathbf{m},\mathbf{W})\right),\\ \dot{\mathbf{m}} &= -\mathbf{R}(\mathbf{m})\mathbf{P}^c\left(\mathbf{B}(\mathbf{m})\mathbf{W} + \mathbf{J}_{cc}(\mathbf{m},\mathbf{W})\right) - \mathbf{R}(\mathbf{m})\mathbf{K}_{cc}(\mathbf{m},\mathbf{W}).\end{aligned} \tag{4.7}$$

Note that if we take the appropriate $L^5[0,T]$ norms of the estimates in Lemmas 3.4–3.7, we arrive at the following estimates for $\mathbf{J}_s$ and $\mathbf{J}_{cc}$.

Lemma 4.2. *There is a constant $C>0$ so that if $\|\mathbf{m}\|_{C^1[0,T]} \le \pi/2$, then*

$$\begin{aligned}\|\mathbf{J}_s\|_{X_{a,0}[T]} \le C\Big\{&\|\mathbf{m}\|^2_{C^1([0,T])} + \|\mathbf{m}\|_{C^1([0,T])}\|\mathbf{W}\|_{X'_{a,0}[T]}\\ &+ N(\mathbf{m}) + e^{-\frac{b_0}{2}\mu_0}\|\mathbf{W}\|_{X'_{a,0}[T]} + \|\mathbf{W}\|^2_{X'_{a,0}[T]}\Big\}\end{aligned} \tag{4.8}$$

and
(4.9)

$$\|\mathbf{J}_{cc}\|_{X_{a,0}[T]} \leq C\left\{\|\mathbf{m}\|_{C^1([0,T])}\|\mathbf{W}\|_{X'_{a,0}[T]} + N(\mathbf{m}) + e^{-\frac{b_0}{2}\mu_0}\|\mathbf{W}\|_{X'_{a,0}[T]} + \|\mathbf{W}\|^2_{X'_{a,0}[T]}\right\}.$$

Here

$$N(\mathbf{m}) := \left(\int_0^T \left|e^{at}e^{-\frac{b_0}{2}\mu(t)}\right|^5 dt\right)^{1/5} = \left\|\exp\left(a\diamond - \frac{b_0}{2\mu(\diamond)}\right)\right\|_{L^5[0,T]}.$$

The constant C does not depend on $0 \leq T \leq \infty$.

4.2. Stability of exits. We now can prove Theorem 1.7. We must first show that (given appropriate initial pulse positions and orientations) smallness of the modulation parameters $\mathbf{m}(t)$ implies that $\mu(t)$ is large. This in turn implies that $\mathbf{J}_s$, $\mathbf{J}_{cc}$, and $\mathbf{K}_{cc}$ will be small. Recall that $\mu^\star(t)$ is the separation distance between pulses if $\mathbf{m} = 0$ and that $\mu_0^\star$ is the minimum separation distance in this situation. We need the following result.

Lemma 4.3. *For all $\Delta\mathbf{v}_0 \neq 0$ and $\Delta\mathbf{r}_0$ with $\mu_0^\star > 0$, there exists $\delta_1 > 0$ so that $|\mathbf{m}(t)| \leq \delta_1$ (for $t \geq 0$) implies*

$$\mu(t) \geq \frac{1}{2}\mu^\star(t)$$

for all $t \geq 0$. Additionally, with $k^\star := c|\Delta\mathbf{v}_0|\frac{\mu_0^\star}{|\Delta\mathbf{r}_0|}$,

$$N(\mathbf{m}) = \left|\int_0^\infty (e^{\frac{b_0}{8}k^\star t}e^{-\frac{b_0}{2}\mu(t)})^5 dt\right|^{1/5} \leq C|\Delta\mathbf{v}_0|^{-1/5}e^{-\frac{b_0}{4}\mu_0^\star}.$$

Proof of Lemma 4.3. Recall from the introduction that

$$\mu^\star(t) \geq k^\star t,$$

where $k^\star$ is as in the statement of the lemma.

Take

$$\delta_1 = \min\left\{\frac{\mu_0^\star}{4}, \frac{k^\star}{8c}\right\}.$$

The definition of $\mathbf{p}_j$ and ϕ_j together with the triangle inequality give

$$\begin{aligned}
\mu(t) &\geq \mu^\star(t) - \sum_{j=1}^2\left(|\mathbf{p}_j(t)| + c\left|\int_0^t [\mathcal{R}[\phi_j(s)+\theta_{j0}] - \mathcal{R}[\theta_{j0}]]\,\mathbf{i}ds\right|\right)\\
&\geq \mu^\star(t) - \sum_{j=1}^2\left(|\mathbf{p}_j(t)| + ct\sup_{0\leq s\leq t}|\mathcal{R}[\phi_j(s)+\theta_{j0}] - \mathcal{R}[\theta_{j0}]|\right)\\
&\geq \mu^\star(t) - \delta_1 - 2ct\delta_1\\
&\geq \frac{1}{2}\mu^\star(t) + \frac{1}{4}\mu_0^\star - \delta_1 + \frac{1}{4}(\mu^\star(t) - 8ct\delta_1)\\
&\geq \frac{1}{2}\mu^\star(t).
\end{aligned}$$

The integral estimate follows from this. Here is the calculation:

$$\begin{aligned}\int_0^\infty (e^{\frac{b_0}{8}k^\star t} e^{-\frac{b_0}{2}\mu(t)})^5 dt &\le e^{-\frac{5b}{4}\mu_0^\star} \int_0^\infty e^{\frac{5b_0}{8}k^\star t - \frac{5b_0}{4}\mu^\star(t)} dt \\ &\le e^{-\frac{5b_0}{4}\mu_0^\star} \int_0^\infty e^{-\frac{5b_0}{8}k^\star t} \\ &\le C|\Delta \mathbf{v}_0|^{-1} e^{-\frac{5b_0}{4}\mu_0^\star}. \qquad \blacksquare\end{aligned}$$

Proof of Theorem 1.7. If the operator $\mathbf{B}$ was small in norm on $X_{a,0}$, we could use this and a straightforward perturbation argument to prove our result. Instead we must make one change to (4.7), which is motivated by (1.4). Consider the system

$$\begin{aligned}\mathbf{W}^{ic}_t &= \mathbf{A}\mathbf{W}^{ic} + \mathbf{P}^s \mathbf{J}_s(\mathbf{m}, \mathbf{W}^{ic} + \mathbf{W}^{loc}),\\ \mathbf{W}^{loc}_t &= \mathbf{A}\mathbf{W}^{loc} + \mathbf{P}^s \left(\mathbf{B}(\mathbf{m}) \left(\mathbf{W}^{ic} + \mathbf{W}^{loc}\right)\right),\\ \dot{\mathbf{m}} &= -\mathbf{R}(\mathbf{m})\mathbf{P}^c \left(\mathbf{B}(\mathbf{m}) \left(\mathbf{W}^{ic} + \mathbf{W}^{loc}\right) + \mathbf{J}_{cc}(\mathbf{m}, \mathbf{W}^{ic} + \mathbf{W}^{loc})\right)\\ &\quad - \mathbf{R}(\mathbf{m})\mathbf{K}_{cc}(\mathbf{m}, \mathbf{W}^{ic} + \mathbf{W}^{loc}),\end{aligned} \tag{4.10}$$

with initial conditions

$$\mathbf{W}^{ic}(\mathbf{y},0) = \mathbf{W}_0 \in (E^s)^2 \cap W^{8/5,5}, \quad \mathbf{W}^{loc}(\mathbf{y},0) = 0, \quad \text{and} \quad \mathbf{m}(0) = 0.$$

If $\mathbf{W} = \mathbf{W}^{ic} + \mathbf{W}^{loc}$, then $\mathbf{W}$ solves (4.7). Notice that the right-hand side of the equation for $\mathbf{W}^{ic}$ consists of terms which are small in $X_{a,0}[T]$ due to Lemma 4.2 and estimate (4.5). The right-hand side of the equation for $\mathbf{W}^{loc}$ is in $X_{a,b_0}[T]$ even if $\mathbf{W}^{ic} \in X_{a,0}[T]$, due to the localizing property of $\mathbf{B}$ seen in Lemma 3.3. (The superscript "ic" represents "initial condition," and the superscript "loc" stands for "localized.")

In Theorem 6.1 in the appendix we establish that $\|\mathbf{W}_0\|_{W^{8/5,5}} \le \delta_2$ implies that (4.10) has a unique solution

$$(\mathbf{W}^{ic}, \mathbf{W}^{loc}, \mathbf{m}) \in X'_{a,0}[T_1] \times X'_{a,b_0}[T_1] \times C^1[0,T_1]$$

for some $0 < T_1 \le \infty$, where we have taken $a = \min\{\alpha/2, b_0 k^\star/8\}$. $\delta_2 > 0$ is a universal constant. Such a solution satisfies the integral equation

$$\begin{aligned}\mathbf{W}^{ic}(t) &= e^{\mathbf{A}t}\mathbf{W}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{P}^s \mathbf{J}_s(\mathbf{m}(\tau), \mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau))d\tau,\\ \mathbf{W}^{loc}(t) &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{P}^s \mathbf{B}(\mathbf{m}(\tau)) \left(\mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau)\right) d\tau,\\ \mathbf{m}(t) &= -\int_0^t \Big\{\mathbf{R}(\mathbf{m}(\tau))\mathbf{P}^c \left(\mathbf{B}(\mathbf{m}(\tau)) \left(\mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau)\right) + \mathbf{J}_{cc}(\mathbf{m}(\tau), \mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau))\right)\\ &\qquad + \mathbf{R}[\mathbf{m}(\tau)]\mathbf{K}_{cc}(\mathbf{m}(\tau), \mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau))\Big\} d\tau.\end{aligned} \tag{4.11}$$

Let

$$\mathcal{K}(T) := \|\mathbf{W}^{ic}\|_{X'_{a,0}[T]} + \|\mathbf{W}^{loc}\|_{X'_{a,b_0}[T]} + \|\mathbf{m}\|_{C^1[0,T]}.$$

Here, $0 < T \le T_1$, and we assume that $\mathcal{K}(T) \le \delta_1$, which allows us to apply Lemma 4.3 when needed.

Applying Theorem 2.1 to (4.11) tells us that

$$(4.12)\quad \begin{aligned} \|\mathbf{W}^{ic}\|_{X'_{a,0}[T]} &\le C\left(\|\mathbf{W}_0\|_{W^{8/5,5}} + \left\|\mathbf{J}_s(\mathbf{m},\mathbf{W}^{ic}+\mathbf{W}^{loc})\right\|_{X_{a,0}[T]}\right),\\ \|\mathbf{W}^{loc}\|_{X'_{a,b_0}[T]} &\le C\left\|\mathbf{B}(\mathbf{m})\left(\mathbf{W}^{ic}+\mathbf{W}^{loc}\right)\right\|_{X_{a,b_0}[T]}. \end{aligned}$$

The estimates in Lemmas 4.2 and 4.3 yield

$$(4.13)\quad \begin{aligned} \|\mathbf{W}^{ic}\|_{X'_{a,0}[T]} \le C\Big\{&\|\mathbf{W}_0\|_{W^{8/5,5}} + \|\mathbf{m}\|^2_{C^1([0,T])}\\ &+\left(\|\mathbf{m}\|_{C^1([0,T])} + e^{-\frac{b_0}{4}\mu_0^\star}\right)\left(\|\mathbf{W}^{ic}\|_{X'_{a,0}[T]} + \|\mathbf{W}^{loc}\|_{X'_{a,b_0}[T]}\right)\\ &+C|\Delta\mathbf{v}_0|^{-1/5}e^{-\frac{b_0}{4}\mu_0^\star} + \|\mathbf{W}^{ic}\|^2_{X'_{a,0}[T]} + \|\mathbf{W}^{loc}\|^2_{X'_{a,b_0}[T]}\Big\}\\ \le C_1^*\,|\Delta\mathbf{v}_0|^{-1/5}&\left(\|\mathbf{W}_0\|_{W^{8/5,5}} + e^{-\frac{b_0}{4}\mu_0^\star} + e^{-\frac{b_0}{4}\mu_0^\star}\mathcal{K}(T) + \mathcal{K}^2(T)\right). \end{aligned}$$

Note that the constant $C_1^* > 0$ does not depend upon $\mathcal{K}(T)$, T, a, b_0, or $\|\mathbf{W}_0\|_{W^{8/5,5}}$.

Similarly if we apply Lemma 3.3, we get

$$(4.14)\quad \|\mathbf{W}^{loc}\|_{X'_{a,b_0}[T]} \le C\left\{e^{-\frac{b_0}{4}\mu_0^\star}\,\|\mathbf{W}^{loc}\|_{X'_{a,b_0}[T]} + \|\mathbf{W}^{ic}\|_{X'_{a,0}[T_1]}\right\},$$

which, if we combine it with the last inequality for $\|\mathbf{W}^{ic}\|_{X'_{a,0}[T]}$, gives

$$(4.15)\quad \|\mathbf{W}^{loc}\|_{X'_{a,b_0}[T]} \le C_2^*|\Delta\mathbf{v}_0|^{-1/5}\left\{\|\mathbf{W}_0\|_{W^{8/5,5}} + e^{-\frac{b_0}{4}\mu_0^\star} + e^{-\frac{b_0}{4}\mu_0^\star}\mathcal{K}(T) + \mathcal{K}^2(T)\right\}.$$

$C_2^* > 0$ does not depend upon $\mathcal{K}(T)$, T, a, b_0, or $\|\mathbf{W}_0\|_{W^{8/5,5}}$.

We wish to estimate $\mathbf{m}$ in $C^1[0,T]$. We begin with

$$(4.16)\quad \begin{aligned} \|\mathbf{m}\|_{C[0,T]} \le C\sup_{0\le t\le T}\int_0^t\Big\{&\left\|\mathbf{B}(\mathbf{m}(\tau))\left(\mathbf{W}^{ic}(\tau)+\mathbf{W}^{loc}(\tau)\right)\right\|_{L^5}\\ &+\left\|\mathbf{J}_{cc}(\mathbf{m}(\tau),\mathbf{W}^{ic}(\tau)+\mathbf{W}^{loc}(\tau))\right\|_{L^5} + \left|\mathbf{K}_{cc}(\mathbf{m}(\tau),\mathbf{W}^{ic}(\tau)+\mathbf{W}^{loc}(\tau))\right|\Big\}\,d\tau. \end{aligned}$$

To estimate (4.16) we first apply Lemma 3.3:

$$\begin{aligned} |\mathbf{m}(t)| \le C\sup_{0\le t\le T}\int_0^t\Big\{&\|\mathbf{W}^{ic}(\tau)\|_{L^5} + e^{-\frac{b_0}{4}\mu_0^\star}\,\|\mathbf{W}^{loc}(\tau)\|_{L^5_{b_0}}\\ &+\|\mathbf{J}_{cc}(\mathbf{m}(\tau),\mathbf{W}^{ic}(\tau)+\mathbf{W}^{loc}(\tau))\|_{L^5} + \left|\mathbf{K}_{cc}(\mathbf{m}(\tau),\mathbf{W}^{ic}(\tau)+\mathbf{W}^{loc}(\tau))\right|\Big\}\,d\tau. \end{aligned}$$

We want a bound independent of t. We have by Hölder's inequality for a general function $\mathbf{F}$ and $0 \le b \le b_0$

$$\begin{aligned}\int_0^t \|\mathbf{F}(\tau)\|_{L^5_b}\, d\tau &= \int_0^t e^{-b_0 k^\star t/8} e^{b_0 k^\star t/8} \|\mathbf{F}(\tau)\|_{L^5_b}\, d\tau \\ &\le |b_0 k^\star/8|^{-4/5} \|\mathbf{F}\|_{X_{a,b}[T]} \\ &\le C\, |\Delta \mathbf{v}_0|^{-4/5} \|\mathbf{F}\|_{X_{a,b}[T]}.\end{aligned} \tag{4.17}$$

Applying this to all but the $\mathbf{K}_{cc}$ term, we have

$$\begin{aligned}|\mathbf{m}(t)| \le C\, |\Delta \mathbf{v}_0|^{-4/5} &\left\{ \left\|\mathbf{W}^{ic}\right\|_{X_{a,0}[T]} + e^{-\frac{b_0}{4}\mu^\star} \left\|\mathbf{W}^{loc}\right\|_{X_{a,b_0}[T]} + \left\|\mathbf{J}_{cc}(\mathbf{m}, \mathbf{W}^{ic} + \mathbf{W}^{loc})\right\|_{X_{a,0}[T]} \right\} \\ &+ \sup_{0\le t\le T} \int_0^t \left| \mathbf{K}_{cc}(\mathbf{m}(\tau), \mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau)) \right| d\tau.\end{aligned}$$

Using (4.9), Lemma 4.3 and the estimate (4.13) tell us that the first line above is bounded by

$$C\, |\Delta \mathbf{v}_0|^{-1} \left\{ \|\mathbf{W}_0\|_{W^{8/5,5}} + e^{-\frac{b}{4}\mu_0^\star} + e^{-\frac{b}{4}\mu_0^\star} \mathcal{K}(T) + \mathcal{K}^2(T) \right\}.$$

For the second line, applying (4.5) gives (where INT is the right-hand-side integral)

$$\begin{aligned}INT := &\sup_{0\le t\le T} \int_0^t \left| \mathbf{K}_{cc}(\mathbf{m}(\tau), \mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau)) \right| d\tau \\ &\le C \sup_{0\le t\le T} \int_0^t \left(\|\mathbf{m}\|_{C[0,T]} + \left\|\mathbf{W}^{ic}\right\|_{L^5} + \left\|\mathbf{W}^{loc}\right\|_{L^5_{b_0}} \right) \\ &\qquad \left(\left\|\mathbf{W}^{ic}\right\|_{L^5} + \left\|\mathbf{W}^{loc}\right\|_{L^5_{b_0}} + \|\mathbf{J}_{cc}\|_{L^5_{b_0}} \right) d\tau.\end{aligned}$$

The embedding (2.2) and (4.17) then imply

$$\begin{aligned}INT \le C|\Delta \mathbf{v}_0|^{-4/5} &\left(\|\mathbf{m}\|_{C[0,T]} + \left\|\mathbf{W}^{ic}\right\|_{X'_{a,0}[T]} + \left\|\mathbf{W}^{loc}\right\|_{X'_{a,b_0}[T]} \right) \\ &\left(\left\|\mathbf{W}^{ic}\right\|_{X_{a,0}[T]} + \left\|\mathbf{W}^{loc}\right\|_{X_{a,b_0}[T]} + \|\mathbf{J}_{cc}\|_{X_{a,0}[T]} \right).\end{aligned}$$

We consolidate the estimate for the two pieces to get

$$\|\mathbf{m}\|_{C[0,T]} \le C_3^* |\Delta \mathbf{v}_0|^{-1} (1 + \mathcal{K}(T)) \left\{ \|\mathbf{W}_0\|_{W^{8/5,5}} + e^{-\frac{b}{4}\mu_0^\star} + e^{-\frac{b}{4}\mu_0^\star} \mathcal{K}(T) + \mathcal{K}^2(T) \right\}.$$

The constant $C_3^* > 0$ does not depend upon $\mathcal{K}(T)$, T, a, b_0, or $\|\mathbf{W}_0\|_{W^{8/5,5}}$.

We must also estimate $\|\dot{\mathbf{m}}\|_{C[0,T]}$. Since $|\mathbf{P}^s \mathbf{F}| \le C\, \|\mathbf{F}\|_{L^5}$ it is clear from (4.10) and (2.2) that

$$\begin{aligned}\|\dot{\mathbf{m}}\|_{C[0,T]} &\le C \sup_{t\in[0,T]} \left\{ \left\|\mathbf{W}^{ic}\right\|_{L^5} + e^{-\frac{b_0}{4}\mu_0^\star} \left\|\mathbf{W}^{loc}\right\|_{L^5} + \|\mathbf{J}_{cc}\|_{L^5} + |\mathbf{K}_{cc}(t)| \right\} \\ &\le C \left\{ \left\|\mathbf{W}^{ic}\right\|_{X'_{a,0}[T]} + e^{-\frac{b_0}{4}\mu_0^\star} \left\|\mathbf{W}^{loc}\right\|_{X'_{a,b_0}} + \|\mathbf{J}_{cc}\|_{X'_{a,0}[T]} + \sup_{t\in[0,T]} |\mathbf{K}_{cc}(t)| \right\}.\end{aligned} \tag{4.18}$$

With the exception of $\sup_{t\in[0,T]}|\mathbf{K}_{cc}(t)|$ we have estimated each of these pieces already. This last piece can be estimated much like INT, and we omit the details as they are in fact simpler than those for INT.

Therefore

$$\|\mathbf{m}\|_{C^1[0,T]} \leq C_4^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K}(T))\left\{\|\mathbf{W}_0\|_{W^{8/5,5}} + e^{-\frac{b_0}{4}\mu_0^\star} + e^{-\frac{b_0}{4}\mu_0^\star}\mathcal{K}(T) + \mathcal{K}^2(T)\right\}. \tag{4.19}$$

The constant $C_4^* > 0$ does not depend upon $\mathcal{K}(T)$, T, a, b_0, or $\|\mathbf{W}_0\|_{W^{8/5,5}}$.

Putting (4.13), (4.15), and (4.19) together gives

$$\mathcal{K}(T) \leq C^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K}(T))\left\{\|\mathbf{W}_0\|_{W^{8/5,5}} + e^{-\frac{b_0}{4}\mu_0^\star} + e^{-\frac{b_0}{4}\mu_0^\star}\mathcal{K}(T) + \mathcal{K}^2(T)\right\}, \tag{4.20}$$

where $C^* = \max\{C_1^*, C_2^*, C_4^*\}$.

Let

$$\mathcal{K}_0 := \min\left\{1, \delta_1, \tilde{C}\delta_2, \frac{|\Delta\mathbf{v}_0|}{8C^*}\right\},$$

$$\delta_0 := \frac{\mathcal{K}_0|\Delta\mathbf{v}_0|}{4C^*(1+\mathcal{K}_0)},$$

and

$$\epsilon_{20} = \frac{\mathcal{K}_0|\Delta\mathbf{v}_0|}{4C^*(1+\mathcal{K}_0)^2}.$$

(Here, δ_1 is as in Lemma 4.3, δ_2 is as in Theorem 6.1, and $1/\tilde{C}$ is the constant from the embedding (2.2).) Suppose that

$$\|\mathbf{W}_0\|_{W^{8/5,5}} \leq \delta_{exit}$$

and

$$\mu_0^\star \geq -\frac{4}{b_0}\ln(\epsilon_{20}) =: M_0.$$

Also set

$$T_2 := \sup\{T \leq T_1 : \mathcal{K}(t) \leq \mathcal{K}_0 \text{ for all } 0 \leq t \leq T\}.$$

Since $\mathcal{K}(0) = 0$, this is either nonnegative or infinite. If it is infinite, then we have proven our main result. If it is finite, then we must have $\mathcal{K}(T_2) = \mathcal{K}_0$. Otherwise, Theorem 6.1 and (2.2) imply that the solution exists for some slightly longer time and, moreover, that $\mathcal{K}(t)$ is continuous on this larger interval.

However, (4.20) and our choices for $\mathcal{K}_0$, δ_0, and ϵ_{20} imply

$$C^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K})\mathcal{K}^2 \leq 2C^*|\Delta\mathbf{v}_0|^{-1}{\mathcal{K}_0}^2 \leq \frac{\mathcal{K}_0}{4},$$

$$C^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K})\,\|\mathbf{W}_0\|_{W^{8/5,5}} \leq C^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K}_0)\delta_0 \leq \frac{\mathcal{K}_0}{4},$$

and

$$C^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K})^2 e^{-\frac{b_0}{4}\mu_0^\star} \leq C^*|\Delta\mathbf{v}_0|^{-1}(1+\mathcal{K}_0)^2\epsilon_{20} \leq \frac{\mathcal{K}_0}{4},$$

which all together imply

$$\mathcal{K}_0 \leq 3/4\mathcal{K}_0,$$

which is impossible. Therefore T_2 is not finite, and we are done. ■

5. The shooting manifold.

5.1. Existence of shooting solutions. We now prove the existence of the shooting solution described in Theorem 1.9.

Proof of Theorem 1.9. Recall from section 3 that for an arbitrary choice $\mathbf{m}$, equations (3.2)–(3.3) describe the evolution of $\mathbf{W} = (W_1, W_2)$. In the previous sections it was desirable to choose $\mathbf{m}$ so that $\mathbf{W} \in E^s \times E^s$ for all $t > 0$. Now it is useful to set $\mathbf{m} = 0$, which is to say we take $\mathbf{r}_j(t) = \mathbf{r}_j^\star(t) = \mathbf{r}_{j0} + ct\mathbf{v}(\theta_{j0})$ and $\theta_j(t) = \theta_{j0}$. We make the assumption that $\Delta\mathbf{r}_0 \cdot \Delta\mathbf{v}_0 \leq 0$, which implies that

$$\mu(t) \geq \max\{|\Delta\mathbf{r}_0|, -c|\Delta\mathbf{v}_0|t\}$$

for $t \leq 0$.

In this case (1.2) becomes

$$\partial_t W_1 = AW_1 + B_1(0)W_2 + J_{1,int} + J_{1,nl},$$
$$\partial_t W_2 = AW_2 + B_2(0)W_1 + J_{2,int} + J_{2,nl},$$

which we write compactly as

$$\mathbf{W}_t = \mathbf{A}\mathbf{W} + \mathbf{B}(0)\mathbf{W} + \mathbf{J}_{shoot}(\mathbf{W}). \tag{5.1}$$

Let $\eta = -b_0 c\,|\Delta\mathbf{v}_0|\,/8$, and consider the map $\Psi : Z'_{\eta,b_0} \to Z'_{\eta,b_0}$ defined by

$$\Psi(\mathbf{W})(t) = \int_{-\infty}^{t} e^{\mathbf{A}(t-\tau)}\left(\mathbf{B}(0)\mathbf{W}(\tau) + \mathbf{J}_{shoot}(\mathbf{W}(\tau))\right) d\tau.$$

We prove that there exists $\mathcal{K}_1 > 0$ with the property that $\|\mathbf{W}\|_{Z_{\eta,b_0}} \leq \mathcal{K}_1$ implies

$$\|\Psi(\mathbf{W})\|_{Z'_{\eta,b_0}} \leq \mathcal{K}_1 \tag{5.2}$$

and

$$\left\|\Psi(\mathbf{W}) - \Psi(\tilde{\mathbf{W}})\right\|_{Z'_{\eta,b_0}} \leq \frac{1}{2}\left\|\mathbf{W} - \tilde{\mathbf{W}}\right\|_{Z'_{\eta,b_0}}. \tag{5.3}$$

This implies that Ψ has a unique fixed point $\mathbf{W}^* = (W_1^*, W_2^*) \in Z'_{\eta,b_0}$ which solves (5.1). These are the functions described in Theorem 1.9, and once we establish their existence, the proof is complete.

To prove the estimate (5.2) we apply Theorem 2.1, part 2:

$$\|\Psi(\mathbf{W})\|_{Z'_{\eta,b_0}} \leq C\left(\|\mathbf{B}(0)\mathbf{W}\|_{Z_{\eta,b_0}} + \|\mathbf{J}_{shoot}(\mathbf{W})\|_{Z_{\eta,b_0}}\right).$$

Lemma 3.3 implies (since $\mu(t) \geq |\Delta\mathbf{r}_0|$)

$$\|\mathbf{B}(0)\mathbf{W}\|_{Z_{\eta,b_0}} \leq Ce^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|}\,\|\mathbf{W}\|_{Z_{\eta,b_0}}.$$

Lemmas 3.5 and 3.6 imply

$$\|\mathbf{J}_{shoot}\|_{Z_{\eta,b_0}} \leq C\left(N + e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|}\|\mathbf{W}\|_{Z_{\eta,b_0}} + \|\mathbf{W}\|^2_{Z_{\eta,b_0}}\right),$$

where

$$N = \left(\int_{\mathbf{R}^-}\left|e^{-\frac{b_0}{2}\mu(t)}e^{\eta t}\right|^5 dt\right)^{1/5} \leq e^{-\frac{b_0}{4}|\Delta\mathbf{r}_0|}\left(\int_{\mathbf{R}^-}\left|e^{\frac{b_0}{4}c|\Delta\mathbf{v}_0|t}e^{\eta t}\right|^5 dt\right)^{1/5}$$
$$\leq e^{-\frac{b_0}{4}|\Delta\mathbf{r}_0|}\left(\int_{\mathbf{R}^-}\left|e^{\frac{b_0}{8}c|\Delta\mathbf{v}_0|t}\right|^5 dt\right)^{1/5} \leq C\,|\Delta\mathbf{v}_0|^{-1/5}e^{-\frac{b_0}{4}|\Delta\mathbf{r}_0|}.$$

Therefore

$$\|\Psi(\mathbf{W})\|_{Z'_{\eta,b_0}} \leq C^*\,|\Delta\mathbf{v}_0|^{-1/5}\left(e^{-\frac{b_0}{4}|\Delta\mathbf{r}_0|} + e^{-\frac{b_0}{4}|\Delta\mathbf{r}_0|}\|\mathbf{W}\|_{Z'_{\eta,b_0}} + \|\mathbf{W}\|^2_{Z'_{\eta,b_0}}\right).$$

Similarly,

$$\left\|\Psi(\mathbf{W}) - \Psi(\tilde{\mathbf{W}})\right\|_{Z'_{\eta,b_0}} \leq C\left(\left\|\mathbf{B}(0)(\mathbf{W}-\tilde{\mathbf{W}})\right\|_{Z_{\eta,b_0}} + \left\|\mathbf{J}_{shoot}(\mathbf{W}) - \mathbf{J}_{shoot}(\tilde{\mathbf{W}})\right\|_{Z_{\eta,b_0}}\right)$$
$$\leq C\left(e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|}\left\|\mathbf{W}-\tilde{\mathbf{W}}\right\|_{Z_{\eta,b_0}} + \sum_j\left\|J_{j,nl}(\mathbf{W}) - J_{j,nl}(\tilde{\mathbf{W}})\right\|_{Z_{\eta,b_0}}\right).$$

To estimate the $J_{j,nl}$ terms, recall that F is C^2, so we have the following pointwise estimate for $J = F(Q+W) - F(Q) - F'(Q)W$ and $\tilde{J} = F(Q+\tilde{W}) - F(Q) - F'(Q)\tilde{W}$. (Here $Q, W, \tilde{W} \in \mathbf{R}^N$.) First $F(Q+W) - F(Q+\tilde{W}) = F'(Q+W_\gamma)(W-\tilde{W})$, where W^γ lies on the line segment connecting W to $\tilde{W}$. Then

$$\left|J-\tilde{J}\right| = \left|F'(Q+W_\gamma)(W-\tilde{W}) - F'(Q)(W-\tilde{W})\right|$$
$$\leq C\,|W_\gamma|\left|W-\tilde{W}\right|$$
$$\leq C\left(|W| + \left|\tilde{W}\right|\right)\left|W-\tilde{W}\right|.$$

This yields

$$\left\|J_{j,nl}(\mathbf{W}) - J_{j,nl}(\tilde{\mathbf{W}})\right\|_{Z_{\eta,b_0}} \leq C\left(\|\mathbf{W}\|_{Z'_{\eta,b_0}} + \left\|\tilde{\mathbf{W}}\right\|_{Z'_{\eta,b_0}}\right)\left\|\mathbf{W}-\tilde{\mathbf{W}}\right\|_{Z'_{\eta,b_0}}.$$

Therefore

$$\left\|\Psi(\mathbf{W}) - \Psi(\tilde{\mathbf{W}})\right\|_{Z'_{\eta,b_0}} \leq C^{**}\left(e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|} + \|\mathbf{W}\|_{Z'_{\eta,b_0}} + \left\|\tilde{\mathbf{W}}\right\|_{Z'_{\eta,b_0}}\right)\left\|\mathbf{W}-\tilde{\mathbf{W}}\right\|_{Z_{\eta,b_0}}. \tag{5.4}$$

Let

$$\mathcal{K}_1 = \min\left\{\frac{|\Delta\mathbf{v}_0|^{1/5}}{3C^*}, \frac{1}{4C^{**}}\right\}$$

and

$$\epsilon_{10} = \min\left\{ \frac{\mathcal{K}_1 |\Delta \mathbf{v}_0|^{1/5}}{3C^*(1+\mathcal{K}_1)}, \frac{1}{4C^{**}} \right\}.$$

If

$$\|\mathbf{W}\|_{Z'_{\eta,b_0}} \leq \mathcal{K}_1 \quad \text{and} \quad |\Delta \mathbf{r}_0| \geq -\frac{4}{b_0}\ln(\epsilon_{10}) =: M_1,$$

then we have (5.2) and (5.3). We are done. ■

Before we move on to the stability of the shooting solution, we make a few remarks about the dependence of the functions $\mathbf{W}^*$ on the trajectory parameters $\mathbf{n} = (\mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20})$. We claim that there exists a constant $C > 0$ so that

$$\|D_{\mathbf{n}} \mathbf{W}^*(-T)\|_{W^{8/5,5}} \leq C e^{\eta T/2}. \tag{5.5}$$

Notice that

$$D_{\mathbf{n}} \Psi(\mathbf{W})(t) = \int_{-\infty}^{t} e^{\mathbf{A}(t-\tau)} \left(D_{\mathbf{n}} \mathbf{B}(0) \mathbf{W}(\tau) + D_{\mathbf{n}} \mathbf{J}_{shoot}(\mathbf{W}(\tau)) \right) d\tau$$

implies

$$\|D_{\mathbf{n}} \mathbf{W}^*\|_{Z'_{\eta/2,0}} \leq C \|D_{\mathbf{n}} \mathbf{B}(0) \mathbf{W}^*\|_{Z_{\eta/2,0}} + C \|D_{\mathbf{n}} \mathbf{J}_{shoot}(\mathbf{W}^*)\|_{Z_{\eta/2,0}}$$

by Theorem 2.1 and (2.2). Provided that we can show that the right-hand side of this is bounded by a constant C, independent of $\mathbf{n}$, the embedding (2.2) will yield the claim. There is one minor complication, which is that $D_{\mathbf{n}} \mathbf{B}(0) \mathbf{W}^*$ contains terms like

$$\begin{aligned} &\left|\chi(\mathbf{x}) D_{\theta_{10}} \left[\left(F'(Q(\mathbf{y}) + Q(\mathbf{y}_2)) - F'(Q(\mathbf{y})) \right) W_2^*(\mathbf{y}_2) \right] \right| \\ &\qquad \leq \left| F''(Q(\mathbf{y}) + Q(\mathbf{y}_2)) \right| |\nabla Q(\mathbf{y}_2)| \, |D_{\theta_{10}} \mathbf{y}_2| \, |W_2^*(\mathbf{y}_2)| + |D_{\theta_{10}} \mathbf{y}_2| \, |\nabla W_2^*(\mathbf{y}_2)| \, . \end{aligned}$$

From Remark 3.2,

$$|D_{\theta_{10}} \mathbf{y}_2| \leq \left(|D\mathcal{R}[\theta_{10} - \theta_{20}] \mathbf{y}| + |\mathbf{r}_2^\star(t)| + ct \, |D\mathcal{R}[\theta_{10}] \mathbf{i}| \right) \leq C \left(1 + |\mathbf{y}_2| + t \right).$$

Likewise, we can control the derivatives with respect to $\mathbf{r}_{10}$, $\mathbf{r}_{20}$, and θ_{20}. We conclude

$$\|D_{\mathbf{n}} \mathbf{B}(0) \mathbf{W}^*(t)\|_{L^5} \leq C \left\| (1 + |\cdot| + t) \mathbf{W}^\star(\cdot, t) \right\|_{W^{8/5,5}} \leq C \left\| (1+t) \mathbf{W}^\star(\cdot, t) \right\|_{W^{8/5,5}_{b_0}}.$$

The secular growth here seems problematic; however, $te^{-\eta t/2}$ is bounded for $t \leq 0$, and therefore

$$\|D_{\mathbf{n}} \mathbf{B}(0) \mathbf{W}^*\|_{Z_{\eta/2,0}} \leq C \|\mathbf{W}^*\|_{Z'_{\eta,b_0}}.$$

In exactly the same way we can show

$$\|D_{\mathbf{n}} \mathbf{J}_{shoot}(\mathbf{W}^*)\|_{Z_{\eta/2,0}} \leq C \|\mathbf{W}^*\|_{Z'_{\eta,b_0}}.$$

Thus we have (5.5).

5.2. Stability of shooting solutions. To study the stability of the shooting solutions to (1.1) we can just as well study the stability of the solution $\mathbf{W}^*$ to (5.1). Letting $\mathbf{W} = \mathbf{W}^* + \mathbf{V}$ and inserting this into (5.1), we find that $\mathbf{V}$ satisfies

$$\mathbf{V}_t = \mathbf{A}\mathbf{V} + \mathbf{B}(0)\mathbf{V} + \mathbf{H}(\mathbf{V}) \tag{5.6}$$

with

$$\mathbf{V}(\mathbf{x}, -T) = \mathbf{V}_0(\mathbf{x}).$$

Here

$$\mathbf{H}(\mathbf{V}) := \mathbf{J}_{shoot}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{shoot}(\mathbf{W}^*).$$

(Since $\mathbf{W}^*$ is defined for $t \le 0$, we choose here to shift the initial data to $-T \le 0$ instead of shifting $\mathbf{W}^*$ forward in time.) The definition of $\mathbf{J}_{shoot}$ gives

$$\mathbf{H}(\mathbf{V}) = \mathbf{J}_{int}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{int}(\mathbf{W}^*) + \mathbf{J}_{nl}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{nl}(\mathbf{W}^*)$$

with $\mathbf{J}_{int} = (J_{1,int}, J_{2,int})$ and $\mathbf{J}_{nl} = (J_{1,nl}, J_{2,nl})$. A direct computation shows that

$$\begin{aligned}\mathbf{J}_{int}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{int}(\mathbf{W}^*) = \big(&\chi_1(\mathbf{x}, t)\left[F'(Q(\mathbf{y}) + Q(\mathbf{y}_2)) - F'(Q(\mathbf{y}))\right] V_1(\mathbf{y}),\\ &\chi_2(\mathbf{x}, t)\left[F'(Q(\mathbf{y}_1) + Q(\mathbf{y})) - F'(Q(\mathbf{y}))\right] V_2(\mathbf{y})\big)\,.\end{aligned}$$

Note that this term is linear in $\mathbf{V}$, unlike $\mathbf{J}_{int}$ on its own, which contains an inhomogeneous term. This means that we can improve the similar estimate in Lemma 3.5 to

$$\|\mathbf{J}_{int}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{int}(\mathbf{W}^*)\|_{L^5} \le Ce^{-\frac{b_0}{2}|\Delta \mathbf{r}_0|} \|\mathbf{V}\|_{L^5}\,.$$

Recalling that $\mathbf{J}_{nl}$ consist of terms which are $O(|\mathbf{W}|^2)$, it is straightforward to conclude the pointwise estimate

$$|\mathbf{J}_{nl}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{nl}(\mathbf{W}^*)| \le C\left(|\mathbf{W}^*|\,|\mathbf{V}| + |\mathbf{V}|^2\right),$$

which implies

$$\|\mathbf{J}_{nl}(\mathbf{W}^* + \mathbf{V}) - \mathbf{J}_{nl}(\mathbf{W}^*)\|_{L^5} \le C\left(\|\mathbf{W}^*\|_{L^\infty(\mathbf{R}^2)} \|\mathbf{V}\|_{L^5} + \|\mathbf{V}\|^2_{W^{8/5,5}}\right).$$

The embedding (2.2) tells us that

$$\sup_{t \le -T_3} \|\mathbf{W}^*\|_{L^\infty(\mathbf{R}^2)} \le Ce^{\eta T_3} \|\mathbf{W}^*\|_{Z'_{\eta,b}} \le Ce^{-\frac{b_0}{8}c|\Delta \mathbf{v}_0|T_3}.$$

Since $\mathbf{B}$ is not small for general L^5 functions, and we wish to allow arbitrary initial data, we introduce, as before, the decomposition $\mathbf{V} = \mathbf{V}^{ic} + \mathbf{V}^{loc}$, where

$$\begin{aligned}\mathbf{V}^{ic}_t &= \mathbf{A}\mathbf{V}^{ic} + \mathbf{H}(\mathbf{V}),\\ \mathbf{V}^{loc}_t &= \mathbf{A}\mathbf{V}^{loc} + \mathbf{B}(0)\mathbf{V}.\end{aligned} \tag{5.7}$$

Given that $\mathbf{A}$ has a triple eigenvalue, the above estimates will not allow us to conclude that $\mathbf{V}$ is exponentially decaying. Note here that we have already specified $\dot{\mathbf{m}} = 0$, and so we cannot demand that $\mathbf{V} \in E^s \times E^s$ as we did when proving the stability of exit solutions. Instead of the Cauchy problem for (5.7), we proceed as in [20] and [22] and consider the boundary value problem

$$\mathbf{P}^s\mathbf{V}^{ic}(\mathbf{x}, t=-T) = \mathbf{V}^s, \quad \mathbf{P}^c\mathbf{V}^{ic}(\mathbf{x}, t=-T_3) = 0,$$
$$\mathbf{P}^s\mathbf{V}^{loc}(\mathbf{x}, t=-T) = 0, \quad \mathbf{P}^c\mathbf{V}^{loc}(\mathbf{x}, t=-T_3) = 0.$$

A solution of the boundary value problem will be a fixed point of the modified Duhamel integral function:

$$\Psi^{ic}(\mathbf{V}^{ic}, \mathbf{V}^{loc}) := e^{\mathbf{A}(t+T)}\mathbf{P}^s\mathbf{V}^s + \int_{-T}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{P}^s\mathbf{H}(\mathbf{V}(\tau))d\tau - \int_{t}^{-T_3} e^{\mathbf{A}(t-\tau)}\mathbf{P}^c\mathbf{H}(\mathbf{V}(\tau))d\tau,$$
$$\Psi^{loc}(\mathbf{V}^{ic}, \mathbf{V}^{loc}) := \int_{-T}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{P}^s\mathbf{B}(0)(\Gamma^{ic}(\tau) + \mathbf{V}^{loc}(\tau))d\tau$$
$$- \int_{t}^{-T_3} e^{\mathbf{A}(t-\tau)}\mathbf{P}^c\mathbf{B}(0)(\Gamma^{ic}(\tau) + \mathbf{V}^{loc}(\tau))d\tau.$$

We compress notation: $X^b = X'_{\alpha/2,0}[T-T_3] \times X'_{\alpha/2,b_0}[T-T_3]$, $\mathbf{V}^b = (\mathbf{V}^{ic}, \mathbf{V}^{loc})$, and $\Psi^b = (\Psi^{ic}, \Psi^{loc})$. The third estimate in Theorem 2.1, along with Lemma 3.3 and those for $\mathbf{H}$ above, gives the following estimates:

$$\left\|\Psi^b(\mathbf{V}^b)\right\|_{X^b} \le C^*\left(\|\mathbf{V}^s\|_{W^{8/5,5}} + e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|}\left\|\mathbf{V}^b\right\|_{X^b} + \left\|\mathbf{V}^b\right\|^2_{X^b}\right),$$
$$(5.8) \qquad \left\|\Psi^b(\mathbf{V}^b) - \Psi^b(\tilde{\mathbf{V}}^b)\right\|_{X^b} \le C^*\left(e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|} + \left\|\mathbf{V}^b\right\|_{X^b} + \left\|\tilde{\mathbf{V}}^b\right\|_{X^b}\right)\left\|\mathbf{V}^b - \tilde{\mathbf{V}}^b\right\|_{X^b},$$

where C^* is a constant independent of T. We have in the above estimate taken T_3 sufficiently large so that $e^{-\frac{b_0}{8}c|\Delta\mathbf{v}_0|T_3} \le e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|}$.

Let $\mathcal{K}_2 = 4C^*$ and $\delta_0 = \frac{1}{32(C^*)^2}$. If $\|\mathbf{V}^s\|_{W^{8/5,5}} = \delta \le \delta_0$ and $e^{-\frac{b_0}{2}|\Delta\mathbf{r}_0|} \le 1/\mathcal{K}_2$, then the last two estimates imply that Ψ^b is a contraction on the ball in X^b of radius $\mathcal{K}_2\delta$. And so there is a fixed point, and the boundary value problem (5.7) has a solution

$$\left\|\mathbf{V}^b\right\|_{X^b} \le \mathcal{K}_2\|\mathbf{V}^s\|_{W^{8/5,5}},$$

where the constant $\mathcal{K}_2$ is independent of T.

Let
$$\mathcal{T}^c(\mathbf{V}^s; \mathbf{n}) = \mathbf{P}^c\left(\mathbf{V}^{ic}(t=-T) + \mathbf{V}^{loc}(t=-T)\right),$$
where $(\mathbf{V}^{ic}, \mathbf{V}^{loc})$ is the solution of the boundary value problem. Here

$$\mathbf{n} = (\mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20}).$$

If we were in a situation where $\mathbf{V}_0 = \mathbf{V}^s + \mathcal{T}^c(\mathbf{V}^s, \mathbf{n})$ for some function $\mathbf{V}^s \in E^s \times E^s$, then the solution of the initial value problem for (5.7) would decay exponentially quickly and we would be done.

This will not generally be the case, but it turns out that this intuition is enough to complete the proof. Set

$$\mathbf{Q}(\mathbf{x};\mathbf{n}) = (Q(\mathbf{y}_1(\mathbf{n})), Q(\mathbf{y}_2(\mathbf{n})))$$

and (with a small abuse of notation)

$$\mathbf{W}^*(\mathbf{x};\mathbf{n}) = (W_1^*(\mathbf{y}_1, -T), W_2^*(\mathbf{y}_2, -T)),$$

where $\mathbf{y}_j(\mathbf{n}) = \mathcal{R}[\theta_{j0}](\mathbf{x} - \mathbf{r}_{j0} + \mathbf{v}(\theta_{j0})cT)$. Notice that if we could find $\mathbf{V}^s \in E^s \times E^s$ and $\mathbf{n}_1$ so that

$$\mathbf{Q}(\mathbf{x};\mathbf{n}) + \mathbf{W}^*(\mathbf{x};\mathbf{n}) + \mathbf{V}_0 = \mathbf{Q}(\mathbf{x};\mathbf{n}_1) + \mathbf{W}^*(\mathbf{x},\mathbf{n}_1) + \mathbf{V}^s + \mathcal{T}^c(\mathbf{V}^s;\mathbf{n}_1), \tag{5.9}$$

then we would be done. That is to say, if the slight adjustment $\mathbf{n}_1$ of $\mathbf{n}$ can be found so that we have the above, then our initial conditions would be such that the boundary value problem and the initial value problem coincide and we have the exponential decay. (Note that in the above we make sure that the components of $\mathbf{V}_0$, $\mathbf{V}^s$, and $\mathcal{T}^c(\mathbf{V}^s;\mathbf{n}_1)$ are evaluated at the appropriate $\mathbf{y}_j(\mathbf{n})$.)

The proof of the claim follows from the application of the implicit function theorem to the map

$$P(\mathbf{V}^s, \mathbf{n}_1; \mathbf{V}^0) = \mathbf{Q}(\mathbf{x};\mathbf{n}_1) + \mathbf{W}^*(\mathbf{x},\mathbf{n}_1) + \mathbf{V}^s + \mathcal{T}^c(\mathbf{V}^s;\mathbf{n}_1) - \mathbf{Q}(\mathbf{x};\mathbf{n}) - \mathbf{W}^*(\mathbf{x};\mathbf{n}) - \mathbf{V}_0.$$

Clearly $P(0,0;\mathbf{n}) = 0$. Notice $D_{\mathbf{V}^s}\mathbf{P}^s P(0,0;\mathbf{n}) = id_{E^s \times E^s}$ and $D_{\mathbf{V}^s}\mathbf{P}^c P(0,0;\mathbf{n}) = D_{\mathbf{V}^s}\mathcal{T}^c(0;\mathbf{n})$ since $\mathbf{V}^s \in E^s \times E^s$ and $\mathcal{T}^c \in E^c \times E^c$. It is the derivatives of Q with respect to the parameters $\mathbf{r}_{j0}$ and θ_{j0} which give the center eigenspace, and so

$$D_{\mathbf{n}_1}\mathbf{P}^c\mathbf{Q}(\mathbf{x},\mathbf{n}_1) = id_6,$$

where id_6 is the six by six identity matrix and

$$D_{\mathbf{n}_1}\mathbf{P}^s\mathbf{Q}(\mathbf{x},\mathbf{n}_1) = 0.$$

From (5.5) we conclude that $\|D_{\mathbf{n}_1}\mathbf{W}^*(\mathbf{x},\mathbf{n}_1)\|_{W^{8/5,5}} \le Ce^{-\frac{b_0}{16}c|\Delta\mathbf{v}_0|T}$. Finally, $\mathcal{T}^c(0;\mathbf{n}) = 0$ for all choices $\mathbf{n}$, and so $D_{\mathbf{n}}\mathcal{T}^c(0;\mathbf{n}) = 0$. Decomposing P into $(\mathbf{P}^s P, \mathbf{P}^c P)$ and arranging all the derivatives computed above accordingly, we see

$$D_{(\mathbf{V}^s,\mathbf{n})}P(0,\mathbf{n};0) = \begin{bmatrix} id_{E^s\times E^s} & D_{\mathbf{V}^s}\mathcal{T}^c(0;\mathbf{n}) \\ 0 & id_6 \end{bmatrix} + O(e^{-\frac{b_0}{8}c|\Delta\mathbf{v}_0|T}).$$

This is invertible if T is sufficiently large, which it is. This completes the proof. ■

We conclude this section with the proof of Corollary 1.12.

Proof of Corollary 1.12. Suppose that $\Phi_j(\mathbf{x},t;\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20})$ for $j=1,2$ are two different solutions of (1.1) meeting the conclusions of Theorem 1.9. Fix $T > T_3$, and let

$$h = \|\Phi_1(\cdot,-T,\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20}) - \Phi_2(\cdot,-T,\mathbf{r}_{10},\mathbf{r}_{20},\theta_{10},\theta_{20})\|_{W^{8/5,5}} \neq 0.$$

We know that there is a $T_2 > T$ so that for $j = 1, 2$ and any $\gamma > 0$

$$\left\| \Phi_j(\cdot, -T_2, \mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20}) - \sum_{j=1}^{2} Q\left(\mathcal{R}[-\theta_{j0}]\left(\cdot - \mathbf{r}_j^\star(-T_2)\right)\right) \right\|_{W^{8/5,5}} \le \gamma,$$

which then implies by the triangle inequality

$$\|\Phi_1(\cdot, -T, \mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20}) - \Phi_2(\cdot, -T, \mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20})\|_{W^{8/5,5}} \le 2\gamma.$$

Applying Theorem 1.11 gives

$$\|\Phi_1(\cdot, -T_2, \mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20}) - \Phi_2(\cdot, -T_2, \mathbf{r}_{10}, \mathbf{r}_{20}, \theta_{10}, \theta_{20})\|_{W^{8/5,5}} \le 2\mathcal{K}_3 \gamma e^{\eta(T_2 - T)} \le 2\mathcal{K}_3\gamma.$$

Since γ is free to choose, we can take it to be $h/20\mathcal{K}_3$. This is a contradiction. ∎

6. Appendix: Local existence of solutions for (4.10).

Let

$$\mathbf{S} = (\mathbf{W}^{ic}, \mathbf{W}^{loc}, \mathbf{m}),$$

$$Y[T_1] := X'_{a,0}[T_1] \times X'_{a,b}[T_1] \times C^1[0, T_1],$$

and

$$Y_0 := (W^{4/3,3} \cap (E^s)^2) \times (W_b^{4/3,3} \cap (E^s)^2) \times \mathbf{R}^3.$$

Theorem 6.1. *There exist $C > 0$ and $\delta_2 > 0$ so that for any $a > 0$, $0 \le b \le b_0$, and*

$$\|\mathbf{S}_0\|_{Y_0} \le \delta_2$$

there exist $T_1 > 0$ and a unique function and $\mathbf{S} \in Y[T_1]$ so that

$$\|\mathbf{S}\|_{Y[T_1]} \le C$$

and $\mathbf{S}$ solves (4.10) for $t \in [0, T_1]$ a.e. and $\mathbf{S}(0) = \mathbf{S}_0$.

Proof. Let $\Gamma(\mathbf{S}) := (\Gamma^{ic}(\mathbf{S}), \Gamma^{loc}(\mathbf{S}), \Gamma^m(\mathbf{S}))$ be the map defined by the following, which is formally equivalent to the the right-hand side of (4.11):

$$\begin{aligned}
\Gamma^{ic} &= e^{\mathbf{A}t}\mathbf{W}_0^{ic} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{P}^s\mathbf{J}_s(\mathbf{m}(\tau), \mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau))d\tau,\\
\Gamma^{loc} &= e^{\mathbf{A}t}\mathbf{W}_0^{loc} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{P}^s\mathbf{B}(\mathbf{m}(\tau))\left(\mathbf{W}^{ic}(\tau) + \mathbf{W}^{loc}(\tau)\right)d\tau,\\
\Gamma^{\mathbf{m}} &= \mathbf{m}_0 - \int_0^t \Big\{\mathbf{R}(\mathbf{m}(\tau))\mathbf{P}^c\left(\mathbf{B}(\mathbf{m}(\tau))\left(\Gamma^{ic}(\tau) + \Gamma^{loc}(\tau)\right)\right.\\
&\quad \left. + \mathbf{J}_{cc}(\mathbf{m}(\tau), \Gamma^{ic}(\tau) + \Gamma^{loc}(\tau))\right) + \mathbf{R}[\mathbf{m}(\tau)]\mathbf{K}_{cc}(\mathbf{m}(\tau), \Gamma^{ic}(\tau) + \Gamma^{loc}(\tau))\Big\}\, d\tau.
\end{aligned} \tag{6.1}$$

We will show that that there exist $T_1 > 0$, $K^{**} > 0$, and $\delta_2 > 0$ so that $\|\mathbf{S}_0\|_{Y_0} \le \delta_2$ and $\|\mathbf{S}\|_Y \le K^{**}$ imply

$$\|\Gamma(\mathbf{S})\|_Y \le K^{**} \tag{6.2}$$

and

$$\left\|\Gamma(\mathbf{S})-\Gamma(\tilde{\mathbf{S}})\right\|_Y \leq \frac{1}{2}\left\|\mathbf{S}-\tilde{\mathbf{S}}\right\|_Y, \tag{6.3}$$

which by the contraction mapping theorem implies the result.

First we estimate Γ^{ic}. Theorem 2.1 gives

$$\left\|\Gamma^{ic}(\mathbf{S})\right\|_{X_{a,0}[T]} \leq C\left(\|\mathbf{S}_0\|_{Y_0}+\|\mathbf{J}\|_{X_{a,0}[T]}\right).$$

Lemmas 3.4–3.7 imply

$$\|\mathbf{J}\|_{X_{a,0}[T]} \leq C\left(\left(\int_0^T e^{-3b\mu(t)/2}dt\right)^{1/3}+\left\|\mathbf{W}^{ic}\right\|_{L^5([0,T];W^{4/3,3})}+\left\|\mathbf{W}^{loc}\right\|_{L^5([0,T];W_b^{4/3,3})}\right.$$
$$\left.+\|\mathbf{m}\|^2_{C^1[0,T]}+\left\|\mathbf{W}^{ic}\right\|^2_{X'_{a,0}[T]}+\left\|\mathbf{W}^{loc}\right\|^2_{X'_{a,b}[T]}\right). \tag{6.4}$$

To estimate the linear terms above we use (2.2) and Hölder's inequality:

$$\begin{aligned}\|\mathbf{F}\|_{L^5([0,T],W_b^{4/3,3})} &= \left(\int_0^T \|\mathbf{F}(\tau)\|^3_{W_b^{4/3,3}}\,d\tau\right)^{1/3}\\ &\leq \|\mathbf{F}\|^{2/3}_{L^\infty([0,T];W_b^{4/3,3})}\left(\int_0^T \|\mathbf{F}(\tau)\|_{W_b^{4/3,3}}\,d\tau\right)^{1/3}\\ &\leq C\,\|\mathbf{F}\|^{2/3}_{X'_{a,b}}\left(\int_0^T \|\mathbf{F}(\tau)\|_{W_b^{4/3,3}}\,d\tau\right)^{1/3}\\ &\leq CT^{2/9}\,\|\mathbf{F}\|_{X'_{a,b}}.\end{aligned}$$

Thus

$$\left\|\Gamma^{ic}(\mathbf{S})\right\|_{X'_{a,0}} \leq C\left\{\|\mathbf{S}_0\|_{Y_0}+T^{2/9}\left(1+\|\mathbf{S}\|_Y\right)+\|\mathbf{S}\|^2_Y\right\}.$$

In exactly the same fashion we can show

$$\left\|\Gamma^{loc}(\mathbf{S})\right\|_{X'_{a,0}} \leq C\left\{\|\mathbf{S}_0\|_{Y_0}+T^{2/9}\|\mathbf{S}\|_Y\right\}$$

and

$$\|\Gamma^{\mathbf{m}}\|_{C^1[0,T]} \leq C\left\{\|\mathbf{S}_0\|_{Y_0}+T^{2/9}\left(1+\|\mathbf{S}\|_Y\right)+\|\mathbf{S}\|^2_Y+\|\mathbf{S}\|^3_Y\right\}.$$

Therefore

$$\|\Gamma(\mathbf{S})\|_Y \leq C\,\|\mathbf{S}_0\|_{Y_0}+CT^{2/9}\left(1+\|\mathbf{S}\|_Y\right)+C\,\|\mathbf{S}\|^2_Y+C\,\|\mathbf{S}\|^3_Y,$$

which in turn implies (6.2) for T and $\|\mathbf{S}_0\|_{Y_0}$ sufficiently small.

Proving (6.3) is largely similar, though care must be taken with $\mathbf{B}(\mathbf{m})\mathbf{W}$ (as well as some terms in $J_{j,int}$) because it is not immediately obvious that this term is Lipschitz. Of particular concern is the fact that these operators involve spatial translations and thus are nonlocal.

The most difficult term is Γ^{loc},

$$\Gamma^{loc}(\mathbf{S}) - \Gamma^{loc}(\tilde{\mathbf{S}}) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{P}^s\mathbf{B}(\mathbf{m}(\tau))\left(\mathbf{W}(\tau) - \tilde{\mathbf{W}}(\tau)\right)d\tau - \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{P}^s\left(\mathbf{B}(\mathbf{m}(\tau)) - \mathbf{B}(\tilde{\mathbf{m}}(\tau))\right)\mathbf{W}(\tau)d\tau.$$

Here we use the shorthand $\mathbf{W} = \mathbf{W}^{ic}+\mathbf{W}^{loc}$. The first term we can estimate by $CT^{2/9}\|\mathbf{S}-\tilde{\mathbf{S}}\|_Y$ using the same strategy used for (6.2).

We claim that

$$\|(\mathbf{B}(\mathbf{m}(t)) - \mathbf{B}(\tilde{\mathbf{m}}(t)))\,\mathbf{W}(t)\|_{L^5_b} \le C(1+t)\,\|\mathbf{W}\|_{W^{8/5,5}}\,|\mathbf{m} - \tilde{\mathbf{m}}|\,. \tag{6.5}$$

From the definition of $\mathbf{B}$ and the operators B_j we see that we must estimate terms of the form

(6.6)
$$\begin{aligned}&\left|\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right]W_2(\mathbf{y}_2,t)\right.\\ &\quad\left. - \chi_1(\tilde{\mathbf{x}},t)\left[F'(Q(\tilde{\mathbf{z}}_1)+Q(\tilde{\mathbf{z}}_2)) - F'(Q(\tilde{\mathbf{z}}_2))\right]W_2(\tilde{\mathbf{y}}_2,t)\right|\\ &\le \left|\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right](W_2(\mathbf{y}_2,t) - W_2(\tilde{\mathbf{y}}_2,t))\right|\\ &\quad + \left|\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right] - \chi_1(\tilde{\mathbf{x}},t)\left[F'(Q(\tilde{\mathbf{z}}_1)+Q(\tilde{\mathbf{z}}_2)) - F'(Q(\tilde{\mathbf{z}}_2))\right]\right|\left|W_2(\tilde{\mathbf{y}}_2)\right|.\end{aligned}$$

$W_2(t)$ is in $W^{8/5,5}$ and thus is C^1. So

$$\begin{aligned}&\left|\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right](W_2(\mathbf{y}_2,t) - W_2(\tilde{\mathbf{y}}_2,t))\right|\\ &\qquad\le C\left|\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right]\right|\left|\mathbf{y}_2 - \tilde{\mathbf{y}}_2\right|\|W_2\|_{W^{8/5,5}}\,.\end{aligned}$$

From Remark 3.2 (and noting that $\mathbf{m}$ and $\tilde{\mathbf{m}}$ have identical initial data) we have

$$\mathbf{y}_2 - \tilde{\mathbf{y}}_2 = \mathcal{R}[-\theta_{20}]\,(\mathbf{r}_2 - \tilde{\mathbf{r}}_2 - \mathbf{r}_1 + \tilde{\mathbf{r}}_1)\,.$$

Then

$$\begin{aligned}|\mathbf{y}_2 - \tilde{\mathbf{y}}_2| &\le \sum_j |\mathbf{r}_j - \tilde{\mathbf{r}}_j|\\ &\le \sum_j |\mathbf{p}_j - \tilde{\mathbf{p}}_j| + c\int_0^t \left|\mathcal{R}[\theta_{20}+\phi_j(s)] - \mathcal{R}[\theta_{20}+\tilde{\phi}_j(s)]\right|ds\\ &\le C(1+t)\,|\mathbf{m} - \tilde{\mathbf{m}}|\,.\end{aligned}$$

Additionally, we know from the proof of Lemma 3.3 that

$$\left|\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right]\right| \le Ce^{-\beta|\mathbf{y}|},$$

which is in L^5_b. Therefore

$$\begin{aligned}&\left\|\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right](W_2(\mathbf{y}_2,t) - W_2(\tilde{\mathbf{y}}_2,t))\right\|_{L^5_b}\\ &\qquad\le C(1+t)\,\|\mathbf{W}\|_{W^{8/5,5}}\,|\mathbf{m} - \tilde{\mathbf{m}}|\,.\end{aligned}$$

The remaining term in (6.6) is handled in a completely similar way, even though it looks quite awful. Let

$$f(\mathbf{y},t;\mathbf{m}) := e^{b|\mathbf{y}|}\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right].$$

This function is Lipschitz in its arguments (since χ_1, Q, and F' are all Lipschitz) and also in L^∞. Thus

$$\|f(\mathbf{y},t;\mathbf{m}) - f(\mathbf{y},t;\tilde{\mathbf{m}})\|_{L^\infty} \le C(1+t)\,|\mathbf{m}-\tilde{\mathbf{m}}|.$$

From this and Remark 3.2 we can conclude

$$\begin{aligned}\Big\|\big(\chi_1(\mathbf{x},t)\left[F'(Q(\mathbf{z}_1)+Q(\mathbf{z}_2)) - F'(Q(\mathbf{z}_2))\right]&\\ - \chi_1(\tilde{\mathbf{x}},t)\left[F'(Q(\tilde{\mathbf{z}}_1)+Q(\tilde{\mathbf{z}}_2)) - F'(Q(\tilde{\mathbf{z}}_2))\right]\big)\, W_2(\tilde{\mathbf{y}}_2)\Big\|_{L^5_b}&\\ \le C(1+t)\,\|\mathbf{W}\|_{W^{8/5,5}}\,|\mathbf{m}-\tilde{\mathbf{m}}|.&\end{aligned}$$

Thus we have established the claim (6.5), which in turn implies

$$\left\|\Gamma^{loc}(\mathbf{S}) - \Gamma^{loc}(\tilde{\mathbf{S}})\right\|_Y \le C\left(T^{2/9} + (1+T)\left(\|\mathbf{S}\|_Y + \|\tilde{\mathbf{S}}\|_Y\right)\right)\|\mathbf{S}-\tilde{\mathbf{S}}\|_Y.$$

Arguments parallel to this can used to establish similar estimates for Γ^{ic} and $\Gamma^{\mathbf{m}}$ which then implies (6.3). We omit the details. ■

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