Chapter 6.4 (Revisited) - Arc Length for Parametric Curves

Let \[ \begin{align*}
   x &= x(t) & a \leq t \leq b \\
   y &= y(t)
\end{align*} \]

determine a curve in the xy plane.

\( t = a \quad \text{to} \quad t = b \)

Goal: Find the length of the curve from \( t = a \) to \( t = b \).

Idea: Discretize the curve into \( n \) line segments as we did earlier.

\( t = a \quad \text{to} \quad t = b \)

Close up:

\[ dl = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \]

We compute the "tiny" length \( dl \) of one segment by

\[ dl = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \]

Then we take an "infinite sum" of all the \( dl \)'s to get:
Compute the length of the parametric curve \( \begin{cases} x = \cos t + tsin t \\ y = \sin t - t\cos t \end{cases} \quad 0 \leq t \leq \pi \)

\[
\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t
\]

\[
\frac{dy}{dt} = \cos t - [(t\cos t) + \cos t] = t\sin t
\]

\[
\left(\frac{dx}{dt}\right)^2 = t^2\cos^2 t
\]

\[
\left(\frac{dy}{dt}\right)^2 = t^2\sin^2 t
\]

\[
L = \int_0^\pi \sqrt{t^2\cos^2 t + t^2\sin^2 t} \, dt = \int_0^\pi \sqrt{t^2 (\cos^2 t + \sin^2 t)} \, dt = \int_0^\pi t \, dt = \frac{1}{2} t^2 \bigg|_0^{2\pi} = \frac{2\pi^2}{2}
\]

**NOTE:** In general, \( \sqrt{t^2} = |t| \) but for us \( \sqrt{t^2} = t \) b/c we are integrating over \( [0,\pi] \)
In case you were wondering, the curve whose length you just found is shown to the left!