5.6 Discrete Dynamical Systems

\[ \mathbf{x}_{k+1} = A \mathbf{x}_k, \quad \mathbf{x}_0 \text{ given initial vector} \]

\[ A = PDP^{-1}, \quad P = [v_1 \cdots v_n], \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}. \]

Write

\[ \mathbf{x}_0 = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n. \]

Then

\[ \mathbf{x}_1 = c_1 \lambda_1 \mathbf{v}_1 + \cdots + c_n \lambda_n \mathbf{v}_n, \]
\[ \mathbf{x}_2 = c_1 (\lambda_1)^2 \mathbf{v}_1 + \cdots + c_n (\lambda_n)^2 \mathbf{v}_n, \text{ etc.} \]

In general,

\[ \mathbf{x}_k = c_1 (\lambda_1)^k \mathbf{v}_1 + \cdots + c_n (\lambda_n)^k \mathbf{v}_n. \]

- If \(|\lambda_j| < 1\) for all \(j\), then the origin is an attractor of the system.
- If \(|\lambda_j| > 1\) for all \(j\), then the origin is a repellor of the system.
- If \(|\lambda_j| < 1\) for some eigenvalues and \(|\lambda_j| > 1\) for others, then the origin is a saddle point of the system.
**Note:** For complex eigenvalues $\lambda = a + ib$:

$$|\lambda| = \sqrt{a^2 + b^2}.$$
5.7 Applications to Differential Equations.

For a system of differential equations
\[\begin{align*}
  x'_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\
  x'_2 &= a_{21}x_1 + \cdots + a_{2n}x_n \\
  &\vdots \\
  x'_n &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{align*}\]

we have an equivalent matrix system
\[
\begin{bmatrix}
  x'_1(t) \\
  \vdots \\
  x'_n(t)
\end{bmatrix} =
\begin{bmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
  x_1(t) \\
  \vdots \\
  x_n(t)
\end{bmatrix},
\]

or \[x'(t) = Ax(t).\]
For a decoupled system

\[ \mathbf{x}' = A \mathbf{x}, \]

\[ A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix}. \]

General Solution:

\[ \mathbf{x}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n e^{\lambda_n t} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}. \]
For the general case:

\[ x' = Ax, \]

\[ A = PDP^{-1}, \quad P = [v_1 \cdots v_n], \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}. \]

**General Solution:**

\[ x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n. \]

If an initial condition is given, i.e., if \( x(0) \) is specified, then we can solve for the constants \( c_1, \cdots, c_n \) to find a particular solution.
Example. Consider the system

\[
\begin{aligned}
\frac{dx_1}{dt} &= x_1 + 4x_2 \\
\frac{dx_2}{dt} &= 2x_1 - x_2.
\end{aligned}
\]

1. Rewrite this system in matrix form: \( \mathbf{x}' = A\mathbf{x} \).
2. Write down the general solution of the system.
3. Is the origin an attractor, a repellor or a saddle point of the system?
4. Solve the system if \( x_1(0) = 2, \ x_2(0) = -1 \).
For a dynamical system $x' = Ax$, with real eigenvalues $\lambda_1, \cdots, \lambda_n$:

1. If all the eigenvalues are negative, then the origin is called an **attractor** (or **sink**) of the system.
   
   **Direction of greatest attraction:**

2. If all the eigenvalues are positive, then the origin is called a **repellor** (or **source**) of the system.
   
   **Direction of greatest repulsion:**

3. If $A$ has some positive and some negative eigenvalues, then the origin is called a **saddle point** of the system.
Example. Solve $x' = Ax$, 

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & 2 & -1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$ 

Is the origin an attractor, a repellor or a saddle point of the system $x' = Ax$?
What if the eigenvalues are complex (non-real)?

**Answer:** Look at the real parts of the eigenvalues.

- If all the eigenvalues have negative real parts, then the origin is an **attractor** of the system.
- If all the eigenvalues have positive real parts, then the origin is a **repellor** of the system.
- If some eigenvalues have positive real part and others have negative real part, then the origin is a **saddle point** of the system.
Example. Solve $x' = Ax$,

$$A = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix}.$$ 

Eigenvalues: $-1 + 2i, -1 - 2i$

Eigenvectors: $\begin{bmatrix} 3 - i \\ 2 \end{bmatrix}, \begin{bmatrix} 3 + i \\ 2 \end{bmatrix}$

The real part of each eigenvalue is negative. The solutions spiral toward the origin.
Questions about the Review Problems? (or any general questions about the exam?)
If $A$ is 25x25 and rank $A = 25$, then

1. 0 is an eigenvalue of $A$.
2. 0 is not an eigenvalue of $A$. 
If $A$ is $25 \times 32$, then the largest possible value of rank $A$ is

1. 25
2. 32
If $A$ and $B$ are 4x4 and rank $A = 2$, can rank $AB$ be greater than 2?

1. Yes
2. No
For a dynamical system $x_{k+1} = Ax_k$, where $A$ is 2x2 with eigenvalues $\frac{1}{2} + \frac{1}{2}i$, $\frac{1}{2} - \frac{1}{2}i$,

1. the origin is an attractor of the system
2. the origin is a repellor of the system
3. the origin is a saddle point of the system
For a system \( x' = Ax \), where \( A \) is 2x2 with eigenvalues \( \frac{1}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i \),

1. the origin is an attractor of the system
2. the origin is a repellor of the system
3. the origin is a saddle point of the system