MATH 723 **Spring 2008**

Mathematical Neuroscience

Homework Project 2. Due April 28 in class.

Read:

- §§ 2.1, 2.2, 2.4, and 2.5 in the course notes by P. Eckhoff and P. Holmes,
- §§ 4.1, 4.2 of the handout 'Elements of the bifurcation theory'.

Problems:

1. Exercise 7 (p. 53) Lecture notes by P. Eckhoff and P. Holmes.

2. **Lyapunov function.**

   a. (*Warm-up*). Consider

$$\begin{cases} \dot{x} &= -y - x^3; \\ \dot{y} &= x - y^3. \end{cases}$$

Note that the stability of the equilibrium at the origin can not be determined from the linearization. Show that

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$$

is a Lyapunov function. Therefore, the origin is asymptotically stable. Sketch a phase portrait near the origin.

   b. (*The Lorenz system*) is given by the following system of ODEs

$$\begin{cases} \dot{x} &= \sigma\left(y - x\right), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{cases}$$

where r, σ and b are positive parameters. Note that there is an equilibrium at the origin. Use linearization to show that the origin is asymptotically stable for $0 < r < 1$ and is unstable for $r > 1$. Linearization fails to tell us about the stability for $r = 1$. Construct a Lyapunov function to conclude that the origin remains asymptotically stable for $r = 1$. For this, look for the Lyapunov function in the form

$$L(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2$$

and choose suitable values for positive α, β, γ. Identify the bifurcation at $r = 1$.

3. Follow the steps of the analysis of the saddle node bifurcation in the handout (cf. §4.1), to discuss the *transcritical* and the *pitchfork* bifurcations for the following examples:

transcritical: $\dot{x} = rx - x^2$,

pitchfork: $\dot{x} = rx - x^3$.

Here, r is a control parameter. Plot the bifurcation diagram for each case. Identify stable and unstable branches of equilibria.

4. see Homework Project in handout 'The Hodgkin-Huxley model'. Two sample matlab codes for this problem are posted on the course website.