MATH 723 **Spring 2008-09**

Mathematical Neuroscience

Homework Project 2. Due April 20, in class.

Read:

- §§ 2.1, 2.2, 2.4, and 2.5 in the course notes by P. Eckhoff and P. Holmes,

Problems:

1. Exercise 7 (p. 53) Lecture notes by P. Eckhoff and P. Holmes.
2. **Lyapunov function.**

   a. (*Warm-up*). Consider

$$\begin{cases} \dot{x} &= -y - x^3; \\ \dot{y} &= x - y^3. \end{cases}$$

   Note that the stability of the equilibrium at the origin can not be determined from the linearization. Show that

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$$

   is a Lyapunov function. Therefore, the origin is asymptotically stable. Sketch a phase portrait near the origin.

   b. (*The Lorenz system*) is given by the following system of ODEs

$$\begin{cases} \dot{x} &= \sigma(y - x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - bz, \end{cases}$$

   where r, σ and b are positive parameters. Note that there is an equilibrium at the origin. Use linearization to show that the origin is asymptotically stable for $0 < r < 1$ and is unstable for $r > 1$. Linearization fails to tell us about the stability for $r = 1$. Construct a Lyapunov function to conclude that the origin remains asymptotically stable for $r = 1$. For this, look for the Lyapunov function in the form

$$L(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2$$

   and choose suitable values for positive α, β, γ. Identify the bifurcation at $r = 1$.

3. Consider

$$\begin{aligned} \dot{x} &= -x - y + x(x^2 + y^2) \\ \dot{y} &= x - y + y(x^2 + y^2) \end{aligned} \tag{1}$$

(a) Rewrite the equations in (1) in polar cordinates.

(b) Locate the equilibrium and the periodic orbit and determine their stability.

(c) Sketch the phase portrait.

4. Consider

$$\begin{aligned} \dot{x} &= y - \epsilon(x - x^3) \\ \dot{y} &= -x - \epsilon(y - y^3) \end{aligned} \tag{2}$$

For questions **(a)**-**(c)** assume that $\epsilon > 0$ is a small parameter.

(a) Linearize about the equilibrium at the origin. Determine stability of the origin. Sketch the phase portrait near the origin.

(b) Apply the method of Section 3.4 of the lecture notes to locate the limit cycle and determine it's stability. Sketch the first return map.

(c) Sketch the phase portrait of (2).

(d) Let $\epsilon = 1$ (no longer small). Show that (2) still has a limit cycle.