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The Nonlinear Heat Equation on W-Random Graphs

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Abstract

For systems of coupled differential equations on a sequence of W-random graphs, we derive the continuum limit in the form of an evolution integral equation. We prove that solutions of the initial value problems (IVPs) for the discrete model converge to the solution of the IVP for its continuum limit. These results combined with the analysis of nonlocally coupled deterministic networks in Medvedev (The nonlinear heat equation on dense graphs and graph limits. ArXiv e-prints, 2013) justify the continuum (thermodynamic) limit for a large class of coupled dynamical systems on convergent families of graphs.

1. Introduction

In this paper, we study coupled dynamical systems on a sequence of graphs $\{G_n\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}u_{ni}(t) = n^{-1} \sum_{j:(i,j)\in E(G_n)} D(u_{nj} - u_{ni}), \quad i \in [n] := \{1, 2, \dots, n\}, \quad (1.1)$$

where $G_n = \langle V(G_n), E(G_n)\rangle$ is a graph on n nodes, and $V(G_n) = [n]$ and $E(G_n)$ stand for the sets of nodes and edges of G_n respectively. D is a Lipschitz continuous function. The operator on the right-hand side of (1.1) models the nonlinear diffusion across edges of G_n. Thus, we refer to (1.1) as a nonlinear heat equation on G_n.

The evolution equations like (1.1) are used in modeling diverse systems ranging from neuronal networks in biology [7,9,22,32], to Josephson junctions and coupled lasers in physics [16,30], to communication, sensor, and power networks in technology [6,19]. The Kuramoto model, a prominent example of (1.1), is widely used as a paradigm for studying collective dynamics of coupled oscillators of diverse nature [6,8,11,12,35].

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In this paper, we are interested in the case when $\{G_n\}$ is a sequence of dense graphs, that is, $|E(G_n)| = O(n^2)$. This corresponds to the nonlocal diffusion operator in (1.1). Nonlocally coupled systems have attracted much attention in nonlinear science recently [8,14,29,31,33–35]. They arise as models of diverse phenomena throughout physics and biology and feature several remarkable effects, such as chimera states and coherence-incoherence transition (see, for example, [13–15,25–28,31]). Overall, nonlocally coupled dynamical systems are less understood than systems with local coupling.

For analyzing nonlocally coupled systems, the continuum (thermodynamic) limit proved to be a very useful tool [8,14,29,35]. As $n \to \infty$, one can formally interpret the right-hand side of (1.1) as a Riemann sum to obtain

$$\frac{\partial}{\partial t} u(x,t) = \int_I W(x,y) D(u(y,t) - u(x,t))\, dy, \tag{1.2}$$

where $u(x,t)$ now describes a continuum of (local) dynamical systems distributed along $I := [0,1]$. For some patterns of connectivity, the kernel W in (1.2) can be guessed from the pixel picture of the adjacency matrix of G_n [4,17]. For example, let G_n be a graph on n nodes distributed uniformly along a circle, and let $k = \lfloor rn \rfloor$ for fixed $r \in (0,1)$. Suppose each node of G_n is connected to k of its nearest neighbors from each side, that is, G_n is a k-nearest-neighbor graph. The pixel picture of G_n is shown in Fig. 1a. Specifically, Fig. 1a shows the support of the $\{0,1\}$-valued function $W_{G_n} : I^2 \to \{0,1\}$ such that

$$W_{G_n}(x,y) = 1 \text{ if } (i,j) \in E(G_n) \text{ and } (x,y) \in [(i-1)n^{-1}, in^{-1}) \\ \times [(j-1)n^{-1}, jn^{-1}), \ (i,j) \in [n]^2.$$

Function W_{G_n} provides the geometric representation of the adjacency matrix of G_n. It is easy to see that as $n \to \infty$, $\{W_{G_n}\}$ converges to the $\{0,1\}$-valued function, whose support is shown in Fig. 1b. This is the limit of the k-nearest-neighbor family of graphs $\{G_n\}$.

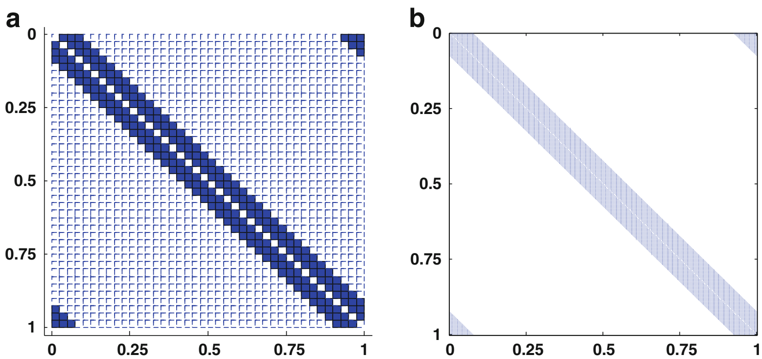

Fig. 1. The plot of the support of W_{G_n} (**a**) and that of the support of its limit W_{G_∞} (**b**). Each function is defined on a unit square and is equal to 1 on the colored regions and 0 otherwise. The direction of the vertical axis was chosen to emphasize the relation of W_{G_n} (**a**) to the adjacency matrix of the corresponding graph

The formally derived continuum limit (1.2) was used to study the discrete model (1.1) for large n in many papers [8,14,29,34,35]. In [21], we provided a rigorous justification of the continuum limit (1.2). The analysis of the continuum limit in [21] uses the ideas from the theory of graph limits [5,17,18], which for every convergent family of dense graphs defines the limiting object, a measurable symmetric function W. This function is called a graphon. It captures the connectivity of G_n for large n. In [21], for convergent sequences of deterministic graphs $\{G_n\}$, it was shown that with the kernel of the integral operator on the right-hand side of (1.2) taken to be the limit of $\{G_n\}$, the solution of the IVP for (1.2) approximates those of the IVPs for (1.1) for large n.

The analysis in [21] does not cover dynamical systems on random graphs. The latter have many important applications [34,35]. Thus, in this paper, we focus on systems on random graphs. Specifically, we prove convergence of solutions of the IVPs for (1.1) on W-random graphs G_n to the solution of the IVP for (1.2). A W-random graph is constructed from a graphon W [17,18]. This construction provides a convenient general analytical model for random graphs, which includes many random graphs that are important in applications, such as Erdős–Rényi and small-world (SW) graphs (see Figs. 1c and 2b, c) [3,10,34]. At the same time, W-random graphs fit naturally into the convergence analysis of the families of discrete models like (1.1).

The remainder of this paper is organized as follows. In the next section, we formulate the IVPs for the discrete model and its continuum limit. In Sections 3 and 4, we prove convergence of solutions of discrete models for two different variants of W-random graphs. In the variant, analyzed in Section 3, the right-hand side of (1.1) can be interpreted as the Monte-Carlo approximation of the integral on the right-hand side of (1.2). Consistent with this interpretation, we find that the rate of convergence of the solutions of discrete problems (in $C(0,T;L^2(I))$ norm) is $O(n^{-1/2})$. In the variant of the random network model considered in Section 4, which was included for the sake of convenience in applications, the rate of convergence also depends on the regularity of the graphon W. As an application of our results, in Section 5 we derive the continuum limit for dynamical systems on SW graphs [34,35] (see Fig. 2). We conclude with the discussion of our results in Section 6.

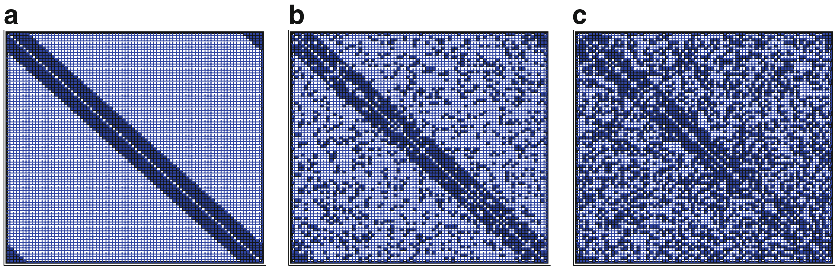

Fig. 2. The pixel pictures of the k-nearest-neighbor network on a ring (**a**) and two small-world graphs (**b**, **c**) that were obtained from the network in (**a**) by replacing local connections with random long-range connections

2. The Discrete Model and Its Continuum Limit

Throughout this paper, we assume that $W(x, y)$ belongs to $\mathcal{W}_0$, a class of symmetric measurable functions on I^2 with values in I. W represents the limit of a convergent family of dense graphs $\{G_n\}$ (see [17], for an exposition of the theory of graph limits; see also Section 2 in [21] for a brief review of facts from this theory that are relevant for constructing continuum limits of dynamical networks.)

Let $X_n = \{x_{n1}, x_{n2}, \ldots, x_{nn}\}$ be a set of distinct points from I and $W \in \mathcal{W}_0$. In this section, we introduce IVPs for the nonlinear heat equation on $G_n = \langle V(G_n), E(G_n)\rangle$, a certain graph on n nodes, constructed using W and X_n.

The sequence of graphs $\{G_n\}$ will be defined below. Suppose G_n is given. By the IVP for the nonlinear heat equation on G_n, we mean

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{ni}(t) = n^{-1} \sum_{j:(i,j)\in E(G_n)} D(u_{nj} - u_{ni}), \tag{2.1}$$

$$u_{ni}(0) = g(x_i), \quad i \in [n], \tag{2.2}$$

where $u_n(t) = (u_{n1}(t), u_{n2}(t), \ldots, u_{nn}(t))$ is the unknown function. Here, $D(\cdot)$ is a Lipschitz function on $\mathbb{R}$ and g is a bounded measurable function on I.

The solution of the IVP for the discrete model (2.1), (2.2) will be compared with the solution of the IVP for the continuum limit

$$\frac{\partial}{\partial t} u(x, t) = \int_I W(x, y) D\left(u(y, t) - u(x, t)\right) \, \mathrm{d}y, \tag{2.3}$$

$$u(x, 0) = g(x), \quad x \in I. \tag{2.4}$$

For $W \in \mathcal{W}_0$, $g \in L^\infty(I)$, and a Lipschitz continuous D, there is a unique strong solution of (2.3), (2.4) $\mathbf{u} \in C^1(\mathbb{R}; L^\infty(I))$ [21]. Here and below, we use bold font to denote vector-valued functions, for example, $\mathbf{u}(t) = u(\cdot, t) \in L^\infty(I)$.

Denote the projection of the solution of the continuous problem (2.3), (2.4), $u(x, t)$ onto X_n by

$$\mathbf{P}_{X_n} u(x, t) = (u(x_{n1}, t), u(x_{n2}, t), \ldots, u_n(x_{nn}, t)).$$

Both functions $u_n(t)$ and $\mathbf{P}_{X_n} u(x, t)$ are defined on the discrete set X_n. For such functions, we will use the weighted Euclidean inner product

$$(u, v)_n = \frac{1}{n} \sum_{i=1}^n u_i v_i, \quad u = (u_1, u_2, \ldots, u_n)^\mathsf{T}, \; v = (v_1, v_2, \ldots, v_n)^\mathsf{T}$$

and the corresponding norm $\|u\|_{2,n} = \sqrt{(u, u)_n}$. Below, we will use $\|\cdot\|_{2,n}$ to study the difference between the solutions of the discrete and continuous problems (2.1) and (2.3) on W-random graphs.

3. Networks on W-Random Graphs Generated by Random Sequences

Denote

$$\tilde{X} = (x_1, x_2, x_3, \ldots) \quad \text{and} \quad \tilde{X}_n = (x_1, x_2, \ldots, x_n), \tag{3.1}$$

where $x_i, i \in \mathbb{N}$ are independent identically distributed (IID) random variables (RVs). RV x_1 has uniform on I distribution, that is, $\mathcal{L}(x_1) = U(I)$.

Definition 3.1. (*cf.* [18]) By a W-random graph on n nodes generated by the random sequence $\tilde{X}$, denoted $\tilde{G}_n = \mathbb{G}(\tilde{X}_n, W)$, we mean $\tilde{G}_n = \langle [n], E(\tilde{G}_n)\rangle$ such that the edges of $\tilde{G}_n$ are selected at random and

$$\mathbb{P}\{(i, j) \in E(\tilde{G}_n)\} = W(x_i, x_j), \quad \text{for each } (i, j) \in [n]^2,\ i \neq j.$$

The decision whether to include a pair $(i, j) \in [n]^2,\ i \neq j$, is made independently from the decisions for other pairs.

Remark 3.2. The graph sequence $\{\tilde{G}_n\}$ converges to graphon W almost surely as $n \to \infty$ [18].

Theorem 3.3. *Suppose $W \in \mathcal{W}_0$, D is a Lipschitz continuous function on $\mathbb{R}$, and $g \in L^\infty(I)$. Let $T > 0$ and suppose that the solution of the IVP (2.3) and (2.4) $u(x, t)$ satisfies the following inequality*

$$\min_{t\in[0,T]} \int_I \left\{ \int_I W(x, y) D\left(u(y, t) - u(x, t)\right)^2 \, \mathrm{d}y \right. \\ \left. - \left(\int_I W(x, y) D\left(u(y, t) - u(x, t)\right) \, \mathrm{d}y \right)^2 \right\} \mathrm{d}x \geqq C_1 \tag{3.2}$$

for some positive constant C_1. Then the solutions of the IVPs for the discrete and continuum models (2.1), (2.2) and (2.3), (2.4) satisfy the following relation

$$\lim_{n\to\infty} \mathbb{P}\left\{ n^{1/2} \sup_{t\in[0,T]} \|u^{(n)}(t) - \mathbf{P}_{\tilde{X}_n} u(x, t)\|_{2,n} \leqq C \right\} = 1$$

for some constant $C > 0$.

Remark 3.4. The integral expression in (3.2) defines a continuous function of t. This follows from $\mathbf{u} \in C(0, T; L^\infty(I))$, $\|W\|_{L^\infty(I^2)=1}$, and Lipschitz continuity of D. This justifies the use of min in (3.2).

For the proof of this theorem we will need the following application of the Central Limit Theorem (CLT) [2].

Lemma 3.5. *Suppose $W \in \mathcal{W}_0,\ f \in L^\infty(I^2)$, and*

$$X = (x_1, x_2, x_3, \ldots),$$

where $x_i, i \in \mathbb{N}$, *are IID RVs with* $\mathcal{L}(x_1) = U(I)$. *Define RVs* $\{\xi_{ij}\}$, $(i, j) \in \mathbb{N}^2$, *such that* $\mathcal{L}(\xi_{ij}|X) = Bin(W(x_i, x_j))$.[1] *Specifically,*

$$\mathbb{P}(\xi_{ij} = 1|X) = W(x_i, x_j) \quad \text{and} \quad \mathbb{P}(\xi_{ij} = 0|X) = 1 - W(x_i, x_j). \tag{3.3}$$

Further, let

$$\eta_{ij} = \xi_{ij} f(x_i, x_j), \quad (i, j) \in \mathbb{N}^2, \tag{3.4}$$

$$z_{ni} = \frac{1}{n}\sum_{j=1}^{n} \eta_{ij} - \int_I f(x_i, y) W(x_i, y)\, \mathrm{d}y, \quad \text{and} \quad S_n = \sum_{i=1}^{n} z_{ni}^2. \tag{3.5}$$

Finally, we assume

$$\sigma^2 := \int_{I^2} f(x, y)^2 W(x, y)\, \mathrm{d}x\, \mathrm{d}y - \int_I \left(\int_I f(x, y) W(x, y)\, \mathrm{d}y\right)^2 \mathrm{d}x > 0. \tag{3.6}$$

Then

$$\frac{S_n - \sigma^2}{n^{-1/2}\sqrt{5\sigma^4 + O(n^{-1})}} \xrightarrow{d} \mathcal{N}(0, 1), \tag{3.7}$$

where $\xrightarrow{d}$ *denotes convergence in distribution, and* $\mathcal{N}(0, 1)$ *stands for the standard normal distribution.*

By construction, $\{\eta_{ij}\}$ are IID RVs. Moreover, from (3.3) and (3.4) we have

$$\mu(x_i) = \mathbb{E}(\eta_{ij}|x_i) = \int_I f(x_i, y) W(x_i, y)\, \mathrm{d}y. \tag{3.8}$$

Therefore,

$$\mu := \mathbb{E}\eta_{ij} = \mathbb{E}\mathbb{E}(\eta_{ij}|x_i) = \int_{I^2} f(x, y) W(x, y)\, \mathrm{d}x\, \mathrm{d}y, \tag{3.9}$$

$$\begin{aligned} \mathrm{V}\eta_{ij} &= \mathbb{E}\mathbb{E}((\eta_{ij} - \mu)^2|x_i) = \mathbb{E}\mathbb{E}((\eta_{ij}^2|x_i) - 2\mu\mathbb{E}(\eta_{ij}|x_i) + \mu^2) \\ &= \int_{I^2} f(x, y)^2 W(x, y)\, \mathrm{d}x\, \mathrm{d}y - \int_I \left(\int_I f(x, y) W(x, y)\, \mathrm{d}y\right)^2 \mathrm{d}x = \sigma^2. \end{aligned} \tag{3.10}$$

Let

$$y_{ni} = \sqrt{n} z_{ni}. \tag{3.11}$$

[1] Bin(p) stands for the binomial distribution with parameter $p \in [0, 1]$.

We prove (3.7) by applying the CLT to $\sum_{i=1}^{n} y_{ni}^2$. To justify the application of the CLT, we need to compute three first moments of y_{ni}^2. To this end,

$$\begin{aligned}
\mathbb{E}y_{ni}^2 &= n^{-1}\mathbb{E}\,\mathbb{E}\left(\sum_{1\leqq j,k\leqq n}(\eta_{ij}-\mu(x_i))(\eta_{ik}-\mu(x_i))|x_i\right)\\
&= \mathbb{E}\,\mathbb{E}\left(\sum_{1\leqq j\leqq n}(\eta_{ij}-\mu(x_i))^2|x_i\right)\\
&\quad +2n^{-1}\mathbb{E}\,\mathbb{E}\left(\sum_{1\leqq j<k\leqq n}(\eta_{ij}-\mu(x_i))(\eta_{ik}-\mu(x_i))|x_i\right). \qquad (3.12)
\end{aligned}$$

The first term on the right hand side of (3.12) is equal to σ^2 [see (3.10)]. The second term is equal to 0, as easy to see using the independence of $\eta_{ij}-\mu(x_i)$ and $\eta_{ik}-\mu(x_i)$ for $k\neq j$. Thus,

$$\mathbb{E}y_{ni}^2 = \sigma^2+2n^{-1}\mathbb{E}\left(\sum_{1\leqq j<k\leqq n}\mathbb{E}(\eta_{ij}-\mu(x_i)|x_i)\mathbb{E}(\eta_{ik}-\mu(x_i)|x_i)\right)=\sigma^2. \qquad (3.13)$$

Recall that $\sigma^2>0$, by (3.6). Similarly, we compute

$$\begin{aligned}
\mathbb{E}(y_{ni}^4) &= n^{-2}\mathbb{E}\,\mathbb{E}\left(\sum_{1\leqq j_1,j_2,j_3,j_4\leqq n}(\eta_{ij_1}-\mu(x_i))\cdots(\eta_{ij_4}-\mu(x_i))|x_i\right)\\
&= 6n^{-2}\mathbb{E}\left(\sum_{1\leqq j<k\leqq n}\mathbb{E}(\eta_{ij}-\mu(x_i)|x_i)^2E(\eta_{ik}-\mu(x_i)|x_i)^2\right)\\
&\quad +n^{-2}\mathbb{E}\left(\sum_{1\leqq j\leqq n}\mathbb{E}(\eta_{ij}-\mu(x_i)|x_i)^4\right)\\
&= \frac{6n(n-1)}{n^2}\sigma^4+O(n^{-1})=6\sigma^4+O(n^{-1}) \qquad (3.14)
\end{aligned}$$

and

$$\mathbb{E}y_{ni}^6 = n^{-3}\mathbb{E}\,\mathbb{E}\left(\sum_{1\leqq j_1,j_2,j_3,j_4,j_5,j_6\leqq n}(\eta_{ij_1}-\mu(x_i))\cdots(\eta_{ij_6}-\mu(x_i))|x_i\right)$$

$$= \binom{6}{2}\binom{4}{2} n^{-3}\mathbb{E}\left(\sum_{1\leqq j<k<l\leqq n} \mathbb{E}(\eta_{ij}-\mu(x_i)|x_i)^2 E(\eta_{ik} - \mu(x_i)|x_i)^2 E(\eta_{il}-\mu(x_i)|x_i)^2\right) + O(n^{-1})$$

$$= \frac{90n(n-1)(n-2)}{n^3}\sigma^6 + O(n^{-1}) = 90\sigma^6 + O(n^{-1}). \tag{3.15}$$

For $n \in \mathbb{N}$, let

$$\zeta_{ni} := \frac{y_{ni}^2 - \mathbb{E}y_{ni}^2}{\sqrt{n\mathrm{V}(y_{in}^2)}} = \frac{y_{ni}^2-\sigma^2}{n^{1/2}\sqrt{5\sigma^4+O(n^{-1})}}, \quad i \in [n], \tag{3.16}$$

where (3.13) and (3.14) were used to obtain the expression on the right hand side.

Consider

$$\zeta_{n1}, \zeta_{n2}, \dots, \zeta_{nn}. \tag{3.17}$$

By construction, ζ_{ni}, $i \in [n]$, are IID RVs. Further,

$$\mathbb{E}\zeta_{ni} = 0 \quad \text{and} \quad \mathrm{V}\left(\sum_{i=1}^n \zeta_{ni}\right) = 1. \tag{3.18}$$

Moreover, the triangular array (3.17) satisfies the Lyapunov condition [2]

$$\sum_{i=1}^n \mathbb{E}|\zeta_{ni}|^3 \leqq \frac{\sum_{i=1}^n \mathbb{E}(y_{ni}^6 + 3y_{ni}^4\sigma^2 + 3y_{ni}^2\sigma^4+\sigma^6)}{n^{3/2}(5\sigma^4+O(n^{-1}))^{3/2}}$$
$$= O(n^{-1/2}) \to 0 \quad \text{as } n\to\infty. \tag{3.19}$$

From (3.18) and (3.19), via the CLT, we conclude that

$$\frac{\sum_{i=1}^n (y_{ni}^2-\sigma^2)}{\sqrt{n(5\sigma^4+O(n^{-1}))}} = \frac{n^{-1}\sum_{i=1}^n y_{ni}^2 - \sigma^2}{n^{-1/2}\sqrt{5\sigma^4+O(n^{-1})}} \xrightarrow{d} \mathcal{N}(0,1) \quad n\to\infty. \tag{3.20}$$

The statement (3.7) follows from (3.20) and the definition of y_{ni} (3.11). □

For the proof of Theorem 3.3, we need to extend Lemma 3.5 to cover the case when f depends on $t \in [0,T]$ in addition to $(x,y) \in I^2$.

Corollary 3.6. *Suppose that f in Lemma 3.5 also depends on $t \in [0,T]$, and $\mathbf{f} \in C(0,T; L^\infty(I^2))$ if viewed as a mapping from $[0,T]$ to $L^\infty(I^2)$, $\mathbf{f}(t) = f(\cdot,t) \in L^\infty(I^2)$. Adding $t-$dependence to all variables defined using f and, otherwise, keeping the notation of Lemma 3.5, we assume that*

$$\min_{t\in[0,T]} \sigma^2(t) \geqq c_1 > 0. \tag{3.21}$$

Then the conclusion of Lemma 3.5 holds for $t-$dependent sums for every $t \in [0,T]$

$$\frac{S_n(t)-\sigma_n^2(t)}{n^{-1/2}\sqrt{5\sigma_n^4(t)+O(n^{-1})}} \xrightarrow{d} \mathcal{N}(0,1) \quad \textit{as } n\to\infty. \tag{3.22}$$

Proof. From the assumption $\mathbf{f} \in C(0, T; L^\infty(I^2))$ and (3.21), for

$$\sigma^2(t) = \int_{I^2} f(x, y, t)^2 W(x, y)\, \mathrm{d}x\, \mathrm{d}y - \int_I \left(\int_I f(x, y, t) W(x, y)\, \mathrm{d}y \right)^2 \mathrm{d}x,$$

we have

$$0 < c_1 \leqq \sigma^2(t) \leqq 2\|\mathbf{f}\|^2_{C(0,T;L^\infty(I^2))}. \tag{3.23}$$

With these bounds, by repeating the steps in the proof of Lemma 3.5, we first show that $t-$dependent moments of $y^2_{ni}(t)$ are bounded uniformly in $t \in [0, T]$; then verify Lyapunov condition for every $t \in [0, T]$ and apply the CLT. This shows (3.22). □

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3. Denote $\zeta_{ni}(t) = u(x_i, t) - u_{ni}(t),\ i \in [n]$ and let

$$\zeta_n(t) = (\zeta_{n1}(t), \zeta_{n2}(t), \ldots, \zeta_{nn}(t)).$$

By subtracting Equation i in (2.1) from the corresponding equation in (2.3) evaluated at $x = x_i$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta_{ni}(t) = z_{ni}(t) + \frac{1}{n}\sum_{j=1}^{n} \xi_{ij}[D(u(x_j, t) - u(x_i, t)) - D(u_{nj}(t) - u_{ni}(t))], \tag{3.24}$$

where

$$z_{ni} = \int_I W(x_i, y) D(u(y, t) - u(x_i, t))\, \mathrm{d}y - \frac{1}{n}\sum_{j=1}^{n} \xi_{ij} D(u(x_j, t) - u(x_i, t)), \tag{3.25}$$

and ξ_{ij} are defined in (3.3).

Next, we multiply both sides of (3.24) by $n^{-1}\zeta_{ni}$ and sum over i to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta_n\|^2_{2,n} = (z_n, \zeta_n)_n + \frac{1}{n^2}\sum_{i,j=1}^{n} \xi_{ij}[D(u(x_j, t) - u(x_i, t)) - D(u_{nj}(t) - u_{ni}(t))]\zeta_{ni}, \tag{3.26}$$

where $z_n = (z_{n1}, z_{n2}, \ldots, z_{nn})$. We estimate the first term on the right-hand side of (3.26) via the Cauchy–Schwarz inequality

$$|(z_n, \zeta_n)_n| \leqq \|z_n\|_{2,n}\|\zeta_n\|_{2,n} \leqq 2^{-1}(\|z_n\|^2_{2,n} + \|\zeta_n\|^2_{2,n}). \tag{3.27}$$

For the second term we use the Lipschitz continuity of D, $|\xi_{ij}| \leqq 1$, the Cauchy–Schwarz inequality, and the triangle inequality to obtain

$$\left| \frac{1}{n^2} \sum_{i,j=1}^{n} \xi_{ij} [D(u(x_j,t) - u(x_i,t)) - D(u_{nj}(t) - u_{ni}(t))] \zeta_{ni} \right| \leqq \frac{L}{n^2} \sum_{i,j=1}^{n} (|\zeta_{nj}(t)| + |\zeta_{ni}(t)|)|\zeta_{ni}(t)| \leqq 2L \|\zeta_n(t)\|_{2,n}^2. \tag{3.28}$$

Using (3.26), (3.27), and (3.28), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\zeta_n\|_{2,n}^2 \leqq (4L+1)\|\zeta_n\|_{2,n}^2 + \|z_n\|_{n,2}^2. \tag{3.29}$$

From (3.29) via the Gronwall's inequality we have

$$\sup_{t\in[0,T]} \|\zeta_n(t)\|_{2,n} \leqq \frac{\sup_{t\in[0,T]} \|z_n(t)\|_{2,n}^2}{4L+1} \exp\{(4L+1)T\}. \tag{3.30}$$

It remains to bound $\sup_{t\in[0,T]} \|z_n(t)\|_{2,n}^2$. To this end, let

$$f(x,y,t) := D(u(y,t) - u(x,t)).$$

Using $\mathbf{u} \in C(0,T;L^\infty(I))$, Lipschitz continuity of D, and the triangle inequality, we have

$$\|\mathbf{f}\|_{C(0,T;L^\infty(I^2))} \leqq L \max_{t\in[0,T]} \operatorname{ess\,sup}_{(x,y)\in I^2} |u(x,t) - u(y,t)| \leqq 2L \|\mathbf{u}\|_{C(0,T;L^\infty(I))}. \tag{3.31}$$

By (3.2) and (3.31), we find that $\sigma^2(t)$ is bounded for $t \in [0,T]$

$$C_1 \leqq \sigma^2(t) \leqq 2L\|\mathbf{u}\|_{C(0,T;L^\infty(I))} =: C_2. \tag{3.32}$$

Using Corollary 3.6, for $z_n = (z_{n1}, z_{n2}, \dots, z_{nn})$ [see (3.25)], we have

$$\frac{n\|z_n\|_{2,n(t)}^2 - \sigma^2(t)}{n^{-1/2}\beta(\sigma^2(t))} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{where } \beta(\sigma^2(t)) = \sqrt{5\sigma^2(t) + O(n^{-1})}.$$

Further, we have

$$\mathbb{P}(|n\|z_n(t)\|_{2,n}^2 - \sigma^2(t)| > 1) = \mathbb{P}\left(\frac{|n\|z_n(t)\|_{2,n}^2 - \sigma^2(t)|}{n^{-1/2}\beta(\sigma^2(t))} > \frac{n^{1/2}}{\beta(\sigma^2(t))} \right) \leqq \mathbb{P}\left(\frac{|n\|z_n(t)\|_{2,n}^2 - \sigma^2(t)|}{n^{-1/2}\beta(\sigma^2(t))} > \frac{n^{1/2}}{C_2} \right) \to 0, \tag{3.33}$$

as $n \to \infty$. We used (3.32) to obtain the last inequality in (3.33). Convergence in (3.33) is uniform for $t \in [0, T]$. Therefore, $\|z_n(t)\|^2_{n,2}$ converges to zero in probability uniformly in t. Moreover,

$$\mathbb{P}(\|z_n(t)\|^2_{n,2} > (C_2 + 1)n^{-1}) \leqq \mathbb{P}(|n\|z_n(t)\|^2_{2,n} - \sigma^2(t)| > 1) \to 0, \quad \text{as } n \to \infty$$

uniformly for $t \in [0, T]$.

Let $\varepsilon > 0$ be arbitrary. Then for $C_3 := C_2 + 1$ and for some $N \in \mathbb{N}$, we have

$$\mathbb{P}\left(\sup_{t \in [0,T]} \|z_n(t)\|^2_{2,n} > C_3 n^{-1} \right) < \varepsilon \quad \text{for } n > N.$$

The combination of this and (3.30) proves the theorem. □

4. Networks on W-Random Graphs Generated by Deterministic Sequences

In this section, we consider the heat equations on W-random graphs generated by deterministic sequences of points from I. To this end, we partition I into n subintervals

$$I_{ni} = [(i-1)n^{-1}, in^{-1}), \quad i \in [n-1], \text{ and } I_{nn} = [(n-1)n^{-1}, 1]. \tag{4.1}$$

Suppose

$$X_n = \{x_{n1}, x_{n2}, \ldots, x_{nn}\}, \quad x_{ni} \in \bar{I}_{ni} \; i \in [n], \tag{4.2}$$

where $\bar{I}_{ni}$ denotes the closure of I_{ni}.

Definition 4.1. Graph $G_n = \langle V(G_n), E(G_n) \rangle$ is called a W-random graph generated by the deterministic sequence X_n and is denoted $G_n = \mathbb{G}(W, X_n)$, if $V(G_n) = [n]$ and for every $(i, j) \in [n]^2$, $i \neq j$,

$$\mathbb{P}\{(i, j) \in E(G_n)\} = W(x_{ni}, x_{nj}).$$

The decision whether to include (i, j) to $E(G_n)$ is made independently for each pair $(i, j) \in [n]^2$, $i \neq j$.

Remark 4.2. If W is continuous on I^2 almost everywhere, then $\{\mathbb{G}(W, X_n)\}$ is convergent with the limit given by graphon W (cf. Lemma 2.5 [4]).

Let $u_n(t) = (u_{n1}(t), u_{n2}(t), \ldots, u_{nn}(t))$ denote the solution of the IVP (2.1), (2.2) for the heat equation on $G_n = \mathbb{G}(W, X_n)$, and define $u_n : I \times \mathbb{R} \to \mathbb{R}$ as follows. For $x \in I_{ni}$, $i \in [n]$, let

$$u_n(x, t) = u_{ni}(t), t \in \mathbb{R}.$$

Theorem 4.3. *Suppose $W \in \mathcal{W}_0$ is almost everywhere continuous on I^2, $D : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, and $g \in L^\infty(I)$. Let $u(x,t)$ denote the solution of the IVP* (2.3), (2.4)*. Suppose further*

$$\min_{t\in[0,T]} \int_{I^2} D(u(y,t)-u(x,t))W(x,y)(1-W(x,y))\,\mathrm{d}x\,\mathrm{d}y > 0 \tag{4.3}$$

for some $T > 0$.[2] *Then*

$$\|\mathbf{u}_n - \mathbf{u}\|_{C(0,T;L^2(I))} \xrightarrow{p} 0 \quad \text{as } n \to \infty. \tag{4.4}$$

The convergence in (4.4) *is in probability.*

For the proof of Theorem 4.3 we need to derive several auxiliary results. The first result is parallel to Lemma 3.5 of the previous section.

Lemma 4.4. *Let $\{W_{nij}\}$ and $\{f_{nij}\}$ be two real arrays defined for $n \in \mathbb{N}$ and $i, j \in [n]$, and*

$$\sigma_{ni}^2 = n^{-1}\sum_{i=1}^n f_{nij}^2 W_{nij}(1-W_{nij}), \quad i \in [n], \tag{4.5}$$

$$\sigma_n^2 = n^{-1}\sum_{i=1}^n \sigma_{ni}^2. \tag{4.6}$$

Assume that $\{f_{nij}\}$, $n \in \mathbb{N}$, $i, j \in [n]$, is a bounded array, $0 \leqq w_{nij} \leqq 1$ and

$$\liminf_{n\to\infty} \sigma_n^2 > 0. \tag{4.7}$$

Let $\{\xi_{nij}\}, n \in \mathbb{N}, (i,j) \in [n]^2$ be independent binomial RVs $\mathcal{L}(\xi_{nij}) = Bin(W_{nij})$. Further, let

$$\begin{aligned} \eta_{nij} &= \xi_{nij} f_{nij}, \quad (i,j) \in [n]^2, \\ z_{ni} &= \frac{1}{n}\sum_{j=1}^n (\eta_{nij} - f_{nij}W_{nij}), \\ S_n &= \sum_{i=1}^n z_{ni}^2. \end{aligned}$$

Then

$$\frac{S_n - \sigma_n^2}{n^{-1/2}\sqrt{5\sigma_n^4 + O(n^{-1})}} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty. \tag{4.8}$$

[2] Because $\mathbf{u} \in C(\mathbb{R}, L^\infty(I))$, D is Lipschitz, and W is bounded, the integral in (4.3) defines a continuous function of t. Thus, the use min in (4.3) is justified.

Proof. First, compute the moments of the independent RVs $\{\eta_{nij}\}$, $n \in \mathbb{N}$, $(i, j) \in [n]^2$,

$$\mathbb{E}\eta_{nij}^k = f_{nij}^k W_{nij}, \quad k \in \mathbb{N}.$$

Thus, for $y_{ni} = \sqrt{n} z_{ni}$, $i \in [n]$, we have $\mathbb{E}y_{ni} = 0$. Further,

$$\begin{aligned}
\mathbb{E}y_{ni}^2 &= n^{-1}\mathbb{E}\left(\sum_{1\leqq j,k\leqq n} (\eta_{nij} - f_{nij}W_{nij})(\eta_{nik} - f_{nij}W_{nij})\right) \\
&= n^{-1}\mathbb{E}\left(\sum_{1\leqq j\leqq n} (\eta_{nij} - f_{nij}W_{nij})^2\right) \\
&\quad +2n^{-1}\mathbb{E}\left(\sum_{1\leqq j<k\leqq n} (\eta_{nij} - f_{nij}W_{nij})(\eta_{ik} - f_{nik}W_{nik})\right) \\
&= \sigma_{ni}^2 + 2n^{-1}\sum_{1\leqq j<k\leqq n} \mathbb{E}(\eta_{nij} - f_{nij}W_{nij})\mathbb{E}(\eta_{nik} - f_{nik}W_{nik}) = \sigma_{ni}^2,
\end{aligned}$$

where

$$\sigma_{ni}^2 := n^{-1}\mathbb{E}\left(\sum_{1\leqq j\leqq n} (\eta_{nij} - f_{nij}W_{nij})^2\right) = n^{-1}\sum_{j=1}^{n} f_{nij}^2 W_{nij}(1 - W_{nij}). \quad (4.9)$$

Similarly, we compute

$$\begin{aligned}
\mathbb{E}y_{ni}^4 &= n^{-2}\mathbb{E}\left(\sum_{1\leqq j_1,j_2,j_3,j_4\leqq n} (\eta_{nij_1} - f_{nij_1}W_{nij_1})\cdots(\eta_{nij_4} - f_{nij_4}W_{nij_4})\right) \\
&= 6n^{-2}\sum_{1\leqq j<k\leqq n} \mathbb{E}(\eta_{nij} - f_{nij}W_{nij})^2 E(\eta_{nik} - f_{nik}W_{nik})^2 \\
&\quad + \ n^{-2}\sum_{1\leqq j\leqq n} \mathbb{E}(\eta_{nij} - f_{nij}W_{nij})^4 \\
&= \frac{6n(n-1)}{n^2}\sigma_{ni}^4 + O(n^{-1}) = 6\sigma_{ni}^4 + O(n^{-1}).
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}y_{ni}^6 &= n^{-3}\mathbb{E}\left(\sum_{1\leqq j_1,j_2,j_3,j_4,j_5,j_6\leqq n} (\eta_{nij_1} - f_{nij_1}W_{nij_1})\cdots(\eta_{nij_6} - f_{nij_6}W_{nij_6})\right) \\
&= \binom{6}{2}\binom{4}{2}n^{-2}\sum_{1\leqq j<k<l\leqq n} \mathbb{E}(\eta_{nij} - f_{nij}W_{nij})^2 \\
&\quad \times E(\eta_{nik} - f_{nik}W_{nik})^2\mathbb{E}(\eta_{nil} - f_{nil}W_{nil})^2 + O(n^{-1}) \\
&= \frac{90n(n-1)(n-2)}{n^3}\sigma_{ni}^6 + O(n^{-1}) = 90\sigma_{ni}^6 + O(n^{-1}).
\end{aligned}$$

For $n \in \mathbb{N}$, let

$$\zeta_{ni} := \frac{y_{ni}^2 - \mathbb{E}y_{ni}^2}{\sqrt{n\mathrm{V}(y_{in}^2)}} = \frac{y_{ni}^2 - \sigma_{ni}^2}{n^{1/2}\sqrt{5\sigma_{ni}^4 + O(n^{-1})}}, \quad i \in [n]. \tag{4.10}$$

Consider

$$\zeta_{n1}, \zeta_{n2}, \ldots, \zeta_{nn}. \tag{4.11}$$

By construction, ζ_{ni}, $i \in [n]$ are independent RVs. Further,

$$\mathbb{E}\zeta_{ni} = 0 \quad \text{and} \quad \mathrm{V}\left(\sum_{i=1}^{n} \zeta_{ni}\right) = 1. \tag{4.12}$$

Moreover, the triangular array (3.17) satisfies the Lyapunov condition [2]

$$\begin{aligned}\sum_{i=1}^{n} \mathbb{E}|\zeta_{ni}|^3 &\leqq \frac{\sum_{i=1}^{n} \mathbb{E}(y_{ni}^6 + 3y_{ni}^4\sigma_{ni}^2 + 3y_{ni}^2\sigma_{ni}^4 + \sigma_{ni}^6)}{n^{3/2}(5\sigma_n^4 + O(n^{-1}))^{3/2}} \\ &= O(n^{-1/2}) \to 0 \quad \text{as } n \to \infty. \end{aligned}\tag{4.13}$$

From (4.12) and (4.13), using the CLT, we conclude that

$$\frac{\sum_{i=1}^{n}(y_{ni}^2 - \sigma_{ni}^2)}{\sqrt{n(5\sigma_n^4 + O(n^{-1}))}} = \frac{n^{-1}\sum_{i=1}^{n} y_{ni}^2 - \sigma_n^2}{n^{-1/2}\sqrt{5\sigma_n^4 + O(n^{-1})}} \xrightarrow{d} \mathcal{N}(0,1), \quad n \to \infty. \tag{4.14}$$

The statement (4.8) follows from (4.14) and the definition of y_{ni}. □

With obvious modifications the proof of Lemma 4.4 can be easily extended to cover the following version of the lemma.

Corollary 4.5. *Suppose f_{nij} in Lemma 4.4 depend on real parameter $t \in [0, T]$ for some T. Keeping the notation of Lemma 4.4, we add $t-$dependence to all variables defined using $f_{nij}(t)$. Assume that functions $f_{nij}(t)$, $n \in \mathbb{N}, i, j, \in [n]$, are uniformly bounded for $t \in [0, T]$ and*

$$\liminf_{n\to\infty} \sigma_n^2(t) = \liminf_{n\to\infty} n^{-1} \sum_{i,j=1}^{n} f_{nij}(t) W_{nij}(1 - W_{nij}) \geqq C_1 > 0 \tag{4.15}$$

for every $t \in [0, T]$.

Then the conclusion of Lemma 4.4 holds for $t-$dependent sums for every $t \in [0, T]$

$$\frac{S_n(t) - \sigma_n^2(t)}{n^{-1/2}\sqrt{5\sigma_n^4(t) + O(n^{-1})}} \xrightarrow{d} \mathcal{N}(0,1), \quad n \to \infty.$$

Having prepared the application of the CLT that will be needed in the proof of Theorem 4.3, we now introduce an auxiliary IVP for the heat equation on a weighted graph $\tilde{G}_n = \mathbb{H}(W, X_n)$. The latter is a complete graph on n nodes, $V(\tilde{G}_n) = [n]$. Each edge of $\tilde{G}_n$ is supplied with the weight

$$W_{nij} = W(x_{ni}, x_{nj}), \quad (i, j) \in [n]^2, \ i \neq j.$$

Consider the IVP for the heat equation on the weighted graph $\tilde{G}_n$

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{ni}(t) = n^{-1} \sum_{j:(i,j)\in E(\tilde{G}_n)} W_{nij} D(v_{nj} - v_{ni}), \tag{4.16}$$

$$v_{ni}(0) = g(x_i), \quad i \in [n]. \tag{4.17}$$

Denote the solution of the IVP (4.16) and (4.17) by $v_n(t) = (v_{n1}(t), v_{n2}(t), \ldots, v_{nn}(t))$. Let $v_n(x, t)$ be a function defined on $I \times \mathbb{R}$ and such that for $x \in I_{ni}$, $i \in [n]$

$$v_n(x, t) = v_n(t), \quad t \in \mathbb{R}.$$

Next, define a step-function W_n on I^2 such that for $(x, y) \in I_{ni} \times I_{nj}$, $i, j \in [n]$,

$$W_n(x, y) = W_{nij}.$$

By construction, $v_n(x, t)$ solves the following IVP

$$\frac{\partial}{\partial t} v_n(x, t) = \int_I W_n(x, y) D(v_n(y, t) - v_n(x, t))\, \mathrm{d}y, \tag{4.18}$$

$$v_n(x, 0) = g(x_{ni}), \quad x \in I_{ni}, i \in [n]. \tag{4.19}$$

It was shown in [21] that for large n, $v_n(x, t)$ approximates the solution of the IVP (2.3), (2.4). Specifically, we have the following lemma.

Lemma 4.6. [21, Theorem 5.2] *Suppose $W \in L^\infty(I^2)$ is almost everywhere continuous on I^2, D is Lipschitz continuous, and $g \in L^\infty(I)$. Then for any $T > 0$*

$$\|\mathbf{u} - \mathbf{v}_n\|_{C(0,T;L^2(I))} \to 0 \quad \text{as } n \to \infty. \tag{4.20}$$

We use Lemma 4.6 to derive the following result.

Lemma 4.7. *Suppose $W \in \mathcal{W}_0$ is almost everywhere continuous on I^2, D is Lipschitz continuous, and $g \in L^\infty(I)$. Let $u(x, t)$ and $v_n(x, t)$ denote the solutions of the IVPs* (2.3), (2.4) *and* (4.18), (4.19), *respectively; and let*

$$\sigma^2(t) = \int_{I^2} D(u(y, t) - u(x, t)) W(x, y)(1 - W(x, y))\, \mathrm{d}x\, \mathrm{d}y,$$

$$\sigma_n^2(t) = \int_{I^2} D(v_n(y, t) - v_n(x, t)) W_n(x, y)(1 - W_n(x, y))\, \mathrm{d}x\, \mathrm{d}y.$$

Then

$$\sup_{t\in[0,T]} |\sigma_n^2(t) - \sigma^2(t)| \leqq C_2[\|\mathbf{v}_n - \mathbf{u}\|_{C(0,T;L^2(I))} + \|W_n - W\|_{L^2(I^2)}],$$

for some $C_2 > 0$. In particular, $\sigma_n^2 \to \sigma^2$ uniformly for $t \in [0, T]$.

Proof. 1. Using Lipschitz continuity of D and the triangle inequality, for any $t \in [0, T]$ we have

$$\begin{aligned}
&\left| \int_{I^2} D(v_n(y,t) - v_n(x,t)) - D(u(y,t) - u(x,t))\, \mathrm{d}x\, \mathrm{d}y \right| \\
&\leqq L \int_{I^2} |v_n(y,t) - u(y,t)| + |v_n(x,t) - u(x,t)|\, \mathrm{d}x\, \mathrm{d}y \\
&\leqq 2L \|\mathbf{v}_n - \mathbf{u}\|_{C(0,T;L^2(I))} \to 0,
\end{aligned} \tag{4.21}$$

as $n \to \infty$. Therefore,

$$\max_{t\in[0,T]} \left| \int_{I^2} D(v_n(y,t) - v_n(x,t))\, \mathrm{d}x\, \mathrm{d}y \right| \leqq C_3, \quad n \in \mathbb{N}, \tag{4.22}$$

for some C_3 independent of n.

2. Denote $q(x) = x(1-x)$. For $x, y \in [0, 1]$, $|q(x) - q(y)| \leqq |x - y|$. Thus,

$$|q(W) - q(W_n)| \leqq |W - W_n|. \tag{4.23}$$

3. Finally, we estimate $|\sigma_n(t) - \sigma(t)|$. For arbitrary $t \in [0, T]$, we have

$$\begin{aligned}
&\left| \int_{I^2} D(v_n(y,t) - v_n(x,t)) q(W_n(x,y))\, \mathrm{d}x\, \mathrm{d}y \right. \\
&\quad \left. - \int_{I^2} D(u(y,t) - u(x,t)) q(W(x,y))\, \mathrm{d}x\, \mathrm{d}y \right| \\
&\leqq \left| \int_{I^2} D(v_n(y,t) - v_n(x,t)) \left[q(W_n(x,y)) - q(W(x,y)) \right] \mathrm{d}x\, \mathrm{d}y \right| \\
&\quad + \left| \int_{I^2} \left[D(v_n(y,t) - v_n(x,t)) - D(u(y,t) - u(x,t)) \right] q(W(x,y))\, \mathrm{d}x\, \mathrm{d}y \right|.
\end{aligned} \tag{4.24}$$

Using the Cauchy–Schwarz inequality, Lipschitz continuity of D, $|q(W)| \leqq 1$, (4.22), and (4.23) from (4.24) we obtain

$$\sup_{t\in[0,T]} |\sigma_n(t) - \sigma(t)| \leqq C_3 \|W - W_n\|_{L^2(I^2)} + L \|\mathbf{v}_n - \mathbf{u}\|_{C(0,T;L^2(I))}. \tag{4.25}$$

Note that $W_n \to W$ as $n \to \infty$ at every point of continuity of W, that is, almost everywhere on I^2. Therefore, by the dominated convergence theorem,

$$\|W - W_n\|_{L^2(I^2)} \to 0 \quad \text{as } n \to \infty. \tag{4.26}$$

The statement of the lemma follows from (4.25), (4.26), and Lemma 4.6.

□

Proof of Theorem 4.3. Denote $\eta_{ni}(t) = u_{ni}(t) - v_{ni}(t)$, $i \in [n]$, and

$$\eta_n(t) = (\eta_{n1}(t), \eta_{n2}(t), \ldots, \eta_{nn}(t)).$$

By subtracting Equation i in (4.16) from the corresponding equation in (2.1) written for $G_n = \mathbb{G}(W, X_n)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_{ni} = \frac{1}{n}\left(\sum_{j=1}^{n}\xi_{nij}D(u_{nj}-u_{ni}) - \sum_{j=1}^{n}W_{nij}D(v_{nj}-v_{ni})\right). \tag{4.27}$$

By rewriting the right-hand side of (4.27), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_{ni} = \frac{1}{n}\sum_{j=1}^{n}\xi_{nij}[D(u_{nj}-u_{ni}) - D(v_{nj}-v_{ni})] + z_{ni}, \tag{4.28}$$

where

$$z_{ni} = \frac{1}{n}\sum_{j=1}^{n}\xi_{nij}D(v_{nj}-v_{ni}) - \frac{1}{n}\sum_{j=1}^{n}w_{nij}D(v_{nj}-v_{ni}). \tag{4.29}$$

By multiplying both sides of (4.28) by $n^{-1}\eta_{ni}$ and summing over i, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_n\|_{2,n}^2 = \frac{1}{n^2}\sum_{i,j=1}^{n}\xi_{ij}[D(u_{nj}-u_{ni}) - D(v_{nj}-v_{ni})]\eta_{ni} + (z_n, \eta_n)_n. \tag{4.30}$$

We bound the first term on the right hand side of (4.30) using the Lipschitz continuity of D, $|\xi_{ij}| \leqq 1$, the Cauchy–Schwarz inequality, and the triangle inequality

$$\begin{aligned}&\left|\frac{1}{n^2}\sum_{i,j=1}^{n}\xi_{ij}[D(u_{nj}-u_{ni}) - D(v_{nj}-v_{ni})]\eta_{ni}\right| \\ &\quad\leqq \frac{L}{n^2}\sum_{i,j=1}^{n}(|\eta_{nj}| + |\eta_{ni}|)|\eta_{ni}| \leqq 2L\|\eta_{ni}\|_{2,n}^2.\end{aligned} \tag{4.31}$$

We bound the second term using the Cauchy–Schwarz inequality

$$|(z_n, \eta_n)_n| \leqq \|z_n\|_{2,n}\|\eta_n\|_{2,n} \leqq \frac{1}{2}(\|z_n\|_{2,n}^2 + \|\eta_n\|_{2,n}^2), \tag{4.32}$$

where $z_n = (z_{n1}, z_{n2}, \ldots, z_{nn})$.

The combination of (4.30), (4.31), and (4.32) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_n\|_{2,n}^2 \leqq (4L+1)\|\eta_n\|_{2,n}^2 + \|z_n\|_{2,n}^2. \tag{4.33}$$

By Gronwall's inequality,

$$\max_{t\in[0,T]}\|\eta\|_{2,n}^2 \leqq \frac{\max_{t\in[0,T]}\|z_n(t)\|_{2,n}^2}{4L+1}\exp\{(4L+1)T\}. \tag{4.34}$$

Thus,

$$\max_{t\in[0,T]}\|\eta\|_{2,n} \leqq \frac{\max_{t\in[0,T]}\|z_n(t)\|_{2,n}}{\sqrt{4L+1}}\exp\{(2L+1)T\}. \tag{4.35}$$

It remains to estimate $\|z_n(t)\|_{2,n}$ [see (4.29)]. To this end, we use Corollary 4.5 with

$$f_{nij}(t) = D(v_{nj}(t) - v_{ni}(t)) \quad \text{and} \quad W_{nij} = W(x_{ni}, x_{nj}).$$

From Lemma 4.7 and (4.3), we have

$$\min_{t\in[0,T]} \sigma_n^2(t) \geqq C_4 > 0, \tag{4.36}$$

for sufficiently large n. In particular, (4.15) holds. Similarly, by Lemma 4.7, we have

$$\max_{t\in[0,T]} \sigma_n^2(t) \leqq C_5, \quad n \in \mathbb{N}. \tag{4.37}$$

By Corollary 4.5, for arbitrary $t \in [0, T]$, we have

$$\begin{aligned}\mathbb{P}\{|n\|z_n(t)\|_{2,n}^2 - \sigma_n^2(t)| > 1\} &= \mathbb{P}\left\{\left|\frac{n\|z_n(t)\|_{2,n}^2 - \sigma_n^2(t)}{n^{-1/2}\sqrt{5\sigma_n^4(t) + O(n^{-1})}}\right| > \frac{n^{1/2}}{\sqrt{5\sigma_n^4(t) + O(n^{-1})}}\right\}\\ &\leqq \mathbb{P}\left\{\left|\frac{n\|z_n(t)\|_{2,n}^2 - \sigma_n^2(t)}{n^{-1/2}\sqrt{5\sigma_n^4(t) + O(n^{-1})}}\right| > \frac{n^{1/2}}{\sqrt{5C_5^2 + O(n^{-1})}}\right\} \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.38}$$

Using (4.37), from (4.38) we have

$$\mathbb{P}\{\|z_n(t)\|_{2,n}^2 \leqq (C_5 + 1)n^{-1}\} \leqq \mathbb{P}\{|n\|z_n(t)\|_{2,n}^2 - \sigma_n^2(t)| > 1\} \to 0 \quad \text{as } n \to \infty. \tag{4.39}$$

Finally, since $t \in [0, T]$ is arbitrary from (4.39) we further have

$$\lim_{n\to\infty} \mathbb{P}\{\max_{t\in[0,T]} \|z_n(t)\|_{2,n} \leqq C_6 n^{-1/2}\} = 0. \tag{4.40}$$

The combination of (4.34) and (4.40) yields that $\|\eta_n\|_{2,n}$ tends to 0 in probability.

Using the definitions of η_n and $\mathbf{u}_n$, we have

$$\|\mathbf{u}_n - \mathbf{u}\|_{C(0,T;L^2(I))} \leqq \max_{t\in[0,T]} \|\eta_n(t)\|_{2,n} + \|\mathbf{v}_n - \mathbf{u}\|_{C(0,T;L^2(I))}. \tag{4.41}$$

Using Lemma 4.6 and (4.40), we show that $\|\mathbf{u}_n - \mathbf{u}\|_{C(0,T;L^2(I))}$ tends to 0 in probability as $n \to \infty$. □

5. Dynamical Models on W-Small-World Graphs

The method developed in the previous sections can be used to derive continuum limits for a large class of dynamical systems on random graphs. As an application, in this section, we consider dynamical systems on SW graphs [34]. The latter are popular in modeling networks of diverse nature, because they exhibit the combination of properties that are characteristic to both regular and random graphs, just as seen in many real-life systems [34].

First, we introduce a convenient generalization of a SW graph. To this end, let X_n be a set of n points from I as defined in (4.2) and let $W \in \mathcal{W}_0$ be a $\{0, 1\}-$valued graphon. We assume that W is almost everywhere continuous on I^2 and its support has a positive Lebesgue measure. Next, define

$$W_p(x, y) = (1-p)W(x, y) + p(1 - W(x, y)), \quad p \in [0, 0.5]. \tag{5.1}$$

Definition 5.1. $G_n = \mathbb{G}(W_p, X_n)$ is called a W-small-world (W-SW) graph.

Remark 5.2. Note that for $p = 0.5$, W_p becomes the Erdős–Rényi graph $G(n, 0.5)$.

Remark 5.3. Using the random set of points from $\tilde{X}_n$ as in (3.1), one constructs a W-SW graph $\tilde{G}_n = \mathbb{G}_n(W, \tilde{X}_n)$ generated by a random set of points.

Remark 5.4. Equation (5.1) implies that in the process of construction of the W-SW graph $G_n = \mathbb{G}(W_p, X_n)$, the new random edges to be added to the deterministic graph $\mathbb{G}(W_p, X_n)$ are selected from the complement of the edge set $E(\mathbb{G}(W, X_n))$.

It is easy to modify (5.1) to imitate other possible variants of the SW model. For instance, for fixed $q \in (0, 1)$,

$$\textbf{A)} \quad W_p = (1-p)W + pq \qquad \text{and} \qquad \textbf{B)} \quad W_p = W + pq, \ \ p \in [0, 1] \tag{5.2}$$

match the descriptions of the SW networks in [34] and [23,24] respectively.

Theorem 4.3 shows that the continuous model (2.3) with $W := W_p$ approximates the discrete network (2.1) on the W-SW graph $G_{n,p}$ for large n, that is, Equation (2.3) with $W = W_p$ is the continuum limit of the discrete heat equation on the SW graph. We illustrate this result with the continuum limit for the Kuramoto model on the SW network [8,35].

Example 5.5. The Kuramoto model of coupled identical phase oscillators on the SW graph $G_{n,p}$ has the following form (cf. [35])

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{ni}(t) = \omega + \sum_{j:(i,j) \in E(G_{n,p})} \sin(2\pi(u_{nj} - u_{ni})), \quad i \in [n], \tag{5.3}$$

where for fixed $n \in \mathbb{N}$ and $i \in [n]$, $u_{ni} : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is interpreted as the phase of oscillator i and ω is its intrinsic frequency.

For this example, let

$$X_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$$

and

$$W(x, y) = \begin{cases} 1, & d(x, y) \leqq r, \\ 0, & \text{otherwise}, \end{cases}$$

where $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$ and parameter $r \in (0, 1)$ is fixed.

With the above definitions, $G_{n,p}$ is a W-SW graph. In particular, $G_{n,0}$ is the k-nearest-neighbor graph ($k = \lfloor rn \rfloor$) (see Fig. 2a), which was used as the underlying deterministic graph in [34], and $G_{n,0.5}$ is the Erdős–Rényi graph $G(n, 0.5)$ (see Fig. 2c). Thus, the family $\{G_{n,p}\}$ interpolates between the k-nearest-neighbor graph and the Erdős–Rényi graph. Furthermore, had we chosen to use (5.2**A**) instead of (5.1), we would have obtained a family of random graph that differs from the original Watts–Strogatz SW model [34] only in minor details.

Theorem 4.3 justifies the following continuum limit for the Kuramoto model on the W-SW graph

$$\frac{\partial}{\partial t}u(x,t) = \omega + \int_I W_p(x,y)\sin(2\pi(u(y,t)-u(x,t)))\,\mathrm{d}y. \tag{5.4}$$

Equation (5.4) can be used to study the stability of $q-$twisted states, a family of steady state solutions of (5.3), just as was done for the k-nearest-neighbor coupled networks in [8,35]. The analysis of this problem is beyond the scope of this paper and will be presented elsewhere [20].

6. Discussion

Coupled dynamical systems on graphs arise in modeling diverse phenomena in physics, biology, and technology [6,11,16,19,22,30,32]. The dynamics of these models is shaped by the properties of the local dynamical systems at the nodes of the graph and the patterns of connections between them. The principal challenge of the mathematical theory of dynamical networks is to elucidate the contribution of the structural properties of the networks to their dynamics. Thus, it is important to develop analytical techniques, which apply to large classes of networks and reveal the interplay between the local dynamics and network topology. For nonlocally coupled dynamical systems, an important (albeit often formal) approach to the analysis of network dynamics has been replacing a discrete model on a large graph with a continuum (thermodynamic) limit. For networks with nonlinear diffusive coupling the continuum limit is an evolution equation with a nonlocal integral operator modeling nonlinear diffusion. This approach has proved very useful for the analysis of nonlocally coupled dynamical systems on deterministic graphs [1,8,14,35].

In applications, one often encounters dynamical networks on random graphs. They are especially important in biology. For example, random graphs are frequently used in computational modeling of neuronal systems, because random connectivity is often consistent with experimental data. For dynamical networks on random graphs, such as SW graphs, even formal continuum limit is not obvious. On the other hand, the theory of graph limits provides many examples of convergent sequences of random graphs with relatively simple deterministic limits [4,17,18]. In [21], we used the ideas of the theory of graph limits to provide a rigorous mathematical justification for taking the continuum limit in a large class of deterministic networks. In this paper, we have shown how to derive the limiting equations for dynamical networks on random graphs. Specifically, we studied coupled dynamical systems on convergent families of W-random graphs [17,18]. The latter provide a convenient analytical framework for modeling random graphs, which include many important examples arising in applications, such as Erdős–Rényi and SW graphs. We have proven that the solutions of the IVPs for discrete models converge in $C(0, T; L^2[0, 1])$ norm to their continuous counterpart as the graph size goes to infinity.

We studied networks for two variants of W-random graphs: those generated by the random and deterministic sequences respectively. For the discrete problems of the first type, the $O(n^{-1/2})$ convergence is shown. The rate of convergence in this case is determined solely by the CLT and holds for all graphons $W \in \mathcal{W}_0$. The proof of convergence of the discrete problems of the second type, in addition to the CLT, involves the analysis of the auxiliary IVPs (4.18), (4.19) [see (4.41), (4.35), (4.40), and Lemma 4.6]. The convergence rate of the auxiliary problems depends on the regularity of the graphon W. For instance, Theorem 4.1 in [21] shows that for a $\{0, 1\}$-valued graphon W, the convergence rate depends on the box-counting dimension of the boundary of the support of W, and may be very slow if the latter is close to 2. Consequently, discrete problems on W-random graphs generated by deterministic sequences may exhibit slower convergence compared to that of their counterparts on W-random graphs generated by random sequences. However, the former are convenient in applications, as they often can be readily related to the existing random graph models. For example, the classical Watts–Strogatz SW graph [34] can be interpreted as a W-random graph generated by a deterministic sequence. In Section 5, we used this fact to drive the continuum limit for dynamical systems on SW networks as an illustration of our method. We believe that the continuum limit analyzed in this paper will become a useful tool for studying coupled dynamical systems on random graphs.

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