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STABILITY OF EQUILIBRIA OF RANDOMLY PERTURBED MAPS

Paweł Hitczenko and Georgi S. Medvedev*

Department of Mathematics
Drexel University
3141 Chestnut Street
Philadelphia, PA 19104, USA

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Abstract. We derive a sufficient condition for stability in probability of an equilibrium of a randomly perturbed map in $\mathbb{R}^d$. This condition can be used to stabilize unstable equilibria by random forcing. Analytical results on stabilization are illustrated with numerical examples of randomly perturbed nonlinear maps in one- and two-dimensional spaces.

1. **Introduction.** The idea of stabilizing unstable equilibria of dynamical systems by noise originates from the pioneering work of Khasminskii on stochastic stability in the nineteen-sixties [24]. Stochastic stabilization has important implications for control theory [7, 28, 5, 6] and for numerical methods for stochastic differential equations [30, 31, 18, 19, 12]. Furthermore, the interplay of stability and noise is important for understanding many dynamical phenomena in applied science including stochastic synchronization [1, 14, 29, 17], stochastic resonance [27, 26, 15], and noise-induced dynamics [8, 13, 20].

To illustrate the mechanism of stabilization in discrete setting, we consider a scalar difference equation

$$x_n = (1 + \epsilon + \sigma\xi_n)x_{n-1},\ n \in \mathbb{N}, \tag{1}$$

where $0 < \epsilon, \sigma \ll 1$ and (ξ_n) are independent copies of the random variable (RV) ξ with zero mean and $\mathbb{E}\xi^2 = 1$. Further, assume $\mathbb{P}(|\xi| > M) = 0$ for some $M > 0$. The last condition is used to simplify the analysis of the introductory example (1). In the main part of the paper, we do not impose this condition, thus allowing for unbounded noise.

For a given $x_0 \in \mathbb{R}$, we have

$$x_n = \left(\prod_{k=1}^{n}(1 + \epsilon + \sigma\xi_k)\right)x_0.$$

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* Corresponding author: Georgi S. Medvedev.

Let $0 < \sigma < M^{-1}$. Then with probability 1

$$1 + \epsilon + \sigma\xi_k > 0 \; \forall k \in \mathbb{N}$$

and we have

$$\log |x_n| = \log |x_0| + \sum_{k=1}^{n} \log(1 + \epsilon + \sigma\xi_k)$$

holding almost surely. By the Strong Law of Large Numbers,

$$n^{-1} \sum_{k=1}^{n} \log(1 + \epsilon + \sigma\xi_k) \to \mathbb{E} \log(1 + \epsilon + \sigma\xi) \text{ as } n \to \infty$$

almost surely. Thus, the asymptotic stability of the origin (in the almost sure sense) will follow if

$$\mathbb{E} \log(1 + \epsilon + \sigma\xi) < 0. \tag{2}$$

Using the Taylor expansion of log and $\mathbb{E}\xi = 0$, we have

$$\mathbb{E} \log(1 + \epsilon + \sigma\xi) = \epsilon - \frac{\sigma^2}{2} + O(\sigma^3, \epsilon^2). \tag{3}$$

Thus, the stabilization of the weakly unstable equilibrium of (1) is achieved if

$$\epsilon - \frac{\sigma^2}{2} < 0 \tag{4}$$

for $0 < \epsilon, \sigma \ll 1$[1].

The stabilization mechanism illustrated by the linear difference equation (1) is relevant to many models. It has been demonstrated for certain nonlinear scalar difference equations by Appleby, Mao, and Rodkina [6] and by Appleby, Berkolaiko, and Rodkina [4] (see also [2, 11, 3, 9]), as well as for some higher dimensional models (see [12] and references therein). The goal of this paper is to establish an analog of (4) in the multidimensional setting under general assumptions on the underlying deterministic system and the form of random perturbation.

Specifically, we study the following difference equation in $\mathbb{R}^d$

$$x_{n+1} = (A + B_n)x_n + q(x_n), \tag{5}$$

where $q(x) = O(|x|^2)$ is a smooth function, A is a $d \times d$ deterministic matrix and (B_n) are independent copies of a $d \times d$ random matrix B. Thus, $x \mapsto Ax$ represents the linearization of the nonlinear map $F(x) = Ax + q(x)$ near the equilibrium at the origin. Our only assumption on F is that the spectral radius of A, $\rho(A) = 1 + \epsilon$ for $0 < \epsilon \ll 1$, i.e., F is unstable, but is close to the stability boundary. We do not make any assumptions on the character of the instability: F may be close to a saddle-node, Hopf, or any other bifurcation. We want to know how to choose a mean zero random matrix $B = B(\epsilon)$, so that the origin, as an equilibrium of the randomly perturbed system (1), becomes stable with high probability.

Randomly perturbed map (5) arises as a numerical discretization of stochastic ordinary differential equations [12, 18, 19] and as a Poincare map of a periodic orbit in randomly perturbed systems of ordinary differential equations [20, 21], to mention two major areas of applications. In both contexts, the effect of the stochastic term on the stability of the system is a key question. In this work, we prove a sufficient condition for stabilization of (5) under fairly general assumptions

[1]Following the lines of the proof of Theorem 3.3, one can arrive at (4) under weaker assumptions on noise. In particular, the assumption of bounded noise that was used in the analysis of (1) can be replaced by $\mathbb{E}|\xi|^3 < \infty$.

on $B(\epsilon) = (b_{ij}(\epsilon))$. Besides $\mathbb{E}\ b_{ij}(\epsilon) = 0$, we ask that the first three moments of $b_{ij}(\epsilon)$ are finite. For such matrices, we formulate the multidimensional analog of the stabilization condition (4) (see Theorem 3.3 for details). Our goal here is not just to show how to achieve stochastic stabilization in (5): the latter can be done by simple means (see Remark 4). Rather, we want to describe a general class of stabilizing perturbations within the framework (5). This is important, for example, in the context of the stabilization of periodic orbits, where matrix B is obtained in the process of the construction of the Poincare map [21], and in practice, one may have only partial control of the entries of B. Stochastic stabilization of periodic orbits remains largely unexplored area of research with many promising applications. We believe that the results of this paper combined with our previous work [21] prepare the way for developing stochastic stabilization of periodic solutions in general systems of ordinary differential equations.

The organization of this paper is as follows. In the next section, we prove a sufficient condition for stability (in probability) of an equilibrium in a d-dimensional map (cf. Theorem 2.2). To prove this theorem, we use the Strong Law of Large Numbers to show that the Lyapunov exponent of a typical trajectory is negative. The rest of the proof follows an argument developed for deterministic dynamical systems [25]. In §3 we apply Theorem 2.2 to the problem of stabilization. In §4, we illustrate our results with several numerical examples using one- and two-dimensional systems.

2. **Stochastic stability.** Consider an initial value problem for the following difference equation

$$x_n = M_n x_{n-1} + q(x_{n-1}),\ n \geq 1. \tag{6}$$

where (M_n) are independent copies of a $d \times d$ random matrix M with $\mathbb{E}\|M\| < \infty$; $q : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function such that

$$|q(x)| \leq C_1 |x|^2,\ x \in B_\delta = \{x : |x| \leq \delta\} \tag{7}$$

for some $C_1, \delta > 0$. Here and below, we will use $|\cdot|$ to denote the Euclidean norm of a vector. The initial condition x_0 is assumed to be deterministic.

Definition 2.1. (cf. [24]) The equilibrium at the origin of (6) is said to be stable in probability if for any $\varepsilon > 0$

$$\lim_{|x_0| \to 0} \mathbb{P}\left\{ \sup_{n \geq 1} |x_n| > \varepsilon \right\} = 0.$$

Theorem 2.2. *Suppose*

$$0 < \lambda = -\mathbb{E} \log \|M\| < \infty. \tag{8}$$

Then the equilibrium at the origin of (6) is stable in probability.[2]

Remark 1. In (8), $\|\cdot\|$ is an arbitrary matrix norm. The same matrix norm is used throughout this section.

Remark 2. If we set $q \equiv 0$ in (6), i.e., consider a linear model, then by the Furstenberg-Kesten theory, (8) implies that $|x_n| \to 0$ exponentially fast with probability 1 (cf. [16]). In this case, the origin is stable in the almost sure sense, as in the one-dimensional model analyzed in the Introduction. However, if $q \not\equiv 0$ in

[2]Note that integrability of $\|M\|$ implies $\mathbb{E} \log^+ \|M\| < \infty$. Here, $\log^+$ stands for the positive part of log. Thus, $\mathbb{E} \log \|M\|$ is well-defined.

(6), (8) implies convergence of (x_n) only in distribution (cf. [23]). Thus, we have to resort to a weaker notion of stability - stability in probability.

Condition (8) implies stability of $x_n \equiv 0$ with high probability. Theorem 2.2 is a stochastic counterpart of the result of Koçak and Palmer for deterministic maps [25, Theorem 4]. It follows immediately from the proof of the following lemma, which also yields the rate of convergence of (x_n) to the origin.

Lemma 2.3. *Let (x_n) denote a trajectory of (6) subject to (8). Then for any $0 < \varepsilon < \min\{1, \lambda/3\}$ there exist $\eta > 0$, $\delta_1 > 0$, and $\mu = \exp\{-\lambda + \varepsilon\} < 1$ such that*

$$|x_i| \le \eta \mu^i, \ i = 0, 1, 2, \dots \tag{9}$$

with probability at least $1 - \varepsilon$ provided $|x_0| \le \delta_1$.

Proof. Suppose $0 < \varepsilon < \min\{1, \lambda/3\}$ is arbitrary but fixed. Let $\lambda_k := \log \|M_k\|$ and note that

$$\frac{1}{n} \sum_{k=1}^{n} \lambda_k \xrightarrow{a.s.} -\lambda < 0 \text{ as } n \to \infty,$$

by the Strong Law of Large Numbers [10, Theorem 22.1]. Thus, there exists $n_0 > 1$ such that

$$\mathbb{P}\left(\bigcup_{n \ge n_0} \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} \lambda_k + \lambda \right| > \varepsilon \right\} \right) < \frac{\varepsilon}{2},$$

i.e., for $n \ge n_0$,

$$-\lambda - \varepsilon \le \frac{1}{n} \sum_{k=1}^{n} \lambda_k \le -\lambda + \varepsilon \tag{10}$$

holds on the set of probability at least $1 - \varepsilon/2$. In the remainder of the proof, we restrict to the realizations (M_k) for which (10) holds.

Using (10), for any $n \ge k \ge n_0 > 1$, we have

$$\prod_{j=k}^{n} \|M_j\| = \frac{\prod_{j=1}^{n} \|M_j\|}{\prod_{j=1}^{k-1} \|M_j\|} \le \exp\{n(-\lambda + \varepsilon) - (k-1)(-\lambda - \varepsilon)\} = \mu^{n-k+1} e^{2(k-1)\varepsilon}. \tag{11}$$

Similarly, for every $1 \le k < n_0$, we have

$$\begin{aligned} \prod_{j=k}^{n} \|M_j\| &= \left(\prod_{j=k}^{n_0 - 1} \|M_j\| \right) \left(\prod_{j=n_0}^{n} \|M_j\| \right) \le \left(\prod_{j=k}^{n_0-1} \|M_j\| \right) \mu^{n-n_0+1} e^{2(n_0-1)\varepsilon} \\ &\le \bar{M}_{n_0} \mu^{n-k+1} e^{2(k-1)\varepsilon}, \end{aligned} \tag{12}$$

where

$$\bar{M}_{n_0} = \max_{1 \le k \le n_0 - 1} \left\{ \mu^{k-n_0} e^{2(n_0-k)\varepsilon} \prod_{j=k}^{n_0-1} \|M_j\| \right\}.$$

Since M_j's are independent and integrable random variables, $\bar{M}_{n_0}$ is integrable as well. Thus, by Markov inequality, we have

$$\mathbb{P}\left(\bar{M}_{n_0} \ge M\right) \le \frac{\mathbb{E}\bar{M}_{n_0}}{M} \quad \forall M > 0.$$

Choosing $M = M(\varepsilon) > 0$ sufficiently large, we have

$$\mathbb{P}\left(\bar{M}_{n_0} \ge M\right) \le \frac{\varepsilon}{2}. \tag{13}$$

The combination of (11), (12), and (13) yields

$$\prod_{j=k}^{n} \|M_j\| \le C_2 \mu^{n-k+1} e^{2(k-1)\varepsilon}, \quad 1 \le k \le n, \tag{14}$$

holding with probability at least $1-\varepsilon$, where $C_2 = \max\{M, 1\}$ depends on ε but not on n or k.

We are now in a position to prove (9). To this end, recall δ defined in (7) and fix $0 < \eta \le \delta$. Choose $0 < \delta_1 \le \eta$ such that

$$C_2 \delta_1 \exp\{C_1 C_2 \eta/(1-\nu)\} \le \eta, \tag{15}$$

where $\nu := e^{-\lambda+3\varepsilon} < 1$. With these constants η and δ, we will show (9) by induction.

The claim in (9) obviously holds for $i = 0$. Let $p \ge 1$ and suppose that

$$|x_i| \le \eta \mu^i \tag{16}$$

holds for $i = 0, 1, \ldots, p-1$. We want to show that this entails

$$|x_p| \le \eta \mu^p.$$

Iterating (6), we have

$$x_p = \left(\prod_{k=0}^{p-1} M_{p-k}\right) x_0 + \sum_{j=1}^{p} \left(\prod_{k=0}^{p-j} M_{p-k}\right) q(x_{j-1}). \tag{17}$$

Using the triangle inequality, submultiplicativity of the matrix norm, and (7), from (17) we obtain

$$|x_p| \le \left(\prod_{k=0}^{p-1} \|M_{p-k}\|\right) |x_0| + C_1 \sum_{j=1}^{p} \left(\prod_{k=0}^{p-j} \|M_{p-k}\|\right) |x_{j-1}|^2.$$

Here, we also used the induction hypothesis (16), which implies that $|x_j| \le \delta$, $j = 0, 1, \ldots, p-1$ so that (7) is applicable. Using (14), we further derive

$$|x_p| \le C_2 \mu^p |x_0| + C_1 C_2 \sum_{j=1}^{p} \mu^{p-j+1} e^{2(j-1)\varepsilon} |x_{j-1}|^2.$$

Using the induction hypothesis (16), we continue

$$|x_p| \le C_2 \mu^p |x_0| + C_1 C_2 \eta \mu^p \sum_{j=1}^{p} e^{2(j-1)\varepsilon} |x_{j-1}|. \tag{18}$$

Next, we rewrite (18) in terms of

$$z_i = \mu^{-i} |x_i|, \; i = 0, 1, 2, \ldots, p, \tag{19}$$

to obtain

$$z_p \le C_2 z_0 + C_1 C_2 \eta \sum_{j=1}^{p} \nu^{j-1} z_{j-1}, \quad \nu = e^{-\lambda+3\varepsilon} < 1.$$

By the discrete Gronwall's inequality (see Lemma 2.4 below), we have

$$z_p \le C_2 z_0 \exp\left\{C_1 C_2 \eta \sum_{k=1}^{p} \nu^{k-1}\right\} \le C_2 \delta_1 \exp\{C_1 C_2 \eta/(1-\nu)\} \le \eta,$$

where we used (15) in the last inequality. Recalling the definition of z_p (19), we conclude that $|x_p| \le \eta \mu^p$. □

Lemma 2.4. *(cf. [25]) Let* $\{z_k\}_{k=0}^{\infty}$ *and* $\{\mu_k\}_{k=1}^{\infty}$ *be two nonnegative sequences such that*

$$z_k \le B + \sum_{j=1}^{k} \mu_j z_{j-1},\ k \in [p] := \{1, 2, \dots, p\}, \tag{20}$$

for some $p \in \mathbb{N}$. *Then for* $k \in [p]$

$$z_k \le B \exp\left\{\sum_{j=1}^{k} \mu_j\right\}.$$

3. **Stabilization.** In this section, we use the results of the previous section to study the problem of stabilization in $\mathbb{R}^d$. Specifically, we consider a twice continuously differentiable map $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $F(0) = 0$. Near the origin,

$$F(x) = Ax + q(x),$$

where $q(x) = O(|x|^2)$ and $A = \partial F(0)$ is the Jacobian matrix of F evaluated at the origin. We assume that $A = A(\epsilon)$ depends on a positive parameter ϵ and

$$\rho(A(\epsilon)) := \max\{|\lambda| :\ \lambda \text{ is an eigenvalue of } A(\epsilon)\} = 1 + \epsilon,\ \ 0 < \epsilon \ll 1. \tag{21}$$

Thus, the origin is an unstable equilibrium of F.

Suppose now $(B_n(\epsilon)), n \in \mathbb{N}$, are independent copies of random $d \times d$ matrix $B(\epsilon)$. We want to describe a class of mean zero stabilizing matrices $B(\epsilon)$ such that the origin becomes a stable equilibrium (with high probability) of the randomly perturbed map

$$x_{n+1} = (A(\epsilon) + B_n(\epsilon))x_n + q(x_n),\ n = 0, 1, 2, \dots. \tag{22}$$

Our construction of stabilizing perturbation for (22) relies on a class of random matrices, which we introduce next.

Definition 3.1. Let Γ denote a class of random matrices $G(\epsilon) = (g_{ij}(\epsilon)) \in \mathbb{R}^{d\times d}$ depending on a positive parameter ϵ that satisfy the following conditions:

$$\mathbb{E} g_{ij}(\epsilon) = 0, \tag{23}$$

$$\sigma_{ij}^2(\epsilon) := \mathbb{E} g_{ij}(\epsilon)^2 < \infty,\ (i,j) \in [d]\times[d],$$

$$\sigma(\epsilon) := (\sigma_{11}(\epsilon), \sigma_{22}(\epsilon), \dots, \sigma_{nn}(\epsilon)),$$

$$\lim_{\epsilon\to 0} |\sigma(\epsilon)| = 0, \tag{24}$$

$$\lim_{\epsilon\to 0} \frac{\sigma_{ij}(\epsilon)}{\sigma_{ii}(\epsilon)^2} = 0,\ i \ne j, \tag{25}$$

$$\left(\mathbb{E}|g_{ij}(\epsilon)|^3\right)^{1/3} \le K\sigma_{ij}(\epsilon), \quad 1 \le i, j \le d, \tag{26}$$

for some $K > 0$ independent of ϵ.

Finally,

$$1 < \limsup_{\epsilon\searrow 0} \frac{|\sigma(\epsilon)|^2}{2\epsilon} < \infty. \tag{27}$$

By Γ_{norm} we denote a subclass of normal matrices from Γ.

Remark 3. As will be clear from the examples in Section 4, conditions (23)-(26) are easy to fulfill. In particular, one may take $g_{ij}(\epsilon) = \epsilon\xi_{ij}$, where ξ_{ij} is a mean–zero random variable with the finite third moment, $1 \le i, j \le d$. Then, for each such i and j, $\left(\mathbb{E}|\xi_{ij}|^3\right)^{1/3} \le K_{ij} \left(\mathbb{E}\xi_{ij}^2\right)^{1/2}$ for some constant K_{ij} and (26) holds with

$K = \max_{1\le i,j\le d} K_{ij}$ for all $\epsilon > 0$. In particular, if (ξ_{ij}) are (arbitrarily dependent) standard normal RVs then (26) holds with $K = 2\sqrt{2/\pi}$.

For matrices in Γ_{norm}, we have the following lemma.

Lemma 3.2. *Let $G \in \Gamma_{norm}$. Then there exist $\epsilon_0 > 0$, $0 < \epsilon_1 \le \epsilon_0$, and $L > 0$ such that*

$$1 + \epsilon_0 \le \frac{|\sigma(\epsilon)|^2}{2\epsilon} \le L \tag{28}$$

and

$$\mathbb{E} \log \|I + G(\epsilon)\| \le \frac{-|\sigma(\epsilon)|^2}{2} \left(1 - \frac{\epsilon}{6L}\right). \tag{29}$$

for all $0 < \epsilon < \epsilon_1$. Here and below, $\|\cdot\|$ stands for the operator norm of a matrix induced by the Euclidean norm $|\cdot|$.

We postpone the proof of the technical Lemma 3.2 until the end of this section. Now we return to the stabilization problem for $x \mapsto F(x)$.

Suppose $G \in \Gamma_{norm}$ has been chosen.[3] Let $0 < \epsilon < \epsilon_1$ be arbitrary but fixed and denote $\kappa = \epsilon \cdot \epsilon_0/3$, where ϵ_0, ϵ_1 are given in Lemma 3.2. By changing coordinates, one can transform A in (22) into the Jordan normal form:

$$A^0 + U, \tag{30}$$

where A^0 is the block-diagonal matrix

$$A^0 = \operatorname{diag}(A_1, A_2, \dots, A_k), \quad k \in [d], \tag{31}$$

and U is the upper triagular matrix subject to[4]

$$\|U\| < \kappa. \tag{32}$$

Block A_i, $i \in [k]$, is (λ_i) if the corresponding eigenvalue of A is real, or

$$\begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix},$$

otherwise. Therefore,

$$\|A^0\| = \rho(A) = 1 + \epsilon. \tag{33}$$

The upper-triangular matrix U is nonzero only if A has multiple eigenvalues. In this case, it has the following form

$$U = \begin{pmatrix} O_{11} & * & * & \dots & * \\ & \dots & * & \dots & * \\ & \dots & & \dots & \\ 0 & & \dots & & O_{kk} \end{pmatrix},$$

where O_{ii} is a $d_i \times d_i$ zero block whose dimension coincides with that of A_i for each $i \in [k]$.

Thus, for the remainder of this section, we assume that matrix A in (22) has the following form

$$A = A^0 + U, \tag{34}$$

where the block-diagonal matrix A^0 and the upper diagonal matrix U are subject to (31), (33), and (32), respectively.

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[3]When we speak of a random matrix G, we mean the probability distribution of G.

[4]Once A is transformed to a Jordan normal form, the entries of the block upper triangular matrix U can be made arbitrarily small via similarity transformation $D_t^{-1}UD$, where $D_t = \operatorname{diag}\left(tI_1, t^2 I_2, \dots, t^k I_k\right)$; I_i, $i \in [k]$, are $d_i \times d_i$ identity matrices, and $t > 0$ is sufficiently small.

We are now in a postion to state the main result of this section.

Theorem 3.3. *Suppose $A(\epsilon)$ is as in (34) subject to (32) and (33); $G \in \Gamma_{norm}$ and $B(\epsilon) = A(\epsilon)G(\epsilon)$. Then for every $0 < \epsilon < \epsilon_1$ the equilibrium at the origin of (22) is stable in probability.*

Corollary 1. *If every eigenvalue of $A(\epsilon)$ is either real or pure imaginary, the statement of Theorem 3.3 remains valid for $G \in \Gamma$.*

Remark 4. As can be easily seen from the proof of Lemma 3.2, for stabilization of the unstable equilibrium in (22) it is sufficient to take a diagonal matrix $G = \mathrm{diag}(g, g, \dots, g)$, where mean zero RV g meets the conditions on the three first moments (24) and (26) as well as (27). In particular, one can take $g = a\xi$, where ξ is a standard normal RV and $a = a(\epsilon) \to 0$, but $\lim_{\epsilon\to 0}(a/\sqrt{2\epsilon}) > 1$. Thus, in practice, it suffices to use a single RV to stabilize a weakly unstable equilibrium in $\mathbb{R}^d$.

Proof. (Theorem 3.3) By Theorem 2.2, for stabilization in (22) it is sufficient to show that the condition

$$\mathbb{E}\log\|A(\epsilon) + B(\epsilon)\| < 0 \tag{35}$$

holds for sufficiently small $\epsilon > 0$. Below, we show that (27) implies (35) for small $\epsilon > 0$.

By the submultiplicativity of the matrix norm and $B = AG$, we have

$$\log\|A + B\| = \log\|A(I + G)\| \le \log\|A\| + \log\|I + G\|. \tag{36}$$

Using (34) and (33), from (36) we further obtain

$$\log\|A + B\| \le \log(1 + \epsilon + \kappa) + \log\|I + G\| \le \epsilon + \kappa + \log\|I + G\|. \tag{37}$$

Recalling $\kappa = \epsilon\epsilon_0/3$ and using (28) and (29), we conclude

$$\log\|A + B\| \le \epsilon + \frac{\epsilon\epsilon_0}{3} - (1 + \epsilon_0)\epsilon + \frac{2L\epsilon^2}{6L} \le \epsilon(-\epsilon_0 + \epsilon/3 + \epsilon/3) < 0.$$

This completes the proof. □

Proof. (Lemma 3.2) By Gershgorin Theorem (cf. [22]),

$$\|I + G\| = \rho(I + G) \le \max_{i\in[d]}\left(|1 + g_{ii}| + \sum_{j\neq i}|g_{ij}|\right), \tag{38}$$

where we used normality of G in the first equality. By the monotonicity of logarithm,

$$\log\|I + G\| \le \max_i \log\left(|1 + g_{ii}| + \sum_{j\neq i}|g_{ij}|\right) \le \sum_{i=1}^{d}\log\left(|1 + g_{ii}| + \sum_{j\neq i}|g_{ij}|\right).$$

Taking expectations on both sides, we get

$$\mathbb{E}\log\|I + G\| \le \sum_{i=1}^{d}\mathbb{E}\log\left(|1 + g_{ii}| + \sum_{j\neq i}|g_{ij}|\right). \tag{39}$$

For each $i \in [d]$,

$$\mathbb{E}\log\left(|1+g_{ii}|+\sum_{j\neq i}|g_{ij}|\right) \le \mathbb{E}\log\left(1+g_{ii}+\sum_{j\neq i}|g_{ij}|\right) I_{|g_{ii}|<1} + \mathbb{E}\log\left(1+|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right) I_{|g_{ii}|\ge 1}. \tag{40}$$

By expanding the logarithm in the first term and using the fact that $\mathbb{E}g_{ii}=0$ we get

$$\begin{aligned}
&\mathbb{E}\left(g_{ii}+\sum_{j\neq i}|g_{ij}| - \frac{(g_{ii}+\sum_{j\neq i}|g_{ij}|)^2}{2} + O\left(\left(|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right)^3\right)\right) I_{|g_{ii}|<1} \\
&= \mathbb{E}\left(g_{ii}+\sum_{j\neq i}|g_{ij}| - \frac{1}{2}\left(g_{ii}+\sum_{j\neq i}|g_{ij}|\right)^2 + O\left(\left(|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right)^3\right)\right) \\
&-\mathbb{E}\left(g_{ii}+\sum_{j\neq i}|g_{ij}| - \frac{1}{2}\left(g_{ii}+\sum_{j\neq i}|g_{ij}|\right)^2 + O\left(\left(|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right)^3\right)\right) I_{|g_{ii}|\ge 1} \\
&= \sum_{j\neq i}\mathbb{E}|g_{ij}| - \frac{1}{2}\mathbb{E}g_{ii}^2 - \sum_{j\neq i}\mathbb{E}g_{ii}|g_{ij}| - \frac{1}{2}\mathbb{E}\left(\sum_{j\neq i}|g_{ij}|\right)^2 \\
&+\mathbb{E}O\left((|g_{ii}|+\sum_{j\neq i}|g_{ij}|)^3\right) + O\left(\sum_{m=1}^{3}\mathbb{E}\left(|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right)^m I_{|g_{ii}|\ge 1}\right).
\end{aligned} \tag{41}$$

We want to estimate the right hand side of (40). Note that since $\log(1+x)\le x$ for $x\ge 0$, the $O((\dots)^m)$ term in (41) (with $m=1$) majorizes the second term on the right hand side of the inequality in (40). Therefore, an upper bound on (41) yields an upper bound on the right hand side of (40).

We estimate the terms above as follows

$$\begin{aligned}
\sum_{j\neq i}\mathbb{E}|g_{ij}| &= \sum_{j\neq i} O(\sigma_{ij}) = o(\sigma_{ii}^2), \quad \text{(by (25))} \\
\left|\sum_{j\neq i}\mathbb{E}g_{ii}|g_{ij}|\right| &\le \sigma_{ii}\sum_{j\neq i}\sigma_{ij} = o(\sigma_{ii}^2), \quad \text{(by the Cauchy-Schwarz inequality and (25))} \\
\mathbb{E}\left(\sum_{j\neq i}|g_{ij}|\right)^2 &= \sum_{j\neq i} O(\sigma_{ij}^2) = o(\sigma_{ii}^2), \quad \text{(by (25))} \\
\mathbb{E}\left(|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right)^3 &= O\left(\mathbb{E}|g_{ii}|^3\right) + \sum_{j\neq i} O\left(\mathbb{E}|g_{ij}|^3\right)
\end{aligned}$$

$$\mathbb{E}\left(|g_{ii}|+\sum_{j\neq i}|g_{ij}|\right)^m I_{|g_{ii}|\geq 1} \quad = \quad O\left(\mathbb{E}|g_{ii}|^m I_{|g_{ii}|\geq 1}\right)+\sum_{j\neq i} O\left(\mathbb{E}|g_{ij}|^m\right).$$

For $m = 1, 2$ and $j \neq i$, $\mathbb{E}|g_{ij}|^m = o(\sigma_{ij}^2)$ as verified above. Further, for $1 \leq m \leq 3$

$$\mathbb{E}|g_{ii}|^m I_{|g_{ii}|\geq 1} \leq \mathbb{E}|g_{ii}|^3 I_{|g_{ii}|\geq 1} \leq \mathbb{E}|g_{ii}|^3.$$

Hence, by (24), (25), and (26) for all $1 \leq i, j \leq d$,

$$\mathbb{E}|g_{ij}|^3 = O\left(\sigma_{ij}^3\right) = o\left(\sigma_{ij}^2\right) = o\left(\sigma_{ii}^2\right).$$

Plugging all of this into (39), we obtain that

$$\mathbb{E}\log\|I+G\| \leq -\frac{1}{2}\sum_{i=1}^{d}\left(\sigma_{ii}^2+o(\sigma_{ii}^2)\right) = \frac{-|\sigma(\epsilon)|^2}{2}+o(|\sigma|^2). \tag{42}$$

Let $L < \infty$ be any number strictly larger than $\limsup(|\sigma(\epsilon)|^2/(2\epsilon))$. By (27) there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$

$$1+\epsilon_0 \leq \frac{|\sigma(\epsilon)|^2}{2\epsilon} \leq L$$

Decreasing ϵ if necessary we may assume that the error term $o(|\sigma(\epsilon)|^2)$ in (42) satisfies

$$\frac{o(|\sigma(\epsilon)|^2)}{|\sigma(\epsilon)|^2} \leq \frac{\epsilon_0}{6L} \quad 0 < \epsilon < \epsilon_1 \tag{43}$$

for some $0 < \epsilon_1 < \epsilon_0$.

The combination of (42) and (43) yields (29). □

Proof. (Corollary 1) Recall the maximum row sum matrix norm of $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ [22]:

$$\|A\|_\infty = \max_{i\in[d]}\sum_{j=1}^{d}|a_{ij}|.$$

Using this norm, (38) can be rewritten as

$$\|I+G\|_\infty = \max_{i\in[d]}\left(|1+g_{ii}|+\sum_{j\neq i}|g_{ij}|\right). \tag{44}$$

Note that in contrast to (38), we do not need to assume that G is normal in (44).

If all eigenvalues of A are either real or pure imaginary then we can replace (33) with

$$\|A^0\|_\infty = \rho(A) = 1+\epsilon.$$

Thus, the proof of the corollary follows by replacing the operator norm by the maximum row sum norm everywhere in the proofs of Lemma 3.2 and Theorem 3.3. □

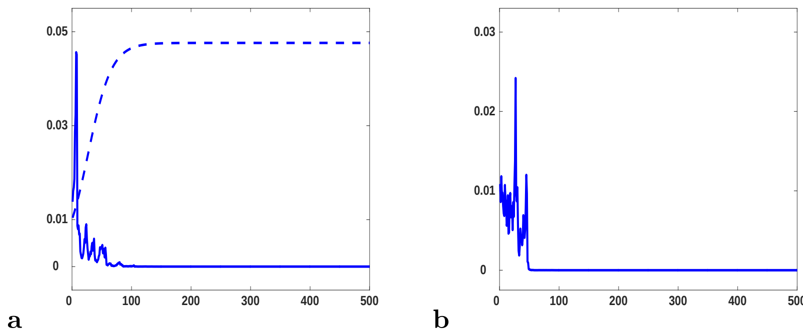

FIGURE 1. **a**) Trajectories of the logistic map $x \mapsto f(x)$ (plotted in dashed line) and that of the randomly perturbed system (1). The former approaches the stable equilibrium of the deterministic system $\bar{x}_2$, while the latter returns to and remains in a small neighborhood of the origin. **b**) A trajectory of the two-dimesional system (46) stays near the orgin after brief transients. All trajectories of the underlying deterministic system $x \mapsto Ax + q(x)$ starting off the $x^{(2)}$-axis tend to infinity. In numerical simulations shown in **a** and **b**, the following parameter values were used: $\epsilon = 0.05$ and $\rho = 3$.

4. **Examples.** In this section, we illustrate our analysis of stabilization with two numerical examples.

First, we consider the following scalar difference equation:

$$x_{n+1} = \left((1+\epsilon)(1-x_n) + \sqrt{\epsilon\rho}\xi_n\right) x_n, \quad n \in \mathbb{N}, \tag{45}$$

where $0 < \epsilon \ll 1, \rho > 0$, and (ξ_n) are IID standard normal RVs. Equation (45) can be viewed as a randomly perturbed logistic map

$$x \mapsto f(x) := (1+\epsilon)x(1-x).$$

For $\epsilon > 0$, the latter has two fixed points:

$$\bar{x}_1 = 0 \quad \text{and} \quad \bar{x}_2 = \frac{\epsilon}{1+\epsilon}.$$

The former is unstable and the latter is stable fixed point of f, if $\epsilon > 0$ is sufficiently small. On the other hand, by Theorem 3.3, $\bar{x}_1$ is a stable in probability equilibrium of the randomly perturbed system (45) provided $0 < \epsilon \ll 1$ and $\rho > 2$. Fig. 1**a** illustrates the difference in qualitative behavior of the deterministic and randomly perturbed systems. It shows two trajectories: one of the deterministic system $x \mapsto f(x)$ and the other of the random system (45), starting from the same initial condition near the origin. The former monotonically approaches $\bar{x}_2$, while the latter after a brief excursion towards $\bar{x}_2$ returns back to a small neighborhood of the origin and remains there.

Our second example illustrates stabilization in the multidimensional setting. Consider a two-dimensional system

$$x_{n+1} = A(I + G_n)x_n + q(x_n), \quad x_n = (x_n^{(1)}, x_n^{(2)})^{\mathsf{T}} \in \mathbb{R}^2,\ n \in \mathbb{N}, \tag{46}$$

where

$$A = \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1/2 \end{pmatrix} \quad q(x) = \begin{pmatrix} 0 \\ {x^{(1)}}^2 \end{pmatrix}, \quad G_n = \sqrt{\epsilon\rho} \begin{pmatrix} \xi_n^{(11)} & \epsilon\xi_n^{(12)} \\ \epsilon\xi_n^{(21)} & \xi_n^{(22)} \end{pmatrix},$$

and $(\xi_n^{(ij)})$, $(i,j) \in [2]^2, n \in \mathbb{N}$, are IID standard normal RVs. The deterministic map $x \mapsto Ax + q(x)$ has an unstable equilibrium at the origin. All trajectories starting off $x^{(2)}$-axis tend to infinity: $|x_n| \to \infty$ as $n \to \infty$. In contrast, the trajectories of the randomly perturbed system (46) starting near the origin with high probability remain near the origin for large times provided $0 < \epsilon \ll 1$ and $\rho > 2$ (cf. Theorem 3.3). The behavior of a representative trajectory of the random system (46) is shown in Fig. 1**b**.

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E-mail address: phitczen@math.drexel.edu
E-mail address: medvedev@drexel.edu