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Inverse Born Series

Abstract: We present a survey of recent results on the inverse Born series. The convergence and stability of the method are characterized in Banach spaces. Applications to inverse scattering problems in various physical settings are described.

1 Introduction

Inverse scattering problems are of fundamental importance in nearly every branch of physics. They also arise in numerous applied fields ranging from biomedical imaging to seismology. Such problems can be formulated in a variety of settings, depending upon the nature of the probing wave field and the length scales of interest. Regardless of such considerations, the fundamental theoretical questions relate to the uniqueness, stability and reconstruction of the solution to the problem. By uniqueness we mean the injectivity of the forward map from the scattering potential to the scattering data. Stability refers to continuity of the inverse map from scattering data to the potential. We note that inverse scattering problems are typically ill-posed, which means that the inverse map must be suitably regularized to achieve stable inversion.

There are a number of approaches to the problem of recovering the scattering potential. See [1–3] for a comprehensive overview of inverse scattering theory. Direct reconstruction methods provide an analytic solution to the inverse problems, principally in one-dimension although higher-dimensional methods are also known. Optimization methods iteratively minimize the distance between the scattering data and the solution to the forward problem, viewed as a functional of the scattering potential. Although such techniques are extremely flexible, the presence of local minima and the computational cost of evaluating the forward map limit their practical utility. Finally, linear sampling and related qualitative methods can be used to recover the support of the scattering potential for obstacle scattering and the inverse medium problem [4].

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The inverse Born series (IBS) is a direct reconstruction method that was initially developed to study the quantum mechanical inverse backscattering problem in one dimension [5, 6] and later extended to higher dimensions [7–11]. The authors analyzed the convergence, stability and error of the IBS [12]. They have also applied the IBS to various inverse scattering problems, including those of optical tomography, electrical impedance tomography, and acoustic and electromagnetic imaging [13–19]. The IBS has also been applied to discrete inverse problems on graphs, independent of the continuous setting in which it was initially proposed [20]. Finally, we note that the inverse of the Bremmer series can be investigated using a related approach [21].

It is important to note that the principal computational advantage of the IBS is that it does not make use of a partial differential equation solver. Instead, the IBS obtains the solution to the inverse problem as an explicitly computable functional of the scattering data. This functional can be expressed in terms of the Green's function for the underlying partial differential equation, whose decay governs the convergence and stability of the method.

In this chapter we present a survey of recent results on the inverse Born series. In section 2, the convergence and stability of the IBS is analyzed in Banach spaces. The results are then applied to a wide range of inverse problems. These include the inverse scattering problem for diffuse waves in section 3, the Calderon problem of electrical impedance tomography in section 4, the inverse radiative transport problem in section 5, and the inverse scattering problem for electromagnetic waves in section 6. Finally, in section 7, we consider the inverse problem for graph diffusion.

We use the following notational conventions throughout this chapter. For $n \geq 2$, Ω denotes a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. If X is a Banach space, X^j indicates the j-fold tensor product $X^j = X \otimes \cdots \otimes X$ equipped with the projective norm [22] for $j > 1$. We note that X^j is generally not a Banach space.

2 Analysis of the inverse Born series

In this section we formulate the IBS in a Banach space setting. This formulation will then be applied to various inverse scattering problems later in the chapter. The presentation closely follows [12], where the case of L^p spaces was considered. The extension to Banach spaces was described in [23]. Let X and Y be Banach spaces. We consider the power series

$$\phi = K_1\eta + K_2\eta \otimes \eta + K_3\eta \otimes \eta \otimes \eta + \cdots \ , \tag{1}$$

where $K_j : X^j \to Y$. The forward problem is to evaluate the map $\mathscr{F} : \eta \mapsto \phi$ defined by (1). We will refer to K_j as forward operators and (1) is called the Born series.

The inverse problem is to determine η assuming that ϕ is known. That is, we seek to construct a map $\mathscr{I} : \phi \mapsto \eta$ which is, in some sense, the inverse of $\mathscr{F}$. Towards this end, we make the ansatz that η may be expressed as a series in tensor powers of ϕ of the form

$$\eta = \mathcal{K}_1\phi + \mathcal{K}_2\phi \otimes \phi + \mathcal{K}_3\phi \otimes \phi \otimes \phi + \cdots \ . \tag{2}$$

Here the inverse operators $\mathcal{K}_j : Y^j \to X$ are to be determined. By substituting (1) into (2) and equating terms with the same tensor power of η, we find that the operators $\mathcal{K}_j$ are given by

$$\mathcal{K}_1 K_1 = I \ , \tag{3}$$

$$\mathcal{K}_2 = -\mathcal{K}_1 K_2 \mathcal{K}_1 \otimes \mathcal{K}_1 \ , \tag{4}$$

$$\mathcal{K}_3 = -\left(\mathcal{K}_2 K_1 \otimes K_2 + \mathcal{K}_2 K_2 \otimes K_1 + \mathcal{K}_1 K_3\right) \mathcal{K}_1 \otimes \mathcal{K}_1 \otimes \mathcal{K}_1 \ , \tag{5}$$

$$\mathcal{K}_j = -\left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m}\right) \mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_1 \ . \tag{6}$$

We will refer to (2) as the inverse Born series (IBS). Here we note several of its properties. (i) The operator K_1 generally does not have a bounded inverse. Thus $\mathcal{K}_1$ is taken to be the regularized pseudoinverse of K_1, which is defined as follows. Consider the Tikhonov functional T which is of the form

$$T(\eta) = \|K_1\eta - \phi\|_Y + \lambda F(\eta) \ , \tag{7}$$

where F is a convex penalty function and $\lambda > 0$ is a regularization parameter [24]. The minimizer of T is denoted $\eta^\dagger$ and is referred to as the regularized pseudoinverse solution of $K_1\eta = \phi$. The operator $\mathcal{K}_1$ is defined as the map $\mathcal{K}_1 : \phi \mapsto \eta^\dagger$. Here we take $\eta \in \tilde{X}$, where $\tilde{X}$ is a uniformly convex subspace of X. If K_1 is bounded, it follows that $\eta^\dagger$ exists and is unique [24]. (ii) The coefficients in the inverse series have a recursive structure. The operator $\mathcal{K}_j$ is determined by the coefficients of the Born series $K_1, K_2, \ldots, K_j$. (iii) Inversion of only the linear term in the Born series is required to compute the IBS to all orders. Thus a nonlinear inverse problem that is often ill-posed is replaced by an ill-posed linear inverse problem plus a well-posed nonlinear problem, namely the computation of the higher order terms in the series.

We now proceed to analyze the convergence and stability of the IBS. Throughout, we assume that the operator K_j is bounded with

$$\|K_j\| \le \nu\mu^{j-1} \ , \tag{8}$$

for suitable constants μ and ν. We immediately see that the Born series (1) converges in norm provided that $\|\eta\|_X < 1/\mu$. The following lemma provides an estimate on the norm of the operator $\mathcal{K}_j$.

Lemma 2.1. *Let $\|\mathcal{K}_1\| < 1/(\mu+\nu)$. Then the operator $\mathcal{K}_j : X^j \to Y$ defined by (6) is bounded and*

$$\|\mathcal{K}_j\| \leq C(\mu+\nu)^j \|\mathcal{K}_1\| \ , \tag{9}$$

where C is independent of j. Moreover, for all $\phi \in Y$

$$\|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_X \leq C(\mu+\nu)^j \|\mathcal{K}_1 \phi\|_X^j \ . \tag{10}$$

Proof. We first prove (9). Using (6) we find that

$$\begin{aligned} \|\mathcal{K}_j\| &\leq \sum_{m=1}^{j-1} \sum_{i_1+\cdots+i_m=j} \|\mathcal{K}_m\| \|K_{i_1}\| \cdots \|K_{i_m}\| \|\mathcal{K}_1\|^j \\ &\leq \|\mathcal{K}_1\|^j \sum_{m=1}^{j-1} \sum_{i_1+\cdots+i_m=j} \|\mathcal{K}_m\| \nu\mu^{i_1-1} \cdots \nu\mu^{i_m-1} \ , \end{aligned} \tag{11}$$

where we have used (8) to obtain the second inequality. Next, we define $\Pi(j,m)$ to be the number of ordered partitions of the integer j into m parts. It can be seen that

$$\Pi(j,m) = \binom{j-1}{m-1} \ , \tag{12}$$

$$\sum_{m=1}^{j-1} \Pi(j,m) = 2^{j-1} - 1 \ . \tag{13}$$

It follows that

$$\begin{aligned} \|\mathcal{K}_j\| &\leq \|\mathcal{K}_1\|^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\| \Pi(j,m) \nu^m \mu^{j-m} \\ &\leq \|\mathcal{K}_1\|^j \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\| \right) \left(\sum_{m=1}^{j-1} \Pi(j,m) \nu^m \mu^{j-m} \right) \\ &\leq \nu \|\mathcal{K}_1\|^j \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\| \right) \left(\sum_{m=0}^{j-1} \binom{j-1}{m} \nu^m \mu^{j-1-m} \right) \\ &= \nu \|\mathcal{K}_1\|^j (\mu+\nu)^{j-1} \sum_{m=1}^{j-1} \|\mathcal{K}_m\| \ . \end{aligned} \tag{14}$$

Thus $\|\mathcal{K}_j\|$ is a bounded operator and

$$\|\mathcal{K}_j\| \leq (\mu+\nu)^j \|\mathcal{K}_1\|^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\| \ . \tag{15}$$

The above estimate for $\|\mathcal{K}_j\|$ has a recursive structure. It can be seen that

$$\|\mathcal{K}_j\| \leq C_j [(\mu+\nu)\|\mathcal{K}_1\|]^j \|\mathcal{K}_1\| \ , \tag{16}$$

where, for $j \geq 2$, C_j obeys the recursion relation

$$C_{j+1} = C_j + [(\mu+\nu)\|\mathcal{K}_1\|]^j C_j \ , \quad C_2 = 1 \ . \tag{17}$$

Evidently

$$C_j = \prod_{m=2}^{j-1} \left(1 + [(\mu+\nu)\|\mathcal{K}_1\|]^m\right) \ . \tag{18}$$

We see that C_j is bounded for all j since

$$\begin{aligned} \ln C_j &\leq \sum_{m=1}^{j-1} \ln\left(1 + [(\mu+\nu)\|\mathcal{K}_1\|]^m\right) \\ &\leq \sum_{m=1}^{j-1} [(\mu+\nu)\|\mathcal{K}_1\|]^m \\ &\leq \frac{1}{1-(\mu+\nu)\|\mathcal{K}_1\|} \ , \end{aligned} \tag{19}$$

where the final inequality follows if $(\mu+\nu)\|\mathcal{K}_1\| < 1$.

To prove (10), we note that the same reasoning as above leads to the inequality

$$\|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_X \leq (\mu+\nu)^j \|\mathcal{K}_1 \phi\|_X^j \sum_{m=1}^{j-1} \|\mathcal{K}_m\|_p \ . \tag{20}$$

Making use of (20) leads to

$$\|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_X \leq C(\mu+\nu)^j \frac{\|\mathcal{K}_1\|}{1-(\mu+\nu)\|\mathcal{K}_1\|} \|\mathcal{K}_1 \phi\|_X^j \ , \tag{21}$$

which completes the proof. □

We now establish a basic result that governs the convergence and approximation error of the IBS.

Theorem 2.1. *Suppose that* $\|\mathcal{K}_1\| < 1/(\mu+\nu)$, $\|\mathcal{K}_1\phi\|_X < 1/(\mu+\nu)$. *Let* $\mathcal{M} = \max\left(\|\eta\|_X, \|\mathcal{K}_1 K_1 \eta\|_X\right)$ *and assume that* $\mathcal{M} < 1/(\mu+\nu)$. *Then the inverse Born series (2) converges in norm and the following error estimate holds:*

$$\left\|\eta - \sum_{j=1}^{N} \mathcal{K}_j \phi \otimes \cdots \otimes \phi\right\|_X \leq C\|(I - \mathcal{K}_1 K_1)\eta\|_X + \tilde{C}\frac{[(\mu+\nu)\|\mathcal{K}_1\phi\|_X]^N}{1-(\mu+\nu)\|\mathcal{K}_1\phi\|_X} ,$$

where C *and* $\tilde{C}$ *are independent of* N *and* ϕ.

Proof. The hypotheses imply that the series

$$\tilde{\eta} = \sum_j \mathcal{K}_j \phi \otimes \cdots \otimes \phi \tag{22}$$

converges. Here $\mathcal{K}_1$ is regularized and we denote by $\tilde{\eta}$ the sum of the corresponding IBS. The Born series (1) also converges by hypothesis, so we can substitute it into (22) to obtain

$$\tilde{\eta} = \sum_j \tilde{\mathcal{K}}_j \eta \otimes \cdots \otimes \eta , \tag{23}$$

where

$$\tilde{\mathcal{K}}_1 = \mathcal{K}_1 K_1 , \tag{24}$$

and

$$\tilde{\mathcal{K}}_j = \left(\sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m}\right) + \mathcal{K}_j K_1 \otimes \cdots \otimes K_1 , \tag{25}$$

for $j \geq 2$. From (6) it follows that

$$\tilde{\mathcal{K}}_j = \sum_{m=1}^{j-1} \mathcal{K}_m \sum_{i_1+\cdots+i_m=j} K_{i_1} \otimes \cdots \otimes K_{i_m} \left(I - \mathcal{K}_1 K_1 \otimes \cdots \otimes \mathcal{K}_1 K_1\right) . \tag{26}$$

We thus obtain the estimate

$$\|\eta-\tilde{\eta}\|_X \leq \sum_j \sum_{m=1}^{j-1} \sum_{i_1+\cdots+i_m=j} \|\mathcal{K}_m\|\|K_{i_1}\|\cdots\|K_{i_m}\|\|\eta\otimes\cdots\otimes\eta - \mathcal{K}_1 K_1 \eta \otimes\cdots\otimes \mathcal{K}_1 K_1 \eta\|_{X^j} . \tag{27}$$

Next, we put

$$\psi = \eta - \mathcal{K}_1 K_1 \eta \tag{28}$$

and make use of the identity

$$\begin{aligned}\eta_1 \otimes \cdots \otimes \eta_1 - \eta_2 \otimes \cdots \otimes \eta_2 &= \zeta \otimes \eta_2 \otimes \cdots \otimes \eta_2 + \eta_1 \otimes \zeta \otimes \eta_2 \otimes \cdots \otimes \eta_2 \\ &+ \cdots + \eta_1 \otimes \eta_1 \otimes \cdots \otimes \zeta \otimes \eta_2 + \eta_1 \otimes \eta_1 \otimes \cdots \otimes \eta_1 \otimes \zeta ,\end{aligned} \tag{29}$$

where $\zeta = \eta_1 - \eta_2$ to obtain

$$\|\eta \otimes \cdots \otimes \eta - \mathcal{K}_1 K_1 \eta \otimes \cdots \otimes \mathcal{K}_1 K_1 \eta\|_{X^j} \leq j\mathcal{M}^{j-1}\|\psi\|_X \ . \tag{30}$$

We then have

$$\begin{aligned} \|\eta - \tilde{\eta}\|_X &\leq \sum_j \sum_{m=1}^{j-1} \sum_{i_1+\cdots+i_m=j} \|\mathcal{K}_m\|\|K_{i_1}\|\cdots\|K_{i_m}\| j\mathcal{M}^{j-1}\|\psi\|_X \\ &\leq \sum_j \sum_{m=1}^{j-1} j\mathcal{M}^{j-1}\|\mathcal{K}_m\|\Pi(j,m)\nu^m\mu^{j-m}\|\psi\|_X \ , \end{aligned} \tag{31}$$

where we have used (8). Making use of (12), we have

$$\begin{aligned} \|\eta - \tilde{\eta}\|_X &\leq \nu \sum_j \|\psi\|_X j\mathcal{M}^{j-1} \left(\sum_{m=1}^{j-1} \|\mathcal{K}_m\|\right) \left(\sum_{m=0}^{j-1} \binom{j-1}{m} \nu^m \mu^{j-1-m}\right) \\ &\leq \|\psi\|_X \sum_j \sum_{m=1}^{j-1} j\mathcal{M}^{j-1}(\mu+\nu)^j \|\mathcal{K}_m\| \ . \end{aligned} \tag{32}$$

We now apply Lemma 2.1 to obtain

$$\|\eta - \tilde{\eta}\|_X \leq C\|\psi\|_X \sum_j \sum_{m=1}^{j-1} j\mathcal{M}^{j-1}(\mu+\nu)^{m+j}\|\mathcal{K}_1\|^m \ , \tag{33}$$

since the constant C from the lemma is independent of j. Performing the sum over m, we have

$$\|\eta - \tilde{\eta}\|_X \leq C\|\psi\|_X \sum_j j\mathcal{M}^{j-1}(\mu+\nu)^j \frac{(\mu+\nu)^j\|\mathcal{K}_1\|^j - 1}{(\mu+\nu)\|\mathcal{K}_1\| - 1} \ , \tag{34}$$

which is bounded since $\mathcal{M}(\mu+\nu) < 1$ and $(\mu+\nu)\|\mathcal{K}_1\| < 1$. Eq. (34) thus becomes

$$\|\eta - \tilde{\eta}\|_X \leq C\|(I - \mathcal{K}_1 K_1)\eta\|_X \ , \tag{35}$$

where C is a new constant which depends on $\mu, \nu, \mathcal{M}$ and $\|\mathcal{K}_1\|$. Finally, using the triangle inequality and (10), we can account for the error which arises from cutting off the tail of the series. We thus obtain

$$\begin{aligned} &\left\|\eta - \sum_{j=1}^N \mathcal{K}_j \phi \otimes \cdots \otimes \phi\right\|_X \\ &\leq \left\|\eta - \sum_j \tilde{\mathcal{K}}_j \eta \otimes \cdots \otimes \eta\right\|_X + \sum_{j=N+1}^{\infty} \|\mathcal{K}_j \phi \otimes \cdots \otimes \phi\|_X \\ &\leq C\|(I - \mathcal{K}_1 K_1)\eta\|_X + \tilde{C} \frac{((\mu+\nu)\|\mathcal{K}_1\phi\|_X)^{N+1}}{1 - (\mu+\nu)\|\mathcal{K}_1\phi\|_X} \ . \end{aligned} \tag{36}$$

□

We make two important remarks concerning Theorem 2.1. (i) The hypothesis $\|K_1\| \leq \nu$, which corresponds to (8) when $j = 1$, is not generally inconsistent with the condition $\|\mathcal{K}_1\| \leq 1/(\mu+\nu)$. In particular, for the case of Tikhonov regularization, $\|\mathcal{K}_1\|$ can be chosen to be arbitrarily small by proper choice of the regularization parameter. (ii) We note that due to regularization of $\|\mathcal{K}_1\|$, the IBS does not converge to η. That is, $\|\mathcal{K}_1\|$ is not the true inverse of K_1. However, if it is known a priori that η belongs to a particular finite-dimensional subspace of X, $\mathcal{K}_1$ can be chosen to be a true inverse on this subspace. Then, provided the hypotheses of Theorem 2.1 hold, the IBS will recover η exactly.

The next result characterizes the stability of the limit of the IBS under perturbations in the data ϕ.

Theorem 2.2. *Let $\|\mathcal{K}_1\| < 1/(\mu+\nu)$ and let ϕ_1 and ϕ_2 be data for which $M\|\mathcal{K}_1\| < 1/(\mu+\nu)$, where $M = \max(\|\phi_1\|_Y, \|\phi_2\|_Y)$. Let η_1 and η_2 denote the corresponding limits of the inverse Born series. Then the following estimate holds*

$$\|\eta_1 - \eta_2\|_X < C\|\phi_1 - \phi_2\|_Y \ ,$$

where C is a constant which is independent of ϕ_1 and ϕ_2.

Proof. We begin with the estimate

$$\|\eta_1 - \eta_2\|_X \leq \sum_j \|\mathcal{K}_j(\phi_1 \otimes \cdots \otimes \phi_1 - \phi_2 \otimes \cdots \otimes \phi_2)\|_X \ . \tag{37}$$

Next, we make use of the identity (29) from which it follows that

$$\begin{aligned} \|\eta_1 - \eta_2\|_X &\leq \sum_j \sum_{k=1}^{j} \|\mathcal{K}_j\| \|\phi_1 \otimes \cdots \otimes \phi_1 \otimes \psi \otimes \phi_2 \otimes \cdots \otimes \phi_2\|_{Y^j} \\ &= \sum_j j\|\mathcal{K}_j\| M^{j-1} \|\psi\|_Y \ , \end{aligned} \tag{38}$$

where $\psi = \phi_1 - \phi_2$ is in the kth position of the tensor product. Using Lemma 2.1, we have

$$\begin{aligned} \|\eta_1 - \eta_2\|_X &\leq C\|\mathcal{K}_1\|\|\psi\|_Y \sum_j j\,[(\mu+\nu)\|\mathcal{K}_1\|M]^j \\ &\leq \|\mathcal{K}_1\|\|\phi_1 - \phi_2\|_Y \frac{C}{[1-(\mu+\nu)\|\mathcal{K}_1\|M]^2} \ . \end{aligned} \tag{39}$$

The above series converges when $(\mu+\nu)\|\mathcal{K}_1\|M < 1$, which holds by hypothesis.

□

3 Diffuse waves

3.1 Forward problem

We consider the propagation of a diffuse wave in an absorbing medium. The energy density u of the wave satisfies the time-independent diffusion equation

$$-\nabla^2 u + k^2(1+\eta(x))u = 0 \quad \text{in} \quad \Omega \ , \tag{40}$$

$$u + \ell\frac{\partial u}{\partial n} = \delta_{x_1} \quad \text{on} \quad \partial\Omega \ . \tag{41}$$

Here x_1 is the position of a point source, the diffuse wave number k is a positive constant, n is the outward unit normal to $\partial\Omega$, and ℓ is a positive constant. The function η, which is the spatially varying part of the absorption coefficient, is assumed to be supported in a closed ball B_a of radius a, with $1+\eta(x)$ nonnegative for all $x \in \Omega$. The energy density u obeys the integral equation

$$u(x) = u_i(x) - k^2 \int\limits_{\Omega} G(x,y)u(y)\eta(y)dy \ , \tag{42}$$

where u_i is the energy density of the incident diffuse wave which satisfies

$$-\nabla^2 u_i + k^2 u_i = 0 \quad \text{in} \quad \Omega \ , \tag{43}$$

$$u_i + \ell\frac{\partial u_i}{\partial n} = \delta_{x_1} \quad \text{on} \quad \partial\Omega \ . \tag{44}$$

Here G is the Green's function for the operator $-\nabla^2 + k^2$, which obey the boundary condition (41). Beginning with the incident wave u_i, we can iterate (42) to obtain the series

$$u(x) = u_i(x) - k^2 \int\limits_{\Omega} G(x,y)\eta(y)u_i(y)dy$$

$$+ k^4 \int\limits_{\Omega\times\Omega} G(x,y)\eta(y)G(y,y')\eta(y')u_i(y')dydy' + \cdots \ . \tag{45}$$

Evidently, we can write (45) in the form of the Born series (1), where $\phi = u_i - u$ and the operator K_j is defined by

$$(K_j f)(x_1,x_2) = (-1)^{j+1}k^{2j} \int\limits_{B_a\times\cdots\times B_a} G(x_1,y_1)G(y_1,y_2)\cdots$$

$$\times G(y_{j-1},y_j)G(y_j,x_2)f(y_1,\ldots,y_j)dy_1\cdots dy_j \ , \tag{46}$$

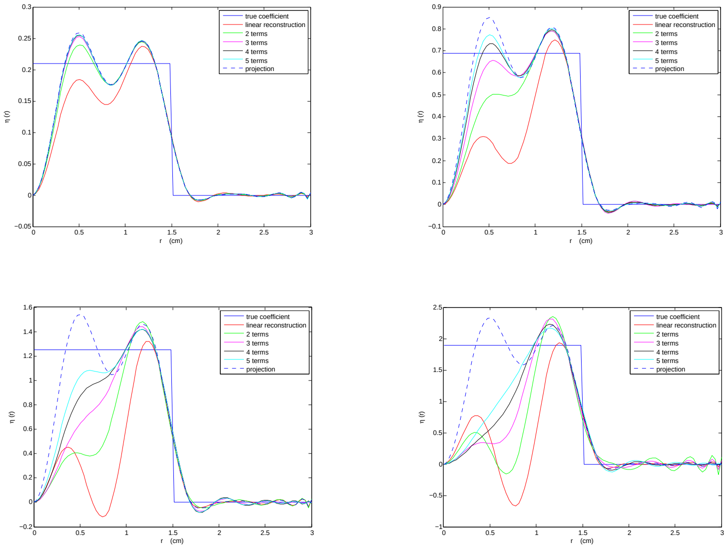

Fig. 1: Reconstructions of inhomogeneities with $R_1 = 1.5$cm and $R = 3$cm. The contrast ranges from: top left $\eta_0 = 1.1$, top right $\eta_0 = 1.3$, bottom left $\eta_0 = 1.5$ and bottom right $\eta_0 = 1.7$

where $x_1, x_2 \in \partial\Omega$. The data $\phi(x_1, x_2)$ is proportional to the intensity measured by a point detector at $x_2 \in \partial\Omega$ due to a point source at $x_1 \in \partial\Omega$. In [12] it was shown that K_j is a bounded operator whose norm obeys the estimate (8) with $X = L^2(B_a)$, $Y = L^2(\partial\Omega \times \partial\Omega)$ and

$$\begin{aligned} \mu &= k^2 \sup_{x \in B_a} \|G(x,\cdot)\|_{L^2(B_a)} \\ &\leq k^2 e^{-ka/2} \left(\frac{\sinh(ka)}{4\pi k}\right)^{1/2}, \end{aligned} \tag{47}$$

$$\begin{aligned} \nu &= k^2 |B_a|^{1/2} \sup_{x \in B_a} \|G(x,\cdot)\|_{L^2(\partial\Omega)} \\ &\leq k^2 |\partial\Omega| |B_a|^{1/2} \frac{e^{-2k\mathrm{dist}(\partial\Omega, B_a)}}{(4\pi \mathrm{dist}(\partial\Omega, B_a))^2}. \end{aligned} \tag{48}$$

Analogous results for L^p spaces were also obtained for $2 \leq p \leq \infty$.

3.2 Inverse problem

The inverse problem is to reconstruct η from boundary measurements of ϕ. We consider a three-dimensional medium which varies only in the radial direction. Here Ω is taken to be a ball of radius R centered at the origin and we consider an absorption coefficient of the form

$$\eta(x) = \begin{cases} \eta_0 & 0 \leq |x| \leq R_1 \ , \\ 0 & R_1 < |x| \leq R \ , \end{cases} \tag{49}$$

which corresponds to a spherical inclusion of radius R_1 and contrast η_0. The data ϕ is obtained by a series solution to the diffusion equation (40). The solution to the linearized inverse problem is obtained by computing $\mathcal{K}_1$ by regularized singular value decomposition. The details are presented in [14].

The parameters for the reconstructions are chosen as follows: $R = 3$cm, $R_1 = 1.5$cm, $\ell = 0.3$cm and $k = 1\text{cm}^{-1}$. Fig. 1 shows a series of experiments where the contrast of the inclusion is varied through a series of values of $\eta_0 = 1.1, 1.3, 1.5, 1.7$. In each of the graphs, we show the reconstruction using up to five terms of the IBS. We also display the projection of η, which is given by $\mathcal{K}_1 K_1 \eta$. In some sense, the projection is the best approximation to η that can be expected. Note that at low contrast, the series appears to converge quite rapidly to a reconstruction that is close to the projection. As the contrast is increased, the higher order terms significantly improve the linear reconstruction.

4 Calderon problem

4.1 Forward problem

The Calderon problem is the inverse problem of electrical impedance tomography. We consider a scalar field u that obeys the equation

$$\nabla \cdot \sigma(x) \nabla u = 0 \quad \text{in} \quad \Omega \ , \tag{50}$$

where the coefficient $\sigma(x) > 0$ for all $x \in \Omega$. The field is also taken to satisfy the Robin boundary condition

$$u + z\sigma \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega \ , \tag{51}$$

where σ and z are nonnegative and constant on $\partial\Omega$. In electrical impedance tomography, the field u is identified with the electric potential and the coefficient

σ with the conductivity. The coefficient z in (86) is the surface impedance and g is the current density. A typical choice for g is a dipole source of unit strength:

$$g = \delta_{x_1} - \delta_{x_2} \ , \quad x_1, x_2 \in \partial\Omega \ , \tag{52}$$

where $\delta_{x_1}, \delta_{x_2}$ are Dirac delta functions at x_1, x_2.

The forward problem is to determine the field u for a given coefficient σ. To proceed, we assume that the conductivity is of the form $\sigma(x) = \sigma_0 (1 + \eta(x))$, where the background coefficient $\sigma_0 = \sigma|_{\partial\Omega}$ is constant and $\eta \in L^\infty(B_a)$ is assumed to be supported in a closed ball B_a of radius a centered at the origin. We thus find that (50) becomes

$$-\Delta u = \nabla \cdot \eta(x) \nabla u \quad \text{in} \quad \Omega \ . \tag{53}$$

The field u obeys the integral equation

$$u(x) = u_0(x) + \int_\Omega G(x,y) \nabla \cdot \eta(y) \nabla u(y) dy \ , \quad x \in \Omega \ , \tag{54}$$

where u_0 obeys (53) with $\eta = 0$ and satisfies the boundary condition (86). Here G is the Green's function for the operator $-\Delta$, which obeys the boundary condition (86) with zero right hand side. Upon integrating (54) by parts, we see that the solution to the forward problem obeys the integral equation

$$u(x) = u_0(x) - \int_\Omega \nabla_y G(x,y) \cdot \nabla u(y) \eta(y) dy \ . \tag{55}$$

Beginning with u_0, we can iterate (55) to obtain a series for u of the form

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \ , \tag{56}$$

where

$$u_{j+1}(x) = -\int_\Omega \nabla_y G(x,y) \cdot \nabla u_j(y) \eta(y) dy \ , \quad j = 0, 1, \dots \ . \tag{57}$$

We now write (56) in the form of the Born series (1), where $\phi = u_0 - u$ and the operator K_j is defined by

$$\begin{aligned}(K_j f)(x) & \\ &= (-1)^j \int_\Omega \nabla_{y_1} G(y_1, x) \cdot \nabla_{y_1} \int_\Omega \nabla_{y_2} G(y_2, y_1) \\ &\cdots \cdot \nabla_{y_{j-1}} \int_\Omega \nabla_{y_j} G(y_j, y_{j-1}) \cdot \nabla_{y_j} u_0(y_j) f(y_1, \dots, y_j) dy_1 \cdots dy_j \ . \end{aligned} \tag{58}$$

It was shown in [17] that K_j is a bounded operator whose norm obeys the estimate (8) with $X = L^\infty(B_a)$, $Y = L^2(\partial\Omega)$ and

$$\mu = 1 , \tag{59}$$
$$\nu = \sup_{x\in\partial\Omega} \|\nabla G(x,\cdot)\|_{L^2(B_a)} \|\nabla u_0\|_{L^2(B_a)} . \tag{60}$$

4.2 Inverse problem

The inverse problem is to reconstruct η from measurements of the data ϕ. The function ϕ depends implicitly upon the position of the source. For example, in the case of the dipole source (52), if we fix the point $x_1 \in \partial\Omega$ and vary $x_2 \in \partial\Omega$, then ϕ will depend upon both x_2 and the point x at which we measure the field on $\partial\Omega$. Accordingly, will assume that $\phi \in L^\infty(\partial\Omega \times \partial\Omega)$. We consider a two-dimensional chest phantom, where Ω is taken to be a disk of radius R centered at the origin.

The data ϕ is obtained by solving (44) by the finite-element method. The solution to the linearized inverse problem is obtained by computing $\mathcal{K}_1$ by regularized singular value decomposition. The details are presented in [17]. The parameters for the reconstructions are chosen as follows: $R = 40$, $\sigma_0 = 1$ and $z\sigma_0 = 1$. Fig. 2 shows reconstructions using up to four terms of the IBS.

5 Radiative transport

5.1 Forward problem

The physical quantity of interest in radiative transport theory is the specific intensity $u(x,\theta)$ at the point $x \in \Omega$ in the direction $\theta \in S^{d-1}$. The specific intensity obeys the radiative transport equation (RTE)

$$\theta \cdot \nabla u + \sigma(x)u = \int_{S^{d-1}} k(\theta,\theta')u(x,\theta')d\theta' \quad \text{in} \quad \Omega \times S^{d-1}, \tag{61}$$
$$u = g \quad \text{on} \quad \Gamma_- . \tag{62}$$

The attenuation coefficient $\sigma(x)$ is assumed to be nonnegative for all $x \in \Omega$. In addition, the scattering kernel $k(\theta,\theta')$ is nonnegative and obeys the reciprocity relation $k(\theta,\theta') = k(-\theta',\theta)$ and is normalized so that

$$\int_{S^{d-1}} k(\theta,\theta')d\theta' = 1 , \quad \theta \in S^{d-1} . \tag{63}$$

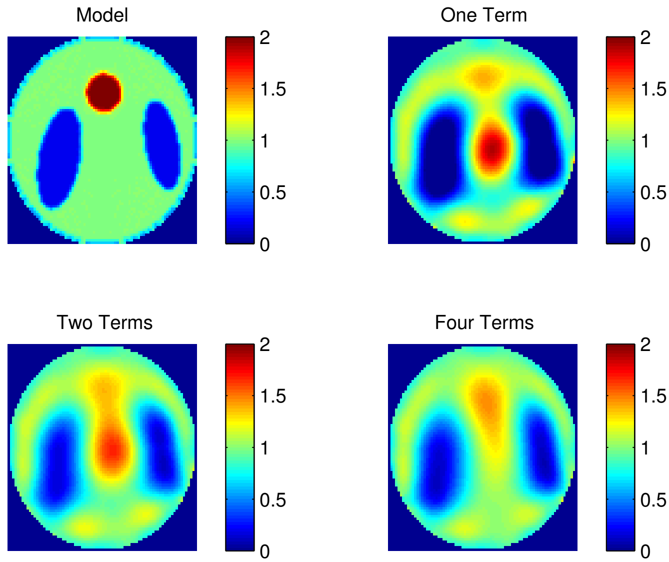

Fig. 2: Reconstruction of the conductivity with $\sigma_{\text{lungs}} = 0.2$ and $\sigma_{\text{heart}} = 2$.

We also introduce the sets $\Gamma_\pm$ which are defined by

$$\Gamma_\pm = \{(x,\theta) \in \partial\Omega \times S^{d-1} : \pm\theta \cdot n(x) > 0\} , \tag{64}$$

with n being the outer unit normal to $\partial\Omega$.

The forward problem is to determine the specific intensity u for a given attenuation σ. We assume that the attenuation coefficient σ is of the form

$$\sigma(x) = \sigma_0(1 + \eta(x)) , \tag{65}$$

where the background attenuation $\sigma_0 = \sigma|_{\partial\Omega}$ is constant and $\eta(x) > -1$ for all $x \in \Omega$. The function η is the spatially varying part of the attenuation coefficient; it is assumed to be supported in a closed ball B_a of radius a, centered at the origin. The specific intensity u obeys the integral equation

$$u(x,\theta) = u_0(x,\theta) - \sigma_0 \int\limits_{\Omega\times S^{d-1}} G(x,\theta;x',\theta')\eta(x')u(x',\theta')dx'd\theta' . \tag{66}$$

Here u_0 obeys (61) with $\eta = 0$ and G is the Green's function for the background medium, which satisfies the equation

$$\theta \cdot \nabla_x G(x,\theta;x',\theta') + \sigma_0 G(x,\theta;x',\theta') = \int\limits_{S^{d-1}} k(\theta,\theta'')G(x,\theta'';x',\theta')d\theta'' + \delta(x-x')\delta(\theta-\theta') ,$$

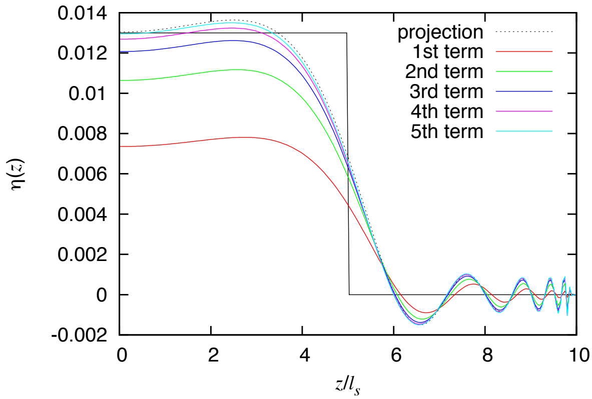

Fig. 3: Reconstruction of the attenuation coefficient.

together with homogeneous boundary conditions on Γ_-. It is easily seen that u_0 is given by the formula

$$u_0(x,\theta) = \int_{\partial\Omega} \int_{\theta'\cdot n<0} G(x,\theta;x',\theta')|\theta'\cdot n(x')|g(x',\theta')dx'd\theta' \ . \tag{67}$$

The integral equation (66) has a unique solution. Upon iteration, beginning with $u = u_0$, we obtain an infinite series for u of the form

$$u(x,\theta) = u_0(x,\theta) + u_1(x,\theta) + u_2(x,\theta) + \cdots \ , \tag{68}$$

where

$$u_{j+1}(x,\theta) = -\sigma_0 \int_{\Omega\times S^{d-1}} G(x,\theta;x',\theta')u_j(x',\theta')\eta(x')dx'd\theta' \ , \quad j = 0,1,\dots \ . \tag{69}$$

The series (68) can be written in the form of the Born series (1), where $\phi = u_0 - u$ and the operator K_j is defined by

$$(K_j f)(x,\theta) = (-1)^{j+1}\sigma_0^j \int_{\Gamma_a\times\cdots\times\Gamma_a} G(x,\theta;x_1',\theta_1')G(x_1',\theta_1';x_2',\theta_2')G(x_2',\theta_2';x_3',\theta_3')\cdots$$
$$\times G(x_{j-1}',\theta_{j-1}';x_j',\theta_j')u_0(x_j',\theta_j')f(x_1',\dots,x_j')dx_1'd\theta_1'\dots dx_j'd\theta_j' \ , \tag{70}$$

where $f \in L^\infty(B_a\times\cdots\times B_a)$. In [19] it was shown that K_j is a bounded operator whose norm obeys the estimate (8) with $X = L^\infty(B_a)$, $Y = L^1(\Gamma_+)$ and

$$\mu = \sigma_0 \sup_{(x',\theta')\in\Gamma_a} \int_{\Gamma_a} G(x,\theta;x',\theta')dxd\theta \ , \tag{71}$$

$$\nu = \sigma_0 \int_{\Gamma_a} u_0 dx d\theta \sup_{(x',\theta')\in\Gamma_a} \int_{\Gamma_+} G(x,\theta;x',\theta') dx d\theta \ , \tag{72}$$

where $\Gamma_a = B_a \times S^{d-1}$.

5.2 Inverse problem

The inverse problem is to reconstruct the coefficient η everywhere within Ω from measurements of the scattering data Φ on Γ_+. We consider a homogeneous isotropically scattering slab-shaped medium with contrast Δ embedded in a homogeneous infinite medium with attenuation σ_0. We suppose the embedded medium occupies the strip $-a \leq z \leq a$ inside of a slab of width $2L$ with $-L \leq z \leq L$. The data ϕ is obtained by solving the RTE by the singular eigenfunction method. The solution to the linearized inverse problem is obtained by computing $\mathcal{K}_1$ by regularized singular value decomposition. The details are presented in [19]. The parameters for the reconstructions are chosen as follows: $L = 10l_s$, $a = 5l_s$, $\sigma_0 = 1$ and $\Delta = 2.3$. Here lengths are measured in units of the scattering length l_s. Fig. 3 shows reconstructions using up to five terms of the IBS.

6 Electromagnetic waves

6.1 Forward problem

We consider the scattering of time-harmonic electromagnetic waves in a nonmagnetic medium. The electric field E obeys the wave equation

$$\nabla \times \nabla \times E - k^2 \epsilon(x) E = 0 \ , \tag{73}$$

where k is the free-space wavenumber and ϵ is the dielectric permittivity. We also impose the radiation condition

$$\lim_{|x|\to\infty} |x| \left[(\nabla \times E^s) \times \frac{x}{|x|} - ikE^s \right] = 0 \ . \tag{74}$$

Here we have decomposed the total field E into its incident and scattered components E^i and E^s according to

$$E = E^i + E^s \ , \tag{75}$$

where E^i obeys (73) with $\epsilon = 1$. Evidently, E^s obeys the equation

$$\nabla \times \nabla \times E^s - k^2 E^s = k^2 \eta(x) E \ , \tag{76}$$

where the susceptibility $\eta = \epsilon - 1$. Eq. (76) is equivalent to the integral equation

$$E^s(x) = (k^2 + \nabla\nabla\cdot) \int\limits_{B_a} G(x,y)\eta(y)E(y)dy \ , \tag{77}$$

where we have assumed that η is supported in B_a, the ball of radius a centered at the origin. In addition, G is the fundamental solution of the Helmholtz equation, which is given by

$$G(x,y) = \frac{e^{ik|x-y|)}}{4\pi|x-y|} \ . \tag{78}$$

The field E thus obeys the integral equation

$$E(x) = E^i(x) + (k^2 + \nabla\nabla\cdot) \int\limits_{B_a} G(x,y)\eta(y)E(y)dy \ . \tag{79}$$

If we iterate (79) beginning with $E = E^i$, we obtain an infinite series for E of the form

$$E = E^i + E^{(1)} + E^{(2)} + \cdots \ , \tag{80}$$

where

$$E^{(j+1)}(x) = \int\limits_{B_a} G(x,y)\eta(y)E^{(j)}(y)dy. \tag{81}$$

We now write (80) in the form of the Born series (1), where $\phi = E^i - E$ and the operator K_j is defined by

$$\begin{aligned}(K_j f)(x) = (k^2 + \nabla\nabla\cdot) \int\limits_{B_a\times\cdots\times B_a} & G(x,y_1)(k^2 + \nabla_{y_1}\nabla_{y_1}\cdot)G(y_1,y_2)\cdots(k^2 + \nabla_{y_{j-1}}\nabla\cdot_{y_{j-1}}) \\ & \times G(y_{j-1},y_n)E_i(y_j)f(y_1,\ldots,y_j)dy_1\cdots dy_j \end{aligned} \tag{82}$$

In the above, the point x will be taken to belong to a compact set which lies outside the support of η. It was shown in [16] that K_j is a bounded operator whose norm obeys the estimate (8) with $X = L^\infty(B_a)$, $Y = [L^2(K)]^3$ and

$$\mu = \frac{17}{2}(ka)^2 + 2\sqrt{74}ka + 105 \ , \tag{83}$$

$$\nu = |B_a|^{1/2} \left\|E^i\right\|_{[L^2(B_a)]^3} \sup_{x\in B_a} \left\|(k^2 + \nabla_{x_1}\nabla_{x_1}\cdot)G(\cdot,x)I\right\|_{[L^2(K)]^3} \ . \tag{84}$$

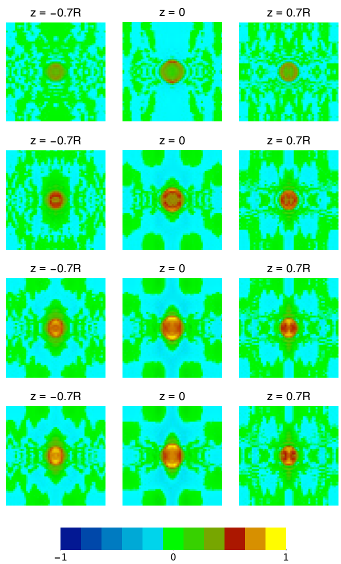

Fig. 4: Tomographic images of the reconstructed susceptibility.

6.2 Inverse problem

The inverse problem is to reconstruct η from measurements of ϕ. We consider a scatterer that consists of a sphere of radius $R = 2\lambda = 4\pi/k$ with index of refraction $n = 1.1$ related to the susceptibility by $\eta_0 = (n^2 - 1)/4\pi$. The data ϕ is computed from the Mie solution to the wave equation [25]. The solution to the linearized inverse problem is obtained by computing $\mathcal{K}_1$ by regularized singular value decomposition. The details are presented in [15]. The incident field is polarized in the $\hat{\mathbf{x}}$ direction and the incident plane wave is in the $\hat{\mathbf{y}}$ direction. The plane of detection was located at a distance $\lambda/3$ from the top of the sphere. Fig. 4 presents the reconstructions obtained using four terms of the IBS. The central column shows the results of reconstructions in the equatorial plane of the sphere. The left and right columns are the results of reconstructions in the planes $0.7R$ above and below the equatorial plane. The first row illustrates the results of linear reconstructions while the second, third and fourth rows show the second, third and fourth order nonlinear reconstructions, respectively. Fig. 5

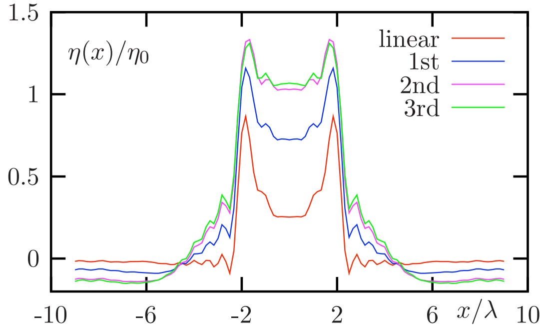

Fig. 5: One-dimensional profiles of the reconstructed susceptibility.

shows the one-dimensional profiles of the reconstructed susceptibility along the central line in the equatorial plane.

7 Diffusion on graphs

7.1 Forward problem

In this section we consider the discrete analog of the inverse problem of optical tomography with diffuse light. We focus on the problem of recovering vertex properties of a graph from boundary measurements. The results presented here are adapted from [20].

Let $G = (V, E)$ be a finite locally connected loop-free graph with vertex boundary δV. We consider the time-independent diffusion equation

$$(Lu)(x) + \alpha_0[1 + \eta(x)]u(x) = f(x), \quad x \in V, \tag{85}$$

$$t\, u(x) + \partial u(x) = g(x), \quad x \in \delta V\ . \tag{86}$$

Here we assume that the absorption of the medium is nearly constant with background absorption α_0 and inhomogeneities represented by the vertex potential η. In place of the Laplace operator, we introduce the combinatorial Laplacian L defined by

$$(Lu)(x) = \sum_{y \sim x} [u(x) - u(y)]\,, \tag{87}$$

where $y \sim x$ if the vertices x and y are adjacent. We make use of the graph analog of Robin boundary conditions, where the normal derivative is defined by

$$\partial u(x) = \sum_{\substack{y \in V \\ y \sim x}} \left[u(x) - u(y) \right], \tag{88}$$

and t is an arbitrary nonnegative parameter, which interpolates between Dirichlet and Neumann boundary conditions. If the potential η is nonnegative, then there exists a unique solution to the diffusion equation (85) satisfying the boundary condition (86).

The forward problem is to determine u, given η. The corresponding inverse problem, which we refer to as graph optical tomography, is to recover the potential η from measurements of u on the boundary of the graph. More precisely, let $G = (V, E)$ be a connected subgraph of a finite graph $\Gamma = (\mathcal{V}, \mathcal{E})$ and let δV denote those vertices in $\mathcal{V}$ adjacent to a vertex in V. In addition, let S, R denote fixed subsets of δV. We will refer to elements of S and R as sources and receivers, respectively. For a fixed potential η, source $s \in S$ and receiver $r \in R$, let $u(r, s; \eta)$ be the solution to (85) with vertex potential η and boundary condition (86), where

$$g(x) = \begin{cases} 1 & x = s, \\ 0 & x \neq s. \end{cases} \tag{89}$$

We define the Robin-to-Dirichlet map Λ_η by

$$\Lambda_\eta(s, r) = u(r, s; \eta). \tag{90}$$

The inverse problem is to recover η from the Robin-to-Dirichlet map Λ_η.

The background Green's function for (85) is the matrix G_0 whose i, jth entry is the solution to (85), with $\eta \equiv 0$, at the ith vertex for a unit source at the jth vertex. Under suitable restrictions this matrix can be used to construct the Robin-to-Dirichlet map Λ_η giving the solution of (85) on $R \subset \delta V$ to unit sources located in $S \subset \delta V$. To write a compact expression for Λ_η in terms of G_0, let D_η denote the matrix with entries given by

$$(D_\eta)_{i,j} = \begin{cases} \eta_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Additionally, for any two sets $U, W \subset V \cup \delta V$, let $G_0^{U;W}$ denote the submatrix of G_0 formed by taking the rows indexed by U and the columns indexed by W. For η sufficiently small we may write the Robin-to-Dirichlet map as the series

$$\Lambda_\eta(s, r) = G_0(r, s) - \sum_{j=1}^{\infty} K_j(\eta, \cdots, \eta)\,(r, s), \quad r \in R,\ s \in S, \tag{91}$$

where $K_j : \ell^p(V^n) \to \ell^p(R \times S)$ is defined by

$$K_j(\eta_1, \cdots, \eta_j)\,(r, s) = (-\alpha_0)^j G_0^{r;V} D_{\eta_1} G_0^{V;V} D_{\eta_2} \cdots G_0^{V;V} D_{\eta_j} G_0^{V;s}. \tag{92}$$

Evidently, (91) has the form of the Born series (1).

In order to establish the convergence and stability of (91), we seek appropriate bounds on the operators $K_j : \ell^p(V \times \cdots \times V) \to \ell^p(\delta V \times \delta V)$. Note that if $|V|$ and $|\delta V|$ are finite then all norms are equivalent. However, since we are interested in the rate of convergence of the IBS it will prove useful to establish bounds for arbitrary ℓ_p norms.

Proposition 7.1. *Let $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$ and define the constants ν_p and μ_p by*

$$\nu_p = \alpha_0 \|G_0^{R;V}\|_{\ell^q(V)\times\ell^p(R)} \|G_0^{V;S}\|_{\ell^q(V)\times\ell^p(S)} \ , \quad \mu_p = \alpha_0 \ , C_{G_0^{V;V}} \ , \tag{93}$$

where

$$C_{G_0^{V;V},q} = \max\nolimits_{v\in V} \|G_0^{V;v}\|_{\ell^q(V)} \ . \tag{94}$$

The Born series (91) converges if

$$\mu_p \|\eta\|_p < 1. \tag{95}$$

Moreover, the N-term truncation error has the following bound,

$$\left\| \Lambda_\eta - \left(G_0 - \sum_{j=N}^{\infty} K_j(\eta, \cdots, \eta) \right) \right\|_{\ell^p(R\times S)} \le \nu_p \|\eta\|_p^{N+1} \mu_p^N \frac{1}{1 - \mu_p \|\eta\|_p}. \tag{96}$$

7.2 Inverse problem

Let $\phi \in \ell^2(R \times S)$ denote the scattering data

$$\phi(r, s) = G_0(r, s) - \Lambda_\eta(r, s) \ , \tag{97}$$

which corresponds to the difference between the measurements in the background medium and those in the medium with the potential present. Note that if the Born series converges, we have

$$\phi(r, s) = \sum_{j=1}^{\infty} K_j(\eta, \dots, \eta). \tag{98}$$

The IBS is now of the form

$$\eta = \mathcal{K}_1(\phi) + \mathcal{K}_2(\phi, \phi) + \mathcal{K}_3(\phi, \phi, \phi) + \cdots \ , \tag{99}$$

where $\mathcal{K}_j$ is given by (6). Note that ϕ can be thought of as an operator from $\ell^2(R)$ to $\ell^2(S)$, in (99) we treat it as a vector of length $|R| \cdot |S|$. Similarly, though it is often convenient to think of η as a (diagonal) matrix, in (99) it should be thought of as a vector of length $|V|$. With a slight abuse of notation, we also use K_1 to denote the $|R||S| \times |V|$ matrix mapping η (viewed as a vector) to $K_1\eta$, once again thought of as a vector.

The following result provides sufficient conditions for the convergence of the IBS for graphs where $|V| = |R \times S|$, corresponding to the case of a formally determined inverse problem.

Theorem 7.1. *Let $|V| = |R \times S|$ and $p \in [1, \infty]$. Suppose that the operator K_1 is invertible. Then the inverse Born series converges to the true potential η if $\|\phi\|_p < r_p$. Here the radius of convergence r_p is defined by*

$$r_p = \frac{C_p}{\mu_p}\left[1 - 2\frac{\nu_p}{C_p}\left(\sqrt{1 + \frac{C_p}{\nu_p}} - 1\right)\right], \tag{100}$$

where

$$C_p = \min_{\|\eta\|_p = 1} \|K_1(\eta)\|_p \tag{101}$$

and ν_p, μ_p are defined in (93).

We now consider the stability of the limit of the inverse scattering series under perturbations in the scattering data. The following stability estimate follows immediately from Theorem 7.1.

Proposition 7.2. *Let E be a compact subset of $\Omega_p = \{\phi \in \mathbb{C}^n \mid \|\phi\|_p < r_p\}$, where r_p is defined in (100) and $p \in [1, \infty]$. Let ϕ_1 and ϕ_2 be scattering data belonging to E and ψ_1 and ψ_2 denote the corresponding limits of the inverse Born series. Then the following stability estimate holds:*

$$\|\psi_1 - \psi_2\|_p \leq M\|\phi_1 - \phi_2\|_p\ ,$$

where $M = M(E, p)$ is a constant which is otherwise independent of ϕ_1 and ϕ_2.

Theorem 7.1 guarantees convergence of the IBS, but does not provide an estimate of the approximation error. Such an estimate is provided in the next theorem.

Theorem 7.2. *Suppose that the hypotheses of Theorem 7.1 hold and $\|\phi\|_p < \tau r_p$, where $\tau < 1$. If η is the true potential corresponding to the scattering data ϕ,* *then*

$$\left\| \eta - \sum_{m=1}^{N} \mathcal{K}_m(\phi, \ldots, \phi) \right\| < M \left(\frac{1}{1-\tau} \right)^n \left(\frac{\|\phi\|_p}{\tau r_p} \right)^N \frac{1}{1 - \frac{\|\phi\|_p}{\tau r_p}}.$$

Remark 7.1. In the analysis of the IBS in the continuous setting it was found that certain smallness conditions on both $\|\mathcal{K}_1\|_p$ and $\|\mathcal{K}_1\phi\|_p$ are sufficient to guarantee convergence. Note that such a condition on $\|\mathcal{K}_1\|_p$ is not present in Theorem 7.1, Proposition 7.2 or Theorem 7.2.

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