Weighted Staircase Tableaux, Asymmetric Exclusion Process, and Eulerian Type Recurrences

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Abstract. We consider a relatively new combinatorial structure called staircase tableaux. They were introduced in the context of the asymmetric exclusion process and Askey–Wilson polynomials; however, their purely combinatorial properties have gained considerable interest in the past few years.

We will be interested in a general model of staircase tableaux in which symbols that appear in staircase tableaux may have arbitrary positive weights. Under this general model we derive a number of results concerning the limiting laws for the number of appearances of symbols in a random staircase tableaux.

One advantage of our generality is that we may let the weights approach extreme values of zero or infinity, which covers further special cases appearing earlier in the literature.

One of the main tools we use are generating functions of the parameters of interests. This leads us to a two–parameter family of polynomials. Specific values of the parameters cover a number of special cases analyzed earlier in the literature including the classical Eulerian polynomials.

Keywords: Staircase tableau, Eulerian polynomial, Asymmetric Exclusion Process

1 Introduction

This note is concerned with a combinatorial structure introduced recently by Corteel and Williams [8, 9] and called *staircase tableaux*. The original motivations were in connections with the asymmetric exclusion process (ASEP) on a one-dimensional lattice with open boundaries, an important model in statistical mechanics. The generating function for staircase tableaux was also used to give a combinatorial formula for the moments of the Askey–Wilson polynomials (see [9, 5] for the details). Further work includes [3], where special situations in which

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the generating function of staircase tableaux took a particularly simple form, were considered. Furthermore, [10] deals with the analysis of various parameters associated with appearances of the Greek letters α, β, γ, and δ in a randomly chosen staircase tableau (see below, or e.g. [9, Section 2], for the definitions and the meaning of these symbols). Moreover, there are natural bijections (see [9, Appendix]) between a class of staircase tableaux (the *α/β–staircase tableaux* defined below) and *permutation tableaux* (see e.g. [4, 6, 7, 15] and the references therein for more information on these objects and their connection to a version of the ASEP) as well as to *alternative tableaux* [20] which, in turn, are in one-to-one correspondence with *tree-like tableaux* [1].

The purpose of this extended abstract is to describe further properties of staircase tableaux, regarding them as interesting combinatorial objects in themselves. We refer to the full paper [16] for more details and proofs.

We recall the definition of a staircase tableau introduced in [8, 9]:

Definition 1. *A staircase tableau of size n is a Young diagram of shape $(n, n-1, \dots, 2, 1)$ whose boxes are filled according to the following rules:*

(Si) *each box is either empty or contains one of the letters α, β, δ, or γ;*
(Sii) *no box on the diagonal is empty;*
(Siii) *all boxes in the same row and to the left of a β or a δ are empty;*
(Siv) *all boxes in the same column and above an α or a γ are empty.*

An example of a staircase tableau is given in Fig. 1(a).

The set of all staircase tableaux of size n will be denoted by $\mathcal{S}_n$. There are several proofs of the fact that the number of staircase tableaux $|\mathcal{S}_n| = 4^n n!$, see e.g. [5, 3, 10] for some of them.

2 Staircase Tableaux and ASEP

As mentioned in the introduction, staircase tableaux were introduced in [8, 9] in connection with the *asymmetric exclusion process (ASEP)*; as a background, we give some details here. In a discrete version, the ASEP is a Markov chain describing a system of particles on a line with n sites $1, \dots, n$; each site may contain at most one particle. Particles jump one step to the right with probability u and to the left with probability q, provided the move is to a site that is empty; moreover, new particles enter site 1 with probability α and site n with probability δ, provided these sites are empty, and particles at site 1 and n leave the system with probabilities γ and β, respectively. See further [9], which also contains references and information on applications and connections to other branches of science.

Explicit expressions for the steady state probabilities of the ASEP were first given in [11]. Corteel and Williams [9] gave an expression for the steady state probabilities using staircase tableaux, their weight, and generating function for them. To describe it we first fill the tableau S by labelling the empty boxes of S with u's and q's as follows: first, we fill all the boxes to the left of a β with

u's, and all the boxes to the left of a δ with q's. Then, we fill the remaining boxes above an α or a δ with u's, and the remaining boxes above a β or a γ with q's. When the tableau is filled, we let $N_\alpha, N_\beta, N_\gamma, N_\delta, N_u, N_q$ be the numbers of symbols $\alpha, \beta, \gamma, \delta, u, q$ in S. We then define its *weight* to be

$$\mathrm{wt}(S) := \alpha^{N_\alpha} \beta^{N_\beta} \gamma^{N_\gamma} \delta^{N_\delta} u^{N_u} q^{N_q}, \tag{1}$$

i.e., the product of all symbols in S; this is thus a monomial of degree $n(n+1)/2$ in α, β, γ, δ, u and q. Figure 1(b) shows the tableau in Fig. 1(a) filled with u's and q's; its weight is $\alpha^5\beta^2\delta^3\gamma^3u^{13}q^{10}$.

Further, we let $Z_n(\alpha, \beta, \gamma, \delta, q, u)$ be the total weight of all filled staircase tableaux of size n, i.e.

$$Z_n(\alpha, \beta, \gamma, \delta, q, u) = \sum_{S \in \mathcal{S}_n} \mathrm{wt}(S).$$

Obviously, Z_n is a homogeneous polynomial of degree $n(n+1)/2$.

To describe the connection to ASEP, define the *type* of a staircase tableau S of size n to be a word of the same size on the alphabet $\{\circ, \bullet\}$ obtained by reading the diagonal boxes from northeast (NE) to southwest (SW) and writing $\bullet$ for each α or δ, and $\circ$ for each β or γ. (Thus a type of a tableau is a possible state for the ASEP.) Figure 1(a) shows a tableau and its type.

(a) (b)

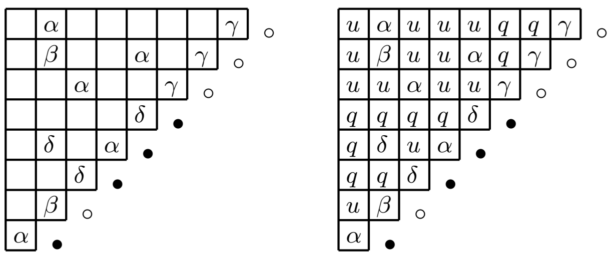

Fig. 1. (a) A staircase tableau of size 8; its type is $\circ\circ\circ\bullet\bullet\bullet\circ\bullet$ (b) the same tableau filled with u's and q's, its weight is $\alpha^5\beta^2\gamma^3\delta^3u^{13}q^{10}$.

As Corteel and Williams [9, 8] have shown that the steady state probability that the ASEP is in state σ is

$$\frac{Z_\sigma(\alpha, \beta, \gamma, \delta, q, u)}{Z_n(\alpha, \beta, \gamma, \delta, q, u)},$$

where $Z_\sigma(\alpha, \beta, \gamma, \delta, q, u) = \sum_{S \text{ of type } \sigma} wt(S)$.

3 Generating Function of the Total Weight

The generating function of the total weight of tableaux in $\mathcal{S}_n$

$$Z_n(\alpha,\beta,\gamma,\delta) := Z_n(\alpha,\beta,\gamma,\delta,1,1) \tag{2}$$

has a particularly simple form, viz., see [5, 3],

$$Z_n(\alpha,\beta,\gamma,\delta) = \prod_{i=0}^{n-1}\Bigl(\alpha+\beta+\gamma+\delta+i(\alpha+\gamma)(\beta+\delta)\Bigr). \tag{3}$$

(A proof is included in [5].) In particular, the number of staircase tableaux of size n is $Z_n(1,1,1,1) = \prod_{i=0}^{n-1}(4+4i) = 4^n n!$.

Note that the symbols α and γ have exactly the same role in the definition above of staircase tableaux, and so do β and δ. (This is no longer true in the connection to the ASEP, which is the reason for using four different symbols in the definition.) We say that a staircase tableau using only the symbols α and β is an *α/β–staircase tableau*, and we let $\overline{\mathcal{S}}_n \subset \mathcal{S}_n$ be the set of all α/β–staircase tableaux of size n. We thus see that any staircase tableau can be obtained from an α/β–staircase tableau by replacing some (or no) α by γ and some (or no) β by δ; conversely, any staircase tableau can be reduced to an α/β–staircase tableau by replacing every γ by α and every δ by β.

We define the generating function of the total weight of α/β–staircase tableaux by

$$Z_n(\alpha,\beta) = \sum_{S\in\overline{\mathcal{S}}_n} \mathrm{wt}(S) = Z_n(\alpha,\beta,0,0),$$

and note that the relabelling argument just given implies

$$Z_n(\alpha,\beta,\gamma,\delta) = Z_n(\alpha+\gamma,\beta+\delta).$$

We let $x^{\overline{n}}$ denote the rising factorial defined by

$$x^{\overline{n}} = x(x+1)\dots(x+n-1) = \Gamma(x+n)/\Gamma(x),$$

and note that by (3),

$$Z_n(\alpha,\beta) = Z_n(\alpha,\beta,0,0) = \prod_{i=0}^{n-1}(\alpha+\beta+i\alpha\beta) = \alpha^n\beta^n(\alpha^{-1}+\beta^{-1})^{\overline{n}} \tag{4}$$

$$= \alpha^n\beta^n\frac{\Gamma(n+\alpha^{-1}+\beta^{-1})}{\Gamma(\alpha^{-1}+\beta^{-1})}. \tag{5}$$

In particular, as noted in [3] and [5], the number of α/β–staircase tableaux is $Z_n(1,1) = 2^{\overline{n}} = (n+1)!$.

4 Main Result: Symbols on the Diagonal

Because of connections with ASEP, the diagonal of the staircase tableau is of natural interest. Dasse–Hartaut and Hitczenko [10] studied random staircase tableaux and in particular the symbols on the diagonal of a tableau obtained by picking a staircase tableau in $\mathcal{S}_n$ uniformly at random. Our purpose here is to consider α/β–staircase tableaux for arbitrary parameters $\alpha, \beta \geq 0$ and generalize several results from [10] to this case. This generality is also useful in studying the structure of random staircase tableaux. See items (ii) and (iii) in Section 7 below for further comments and [16] for more details.

We consider the following probability measure on the set of staircase tableaux of size n with weights α, β.

Definition 2. *Let $n \geq 1$ and let $\alpha, \beta \in [0, \infty)$ with $(\alpha, \beta) \neq (0, 0)$. Then $S_{n,\alpha,\beta}$ is the random α/β–staircase tableau in $\overline{\mathcal{S}}_n$ with the distribution*

$$\mathbb{P}_{\alpha,\beta}(S_{n,\alpha,\beta} = S) = \frac{\mathrm{wt}(S)}{Z_n(\alpha, \beta)} = \frac{\alpha^{N_\alpha(S)}\beta^{N_\beta(S)}}{Z_n(\alpha, \beta)}, \qquad S \in \overline{\mathcal{S}}_n. \tag{6}$$

We also allow the parameters $\alpha = \infty$ or $\beta = \infty$; in this case (6) is interpreted as the limit when $\alpha \to \infty$ or $\beta \to \infty$, with the other parameter fixed. Similarly, we allow $\alpha = \beta = \infty$; in this case (6) is interpreted as the limit when $\alpha = \beta \to \infty$. (In the case $\alpha = \beta = \infty$, we tacitly assume $n \geq 2$ or sometimes even $n \geq 3$ to avoid trivial complications.)

Remark 1. There is a symmetry (involution) $S \mapsto S^\dagger$ of staircase tableaux defined by reflection in the NW–SE diagonal, thus interchanging rows and columns, together with an exchange of the symbols by $\alpha \leftrightarrow \beta$ and $\gamma \leftrightarrow \delta$, see further [3]. This maps $\overline{\mathcal{S}}_n$ onto itself, and maps the random α/β–staircase tableau $S_{n,\alpha,\beta}$ to $S_{n,\beta,\alpha}$; the parameters α and β thus play symmetric roles.

Remark 2. We can similarly define a random staircase tableaux $S_{n,\alpha,\beta,\gamma,\delta}$, with four parameters $\alpha, \beta, \gamma, \delta \geq 0$, by picking a staircase tableau $S \in \mathcal{S}_n$ with probability $\mathrm{wt}(S)/Z_n(\alpha, \beta, \gamma, \delta)$. This is the same as taking a random $S_{n,\alpha+\gamma,\beta+\delta}$ and randomly replacing each symbol α by γ with probability $\gamma/(\alpha + \gamma)$, and each β by δ with probability $\delta/(\beta + \delta)$. Our results can thus be translated to results for $S_{n,\alpha,\beta,\gamma,\delta}$. In particular, the case $\alpha = \beta = \gamma = \delta = 1$ considered in [10] corresponds to picking an α/β–staircase tableau in $\overline{\mathcal{S}}_n$ at random with probability proportional to $2^{N_\alpha+N_\beta}$ and then randomly replacing some symbols; each α is replaced by γ with probability 1/2, and each β by δ with probability 1/2, with all replacements independent. Note that the weight $2^{N_\alpha+N_\beta}$ is the weight (1) if we choose the parameters $\alpha = \beta = 2$.

We are interested in the distribution of the symbols on the diagonal of $S_{n,\alpha,\beta}$. We define $A(S)$ and $B(S)$ as the numbers of α and β, respectively, on the diagonal of an α/β–staircase tableau S, and consider the random variables $A_{n,\alpha,\beta} := A(\mathcal{S}_{n,\alpha,\beta})$ and $B_{n,\alpha,\beta} := B(S_{n,\alpha,\beta})$; note that $A_{n,\alpha,\beta} + B_{n,\alpha,\beta} = n$ by (Sii), so it suffices to consider one of these. Moreover, by (1), $B_{n,\alpha,\beta} \overset{d}{=} A_{n,\beta,\alpha}$.

In order to describe the distribution of $A_{n,\alpha,\beta}$ we need some further notation. Define the numbers $v_{a,b}(n,k)$, for $a,b \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n = 0,1,\dots$, by the recursion

$$v_{a,b}(n,k) = (k+a)v_{a,b}(n-1,k) + (n-k+b)v_{a,b}(n-1,k-1), \qquad n \geq 1, \tag{7}$$

with $v_{a,b}(0,0) = 1$ and $v_{a,b}(0,k) = 0$ for $k \neq 0$ and $v_{a,b}(n,k) = 0$ for $k < 0$ and $k > n$, for all $n \geq 0$. These numbers were defined and studied by Carlitz and Scoville [2]. (Their notation is $A(n-k,k \mid a,b)$.) We give some additional properties below. Furthermore, define polynomials

$$P_{n,a,b}(x) := \sum_{k=0}^{n} v_{a,b}(n,k)x^k = \sum_{k=-\infty}^{\infty} v_{a,b}(n,k)x^k.$$

Thus, $P_{0,a,b}(x) = 1$.

In the case $a = b = 0$, we trivially have $v_{0,0}(n,k) = 0$ and $P_{n,0,0} = 0$ for all $n \geq 1$; in this case we define the substitutes, for $n \geq 2$,

$$\widetilde{v}_{0,0}(n,k) := v_{1,1}(n-2,k-1) \tag{8}$$

and

$$\widetilde{P}_{n,0,0}(x) := \sum_{k=0}^{n} \widetilde{v}_{0,0}(n,k)x^k = xP_{n-2,1,1}(x). \tag{9}$$

We assume the following relation throughout the rest of this abstract: $a = \alpha^{-1}$ and $b = \beta^{-1}$. Our main result is as follows.

Theorem 1. *Let $\alpha,\beta \in (0,\infty]$. If $(\alpha,\beta) \neq (\infty,\infty)$, then the probability generating function $g_A(x)$ of the random variable $A_{n,\alpha,\beta}$ is given by*

$$g_A(x) := \mathbb{E}x^{A_{n,\alpha,\beta}} = \sum_{k=0}^{n} \mathbb{P}(A_{n,\alpha,\beta} = k)x^k = \frac{P_{n,a,b}(x)}{P_{n,a,b}(1)} = \frac{P_{n,a,b}(x)}{(a+b)^{\overline{n}}}$$
$$= \frac{\Gamma(a+b)}{\Gamma(n+a+b)}P_{n,a,b}(x).$$

Equivalently,

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \frac{v_{a,b}(n,k)}{P_{n,a,b}(1)} = \frac{v_{a,b}(n,k)}{(a+b)^{\overline{n}}} = \frac{\Gamma(a+b)}{\Gamma(n+a+b)}v_{a,b}(n,k).$$

In the case $\alpha = \beta = \infty$, and $n \geq 2$, we have instead

$$g_A(x) := \sum_{k=0}^{n} \mathbb{P}(A_{n,\alpha,\beta} = k)x^k = \frac{\widetilde{P}_{n,0,0}(x)}{\widetilde{P}_{n,0,0}(1)} = \frac{\widetilde{P}_{n,0,0}(x)}{(n-1)!},$$

$$\mathbb{P}(A_{n,\alpha,\beta} = k) = \frac{\widetilde{v}_{0,0}(n,k)}{\widetilde{P}_{n,0,0}(1)} = \frac{\widetilde{v}_{0,0}(n,k)}{(n-1)!}.$$

This result has a number of consequences; some of them we describe below. But first, because of their role in Theorem 1 and connections to other parts of mathematics we briefly discuss the polynomials $P_{n,a,b}$ and their coefficients $v_{a,b}(n,k)$.

5 The polynomials $P_{n,a,b}$

For $a = 1$, $b = 0$, the recursion (7) is the standard recursion for *Eulerian numbers* $\left\langle {n \atop k} \right\rangle$, see e.g. [14, Section 6.2], [21, §26.14], [22, A008292]; thus

$$v_{1,0}(n,k) = \left\langle {n \atop k} \right\rangle .$$

These are often defined as the number of permutations of n elements with k descents (or ascents). See e.g. [24, Section 1.3], where also other relations to permutations are given. The corresponding polynomials

$$P_{n,1,0}(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$$

are known as *Eulerian polynomials*. We can thus see $v_{a,b}(n,k)$ and $P_{n,a,b}(x)$ as generalizations of Eulerian numbers and polynomials.

Furthermore, the cases $(a,b) = (0,1)$ and $(1,1)$ also lead to Eulerian numbers, with different indexing:

$$v_{0,1}(n,k) = v_{1,0}(n,n-k) = \left\langle {n \atop n-k} \right\rangle = \left\langle {n \atop k-1} \right\rangle , \qquad n \geq 1$$

(which is non-zero for $1 \leq k \leq n$). Similarly, by (7) and induction,

$$v_{1,1}(n,k) = v_{1,0}(n+1,k) = \left\langle {n+1 \atop k} \right\rangle , \qquad n \geq 0. \tag{10}$$

Equivalently,

$$P_{n,0,1}(x) = xP_{n,1,0}(x), \quad P_{n,1,1}(x) = P_{n+1,1,0}(x). \tag{11}$$

Similarly, by the definition (8) and (10),

$$\widetilde{v}_{0,0}(n,k) = \left\langle {n-1 \atop k-1} \right\rangle , \qquad n \geq 2,$$

and by (9) and (11),

$$\widetilde{P}_{n,0,0}(x) = P_{n-1,0,1}(x) = xP_{n-1,1,0}(x).$$

As mentioned above, in the case $a = b = 0$ we trivially have

$$v_{0,0}(n,k) = 0 \qquad \text{and} \qquad P_{n,0,0}(x) = 0 \qquad \text{for all } n \geq 1.$$

In the case when $a = 0$ or $b = 0$ we have the following simple relations, generalizing the results for Eulerian numbers and polynomials (11).

Proposition 1. *For all* $n \geq 1$,

$$\begin{aligned} v_{a,0}(n,k) &= a v_{a,1}(n-1,k), \\ v_{0,b}(n,k) &= b v_{1,b}(n-1,k-1), \textit{ and, equivalently,} \\ P_{n,a,0}(x) &= a P_{n-1,a,1}(x), \\ P_{n,0,b}(x) &= bx P_{n-1,1,b}(x). \end{aligned}$$

We collect some further properties in the following theorems.

Theorem 2. *For all* a, b *and* $n \geq 0$,

$$\begin{aligned} P_{n,a,b}(1) &= \sum_{k=0}^{n} v_{a,b}(n,k) = (a+b)^{\overline{n}} = \frac{\Gamma(n+a+b)}{\Gamma(a+b)}. \\ P'_{n,a,b}(1) &= \sum_{k=0}^{n} k v_{a,b}(n,k) = \frac{n(n+2b-1)}{2}(a+b)^{\overline{n-1}} \\ P''_{n,a,b}(1) &= \sum_{k=0}^{n} k(k-1) v_{a,b}(n,k) \\ &= \frac{n(n-1)(3n^2+(12b-11)n+12b^2-24b+10)}{12}(a+b)^{\overline{n-2}}. \end{aligned}$$

Furthermore, we have the symmetry

$$v_{a,b}(n,k) = v_{b,a}(n,n-k) \tag{12}$$

and thus

$$P_{n,a,b}(x) = x^n P_{n,b,a}(1/x). \tag{13}$$

Remark 3. The symmetries (12)–(13) between a and b are more evident if we define the homogeneous two-variable polynomials

$$\widehat{P}_{n,a,b}(x,y) := \sum_{k=0}^{n} v_{a,b}(n,k) x^k y^{n-k},$$

which satisfy the recursion

$$\widehat{P}_{n,a,b}(x,y) = \big(bx + ay + xy\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\big)\widehat{P}_{n-1,a,b}(x,y), \qquad n \geq 1$$

and the symmetry $\widehat{P}_{n,a,b}(x,y) = \widehat{P}_{n,b,a}(y,x)$. (Note that $\widehat{P}_{n,a,b}(x,y) = y^n P_{n,a,b}(x/y)$ and $P_{n,a,b}(x) = \widehat{P}_{n,a,b}(x,1)$.)

The following theorem has important consequences for us.

Theorem 3. (i) *If* $a, b > 0$, *then* $v_{a,b}(n,k) > 0$ *for* $0 \leq k \leq n$, *and* $P_{n,a,b}(x)$ *is a polynomial of degree* n *with* n *simple negative roots.*

(ii) *If* $a > b = 0$*, then* $v_{a,b}(n,k) > 0$ *for* $0 \le k < n$*, and* $P_{n,a,b}(x)$ *is a polynomial of degree* $n-1$ *with* $n-1$ *simple negative roots.*
(iii) *If* $a = 0 < b$*, then* $v_{a,b}(n,k) > 0$ *for* $1 \le k \le n$*, and* $P_{n,a,b}(x)$ *is a polynomial of degree* n *with* n *simple roots in* $(-\infty, 0]$*; one of the roots is* 0*, provided* $n > 0$*.*
(iv) *If* $a = b = 0$*, then* $\widetilde{v}_{0,0}(n,k) > 0$ *for* $1 \le k \le n-1$*, and* $\widetilde{P}_{n,0,0}(x)$ *is a polynomial of degree* $n-1$ *with* $n-1$ *simple roots in* $(-\infty, 0]$*; one of the roots is* 0*, provided* $n \ge 2$*.*

The proof that roots are distinct and negative uses an argument of Frobenius [13] for the classical Eulerian polynomials and is based on the recursion

$$P_{n,a,b}(x) = \big((n-1+b)x + a\big)P_{n-1,a,b}(x) + x(1-x)P'_{n-1,a,b}(x), \qquad n \ge 1$$

(which is easily seen to be equivalent to the recursion (7)). The proof also shows that the roots of $P_{n-1,a,b}$ and $P_{n,a,b}$ are interlaced (except that 0 is a common root when $a = 0$). More general results of this kind, can be found in e.g. [25] and [18, Proposition 3.5].

Remark 4. The case $a = b = 1/2$ appeared in [10]. In this case, it is more convenient to study the numbers $B(n,k) := 2^n v_{1/2,1/2}(n,k)$ which are integers and satisfy the recursion

$$B(n,k) = (2k+1)B(n-1,k) + (2n-2k+1)B(n-1,k-1), \qquad n \ge 1; \quad (14)$$

these are called *Eulerian numbers of type B* [22, A060187]. The numbers $B(n,k)$ seem to have been introduced by MacMahon [19] in number theory. They also have combinatorial interpretations, for example as the number of descents in signed permutations. The generating function (in a general symmetric case $a = b$) was found by Franssens [12, Proposition 3.1] who studied numbers $B_{n,k}(c)$ (and the resulting polynomials) given by $B_{n,k}(c) = 2^n v_{c/2,c/2}(n,k)$.

Furthermore, the case $a + b = 1$ yields polynomials $P_{n,a,1-a}(x)$ generalizing the Eulerian polynomials (the case $a = 1$, or $a = 0$); they are sometimes called (generalized) *Euler–Frobenius polynomials* and appear e.g. in spline theory; we refer to [17] for more information and references.

6 Consequences

Theorem 4. *The p.g.f.* $g_A(x)$ *of the random variable* $A_{n,\alpha,\beta}$ *has only simple roots and they are on the negative halfline* $(-\infty, 0]$*. As a consequence, for any given* n, α, β *there exist* $p_1, \ldots, p_n \in (0,1)$ *such that*

$$A_{n,\alpha,\beta} \overset{d}{=} \sum_i \mathrm{Be}(p_i), \qquad (15)$$

where $\mathrm{Be}(p_i)$ *is a Bernoulli random variable with parameter* p_i *and the summands are independent. It follows that the distribution of* $A_{n,\alpha,\beta}$ *and the sequence* $v_{a,b}(n,k)$*,* $k \in \mathbb{Z}$*, are unimodal and log-concave.*

Because of the representation (15), the $A_{n,\alpha,\beta}$ will follow the central (and local) limit theorem as long as the variance $\mathrm{Var}(A_{n,\alpha,\beta}) \to \infty$ (see, e.g. [23]). But from Theorem 2 we see that

$$\mathbb{E}A_{n,\alpha,\beta} = \frac{n(n+2b-1)}{2(n+a+b-1)}$$

and

$$\mathrm{Var}(A_{n,\alpha,\beta}) = n\frac{(n-1)(n-2)(n+4a+4b-1)+6(n-1)(a+b)^2+12ab(a+b-1)}{12(n+a+b-1)^2(n+a+b-2)}.$$

Remark 5. In the symmetric case $\alpha=\beta$ we thus obtain $\mathbb{E}(A_{n,\alpha,\alpha}) = n/2$; this is also obvious by symmetry, since $A_{n,\alpha,\alpha} \overset{d}{=} B_{n,\alpha,\alpha}$ by Remark 1. Regardless of the values of α, β we have $\mathbb{E}(A_{n,\alpha,\beta}) \sim n/2$. Thus, the effects of changing the parameters α and β are surprisingly small. Typically, probability weights of the type (1) (which are common in statistical physics) shift the distributions of the random variables considerably, but here the effects are only second-order.

For the variance we similarly have

$$\mathrm{Var}(A_{n,\alpha,\beta}) \sim \frac{n}{12}.$$

This leads to the following central limit theorem:

Theorem 5. *Let $\alpha,\beta \in (0,\infty]$ be fixed and let $n\to\infty$. Then $A_{n,\alpha,\beta}$ is asymptotically normal:*

$$\frac{A_{n,\alpha,\beta}-n/2}{\sqrt{n}} \overset{d}{\to} N(0,1/12).$$

Moreover, a corresponding local limit theorem holds:

$$\mathbb{P}(A_{n,\alpha,\beta}=k) = \sqrt{\frac{6}{\pi n}}\Big(e^{-6(k-n/2)^2/n}+o(1)\Big),$$

as $n\to\infty$, uniformly in $k\in\mathbb{Z}$.

Remark 6. The proof shows that the (suitably modified to take into account the asymptotics of the expected value and the variance) central limit theorem holds also if α and β are allowed to depend on n, provided only that $\mathrm{Var}(A_{n,\alpha,\beta})\to\infty$, which by the expression for the variance holds as soon as $n^2/(a+b)\to\infty$ or $nab/(a+b)^2\to\infty$; hence this holds except when a or b is ∞ or tends to ∞ rapidly, i.e., unless α or β is 0 or tends to 0 rapidly. It should be noted, however, the asymptotic normality may fail in extreme cases.

7 Further Remarks

(i) We concentrated here on the diagonal of a staircase tableau because of the connections to the ASEP. We can also study the total numbers N_α and N_β of symbols α and β in a random $S_{n,\alpha,\beta}$. This is actually simpler; we refer to [16] for the details. Similarly we can study the joint distribution of N_α and N_β and the joint distribution of N_α and, say, $A_{n,\alpha,\beta}$.

(ii) The notion of weights brings forth the possibility of studying the distribution of the symbols in $S_{n,\alpha,\beta}$. We note that when $\alpha = \beta = \infty$ (i.e. when $\alpha = \beta \to \infty$ in (6) then the probability measure $\mathbb{P}_{\alpha,\beta}$ is concentrated on the tableaux with the maximal total degree in $Z_n(\alpha, \beta)$, i.e. with the maximal number of symbols. As $\alpha = \beta \to \infty$ we have

$$Z_n(\alpha, \beta) \sim (\alpha + \beta) \prod_{i=1}^{n-1} (i\alpha\beta) = (n-1)! \left(\alpha^n \beta^{n-1} + \alpha^{n-1} \beta^n\right).$$

Hence there are $(n-1)!$ tableaux with n α's and $n-1$ β's, and $(n-1)!$ with $n-1$ α's and n β's for the total of $2(n-1)!$ α/β–tableaux with the maximal number of symbols, $2n-1$ (similarly, the corresponding number of staircase tableaux with $2n-1$ symbols $\alpha, \beta, \gamma, \delta$ is $2^{2n}(n-1)!$, see [3]).

(iii) It follows from the previous comment that there are only at most $n-1$ symbols in the $n(n-1)/2$ off–diagonal boxes. So, it is natural to ask where they are. Here is a step towards answering this question; we believe this is the first result in this direction: for a given box of a staircase tableau we give the probability that it contains a given symbol. Let $S_{n,\alpha,\beta}(i,j)$ be a content of the (i,j)th box (enumerated as in a matrix). For the off-diagonal boxes we have

$$\begin{aligned} P(S_{n,\alpha,\beta}(i,j) = \alpha) &= \frac{j-1+b}{(i+j+a+b-1)(i+j+a+b-2)}, \\ P(S_{n,\alpha,\beta}(i,j) = \beta) &= \frac{i-1+a}{(i+j+a+b-1)(i+j+a+b-2)}, \\ P(S_{n,\alpha,\beta}(i,j) \neq \emptyset) &= \frac{1}{i+j+a+b-1}. \end{aligned}$$

For the diagonal boxes we can give a complete description of the distribution of the symbols. To simplify the notation let $S_n(j) := S_{n,\alpha,\beta}(n+1-j, j)$ be the symbol on the diagonal in the jth column and let $1 \le j_1 < \ldots < j_\ell \le n$. Then

$$\mathbb{P}\big(S_n(j_1) = \ldots = S_n(j_\ell) = \alpha\big) = \prod_{k=1}^{\ell} \frac{j_k - k + b}{n - k + a + b}.$$

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