1 Dot and Cross Product Problems

1. Express the vector \( w \) as the sum of a vector \( w \parallel \) parallel to \( v \) and a vector \( w \perp \) orthogonal to \( v \) where (a) \( w = 2i - 4j \) and \( v = i + j \). (b) \( w = 3i + j - 2k \) and \( v = 2i - k \).

Comments: Recall that 
\[
\begin{align*}
\mathbf{w}\parallel &= \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}, \\
\mathbf{w}\perp &= \mathbf{w} - \mathbf{w}\parallel.
\end{align*}
\]

(a) \( \mathbf{w}\parallel = \langle -1, -1 \rangle \), \( \mathbf{w}\perp = \langle 3, 3 \rangle \).
(b) \( \mathbf{w}\parallel = \langle 16/5, 0, -8/5 \rangle \), \( \mathbf{w}\perp = \langle -1/5, 1, -2/5 \rangle \).

2. Consider the two vectors \( \mathbf{v} = 3i - 2j + 4k \) and \( \mathbf{w} = i + j + 2k \). (a) Find a vector \( \mathbf{u} \) that is perpendicular to both \( \mathbf{v} \) and \( \mathbf{w} \). (b) Find the area of the parallelogram that has adjacent edges given by \( \mathbf{v} \) and \( \mathbf{w} \). (c) Find the area of the triangle that has adjacent edges given by \( \mathbf{v} \) and \( \mathbf{w} \).

Comments (a) The vector \( \mathbf{u} \) is given by the cross product \( \mathbf{v} \times \mathbf{w} \) so \( \mathbf{u} = -8i - 2j + 5k \).
(b) The area of the parallelogram is given by \( \|\mathbf{v} \times \mathbf{w}\| = \sqrt{93} \).
(c) The area of the triangle is given by \( \frac{1}{2} \|\mathbf{v} \times \mathbf{w}\| = \sqrt{93}/2 \).

3. (a) Find a angle between a diagonal of a cube and one of its edges.
(b) Show that the diagonals of a rhombus are perpendicular.

Comments: (a) Assume that the cube has side \( a \), the origin is a vertex, the point \( (a, a, a) \) and that it has faces in the coordinate planes. Then a diagonal is given by the vector \( \mathbf{v} = ai + aj + ak \), while \( ai, aj, \) and \( ak \) are all edges. So the cosine of the angle \( \theta \) between \( \mathbf{v} \) and the edge \( ai \) is given by 
\[
\cos \theta = \frac{\mathbf{v} \cdot ai}{\|\mathbf{v}\|\|ai\|} = \frac{a^2}{\sqrt{3}a^2 a} = \frac{1}{\sqrt{3}}.
\]

Hence \( \theta = \arccos(1/\sqrt{3}) \).
(b) A rhombus is a parallelogram whose sides have equal length. Assume that two adjacent edges are given by the vectors \( \mathbf{v} \) and \( \mathbf{w} \). Then the diagonals have the form \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \). To check that they are perpendicular we will show that their dot product is zero:
\[
(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w}
\]
But this reduces to
\[
\mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 0,
\]
since the edges have equal length.

4. Use a triple scalar product to find the volume of the parallelepiped that has \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) as adjacent edges:
\( \mathbf{u} = +2i - 6j + 2k \), \( \mathbf{v} = 4j - 2k \), and \( \mathbf{w} = 2i + 2j - 4k \).
**Comments:** The volume is given as $|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}|$ which is given by evaluating the $3 \times 3$ determinant:

$$
\begin{vmatrix}
2 & -6 & 2 \\
0 & 4 & -2 \\
2 & 2 & -4 \\
\end{vmatrix} = -16.
$$

So the volume is 16.

5. Find a unit vector in 2-space that makes an angle of $\pi/4$ radians with the vector $\mathbf{w} = 4\mathbf{i} + 3\mathbf{j}$.

**Comments:** The algebra is very messy in this problem. We will use the dot product to find the desired vector $\mathbf{v} = \langle v_1, v_2 \rangle$. Since its norm is 1, we know that $v_1^2 + v_2^2 = 1$. Further, by the geometric definition of the dot product, we also have

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \cos(\pi/4) = \frac{\sqrt{2}}{2}.$$

Now $\|\mathbf{w}\| = 5$ and $\|\mathbf{v}\| = 1$ so the reduces to

$$\mathbf{v} \cdot \mathbf{w} = 4v_1 + 3v_2 = 5\sqrt{2}/2, \quad v_1^2 + v_2^2 = 1.$$

Solving this system, we find that there are two solutions: $v_1 = 7\sqrt{2}/10, v_2 = 7\sqrt{2}/10$.

To get these solutions, write $v_1$ in terms of $v_2$ as $v_1 = (5\sqrt{2}/2 - 3v_2)/4$ then substitute back into $v_1^2 + v_2^2 = 1$. This way we get the equation $v_2^2 + (5\sqrt{2}/2 - 3v_2)^2/16 = 1$. Expanding this, we have $25v_2^2/16 - 15\sqrt{2}v_2/16 + 25/32 = 1$. This is a quadratic equation whose solutions give the above values of $v_2$.

### 2 Line Problems

1. Find the intersection of the line $x = -1 + 2t, y = 3 + t, z = 4 - t$ with the $xy$-plane $[z = 0], yz$-plane $[x = 0]$, and $xz$-plane $[y = 0]$.

**Comments:** We just show the method for the $xy$-plane or equivalently, $z = 0$. Then $z = 4 - t = 0$ or $t = 4$. So $x = 1 + 2 \cdot 4 = 9$ and $y = 3 + 4 = 7$. The intersection point is $(9, 7, 0)$.

2. Find the line through $(1, 2, -1)$ and parallel to the vector $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

**Comments:** $x = 1 + 3t, y = 2 - 4t, z = -1 + t$.

3. Are the following three points on the same line? The points are: $P_1(1, 0, 1)$, $P_2(3, -4, -3)$, and $P_3(4, -6, -5)$.

**Comments:** Consider the two vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$. If they are multiples of each other, the lines are the same. Now $\overrightarrow{P_1P_2} = \langle 2, -4, -4 \rangle$ and $\overrightarrow{P_2P_3} = \langle 1, -2, -2 \rangle$. But $\overrightarrow{P_1P_2} = 2\overrightarrow{P_2P_3}$.

4. Find the intersection of the line $L : x = 2 - t, y = 5t, z = -t$ with the plane $2x + y + z = 1$.

**Comments:** The intersection point is determined by $2(2 - t) + (5t) + (-t) = 1$ or $2t = -3$ so $t = -3/2$. The point is $(1/2, 15/2, 3/2)$. 
5. Find all intersection points of the line $x = 1 + t, y = 3 - t, z = 2t$ with the cylinder $x^2 + y^2 = 16$.

**Comments:** The points are determined by $(1 + t)^2 + (3 - t)^2 = 16$ or expanded $t^2 - 2t - 3 = 0$ or $t = -1, 3$. The points are $(0, 4, -2)$ and $(4, 0, 6)$.

6. Show that the lines $L_1$ and $L_2$ are the same where $L_1 : x = 1 + 3t, y = -2 + t, z = 2t$ and $L_2 : x = 4 - 6t, y = -1 - 2t, z = 2 - 4t$.

**Comments:** We need to show that their parallel vectors $v_1$ and $v_2$ are multiplies of each other and that they share a common point. Now $v_1 = \langle 3, 1, 2 \rangle$ and $v_2 = \langle -6, -3, 4 \rangle$. Then $v_2 = -2v_1$. Now $P_0(1, -2, 0)$ lies on the line $L_1$. It also lies on $L_2$ since $z = 0$ implies $t = 1/2$ so $x = -1, y = -2$.

### 3 Plane Problems

1. Find the equation of the plane that contains the points $A(1, 0, 1), B(2, 1, 3)$, and $C(0, 1, 2)$.

**Comments:** Form the two vectors $\overrightarrow{AB} = \langle 1, 1, 2 \rangle$ and $\overrightarrow{AC} = \langle -1, 1, 1 \rangle$. Their cross product is

$$
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{j} \\
1 & 1 & 2 \\
-1 & 1 & 1 \\
\end{vmatrix}
= \langle -1, -3, 2 \rangle
$$

So the plane has the equation $-(x - 1) - 3(y - 0) + 2(z - 1) = 0$ or $-x - 3y + 2z = 1$.

2. Find a unit vector that is parallel to the line of intersection of the two planes: $2(x - 1) + 3(y + 2) + (z - 1) = 0$ and $(x - 1) - 2(y + 1) + 4(z + 1) = 0$. [Hint: find the normal vectors for the two planes, and take cross products].

**Comments:** The normal vectors are $N_1 = \langle 2, 3, 1 \rangle$ and $N_2 = \langle 1, -2, 4 \rangle$. Their cross product is

$$
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{j} \\
2 & 3 & 1 \\
1 & -2 & 4 \\
\end{vmatrix}
= \langle 14, -7, -7 \rangle
$$

A unit vector is $\langle 2, -1, -1 \rangle/\sqrt{6}$.

3. Find the equation of the plane through the point $P_0(-1, 4, 2)$ that contains the line of intersection of the planes $4x - y + z = 2$ and $2x + y - 2z = 3$.

**Comments:** The normal vectors are $N_1 = \langle 4, -1, 1 \rangle$ and $N_2 = \langle 2, 1, -2 \rangle$. Their cross product is

$$
v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{j} \\
4 & -1 & 1 \\
2 & 1 & -2 \\
\end{vmatrix}
= \langle 1, 10, 6 \rangle
$$

which is parallel to the desired plane. To get a second vector parallel to the desired plane, let $P_1(0, 0, 2)$ that lies on the first plane. Then $\overrightarrow{P_0P_1} = \langle 1, -4, 0 \rangle$ is parallel to the
desired plane. Then the cross product $\mathbf{v} \times \overline{P_0P_1}$ is a normal vector to this plane:

$$\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{j} \\
  1 & 10 & 6 \\
  1 & -4 & 0 \\
\end{vmatrix} = \langle 24, 6, -14 \rangle$$

The plane is given by the equation $24(x + 1) + 6(y - 4) - 14(z - 2) = 0$ or $12x + 3y - 7z = -14$.

4. Do the following four points $A(1, 0, -1), B(0, 2, 3), C(-2, 1, 1), D(4, 2, 3)$ all lie on the same plane? If they do, find the equation of the plane.

Comments: