# Appell Polynomials and Their Zero Attractors

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Abstract. A polynomial family $\{p_n(x)\}$ is Appell if it is given by $\frac{e^{xt}}{g(t)} = \sum_{n=0}^{\infty} p_n(x)t^n$ or, equivalently, $p_n'(x) = p_{n-1}(x)$. If $g(t)$ is an entire function, $g(0) \neq 0$, with at least one zero, the asymptotics of linearly scaled polynomials $\{p_n(nx)\}$ are described by means of finitely zeros of $g$, including those of minimal modulus. As a consequence, we determine the limiting behavior of their zeros as well as their density. The techniques and results extend our earlier work on Euler polynomials.

## 1. Introduction

Let $g(t)$ be an entire function such that $g(0) \neq 0$.

Definition 1.1. The Appell polynomials $\{p_n(x)\}$ associated with generating function $g(t)$ are given by

$$\frac{e^{xt}}{g(t)} = \sum_{n=0}^{\infty} p_n(x)t^n. \tag{1.1}$$

Some important examples are: the Taylor polynomials of $e^x$, with $g(t) = 1 - t$; the Euler polynomials, with $g(t) = (e^t + 1)/2$; and the Bernoulli polynomials, with $g(t) = (e^t - 1)/t$; and their higher order analogues.

The asymptotics and limiting behavior of the zeros of these families have been investigated by many people; for example, [**2**], [**6**], and so on.

In this paper, we obtained the asymptotics and the limiting behavior of the zeros for all Appell families provided the generating function $g(t)$ satisfies one further condition: that $g$ must have at least one zero. We use the ideas in our earlier paper [**2**]; furthermore, we simultaneously simplify and generalize some of the techniques there.

We found that the asymptotics in the general case are built from the basic example $g(t) = 1 - t$ which coincides with the classical work of Szegö on the Taylor polynomials of the exponential function. In our paper [**2**], we found that the asymptotics for the Euler and the Bernoulli polynomials are controlled by certain roots of $g(t)$, the ones of minimal modulus. In the general situation, as expected, the minimal modulus roots of $g(t)$ are needed to describe the asymptotics but there may be finitely many other roots needed to determine the asymptotics. These

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additional roots are determined through a geometric condition described in terms of rotated and scaled versions of the Szegö curve: $|xe^{1-x}| = 1$, $|x| \le 1$, $x \in \mathbb{C}$ (see Figure 3).

We frequently use the following notations. Let $Z(g)$ denote the set of all zeros of $g$ and let $r_0 < r_1 < r_2 < \dots$ denote the distinct moduli of these zeros in increasing order.

Recall that if $K_1$ and $K_2$ are two non-empty compact subsets of $\mathbb{C}$, then their Hausdorff distance is the larger of $\sup\{d(x, K_1) : x \in K_2\}$ and $\sup\{d(x, K_2) : x \in K_1\}$.

Definition 1.2. For a family $\{q_n(x)\}$ of polynomials whose degrees are increasing to infinity, their zero attractor is the limit of their set of zeros $Z(q_n)$ in the *Hausdorff metric* on the space of all non-empty compact subsets of the complex plane $\mathbb{C}$.

In the appendix, we discuss how the zero attractor is found in terms of the limsup and liminf of the zero sets.

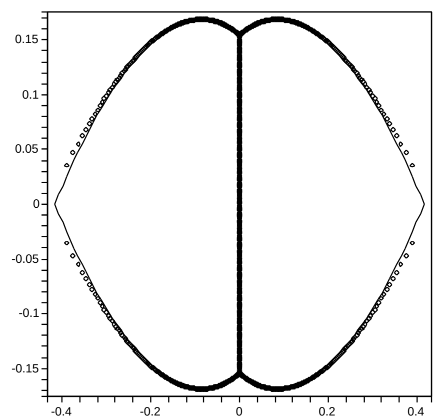


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Figure 1. Zeros for degree 1000 polynomial, with generating function $g(t) = J_0(t)$

There is a related work on the asymptotics and zeros of the Taylor polynomials for linear combinations of exponentials $\sum c_j e^{\lambda_j x}$ where the parameters $\lambda_j$ satisfy a geometric constraint [**1**]. The techniques of proof are very different from our approach.

## 2. The Generalized Szegö Approximations

It is convenient to collect together several results from [**2**] and some extensions of them concerning the asymptotics of $S_n(x) = \sum_{k=0}^{n} x^k/k!$. The domains of where their asymptotics hold are critical in understanding the behavior for the Appell polynomials.

Proposition 2.1. (Left-Half Plane) *Let* $1/3 < \alpha < 1/2$ *and* $1 \le j$. *On any compact subset* $K$ *of* $\{w : \Re w < 1\}$, *we have*

(1) $\dfrac{S_{n-1}(nw)}{e^{nw}} = 1 - \dfrac{(we^{1-w})^n}{\sqrt{2\pi n}(1-w)}\left(1+O(n^{1-3\alpha})\right),$

(2) $D_w^{j-1}(w^{-n}S_{n-1}(nw)) = D_w^{j-1}(w^{-n}e^{nw}) - \dfrac{(j-1)!}{\sqrt{2\pi n}}\dfrac{e^n}{(1-w)^j}\left(1+O(n^{1-3\alpha})\right),$
*where the big O constant holds uniformly for $x \in K$ and $D_w$ is the usual differential operator.*

The proof of part (1) is in [**2**]. Part (2) follows from an application of the saddle point method.

The following Proposition is also from [**2**]:

Proposition 2.2. (Outside Disk) *Let $S$ be a compact subset contained in $|w| > 1$ with distance $\delta > 0$ from the unit circle, and let $\alpha$ be chosen so $1/3 < \alpha < 1/2$. Then*

$$\frac{S_{n-1}(nw)}{e^{nw}} = \frac{(we^{1-w})^n}{\sqrt{2\pi n}(w-1)}\left(1+O(n^{1-3\alpha})\right),$$

*where the big O term holds uniformly for $w \in S$.*

Proposition 2.3. (Evaluations of Integrals) *If $\epsilon < |w|$ and $j \geq 1$, then we have*

(1) $\dfrac{1}{2\pi i}\displaystyle\oint_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n \frac{1}{t-w}\,dt = -w^{-n}S_{n-1}(wxn).$

(2) $\dfrac{1}{2\pi i}\displaystyle\oint_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n \frac{1}{(t-w)^j}\,dt = \frac{-1}{(j-1)!}D_w^{j-1}(w^{-n}S_{n-1}(wxn)),$
*where $D_w$ is the differentiation operator $\frac{d}{dw}$.*

Proof. (1) By expanding $1/(t-z)$ into an infinite geometric series and performing a term-by-term integration, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|t|=\varepsilon}\left(\frac{e^{xt}}{t}\right)^n \frac{1}{t-z}\,dt &= \frac{-1}{z2\pi i}\oint_{|t|=\varepsilon}\left(\frac{e^{xt}}{t}\right)^n \frac{1}{1-\frac{t}{z}}\,dt \\ &= \frac{-1}{z2\pi i}\oint_{|t|=\varepsilon}\left(\frac{e^{xt}}{t}\right)^n \left(\sum_{m\geq 0}\left(\frac{t}{z}\right)^m\right)dt.\end{aligned}$$

By the Cauchy integral theorem the terms correspond to $m \geq n$ vanish. Hence

$$\begin{aligned}\frac{1}{2\pi i}\oint_{|t|=\varepsilon}\left(\frac{e^{xt}}{t}\right)^n \frac{1}{t-z}\,dt &= \frac{-1}{z}\sum_{n-1\geq m\geq 0}\frac{1}{z^m}\left(\frac{1}{2\pi i}\oint_{|t|=\varepsilon} e^{xtn}t^{-n+m}\,dt\right) \\ &= \frac{-1}{z}\sum_{n-1\geq m\geq 0}\frac{1}{z^m}\frac{(xn)^{n-m-1}}{(n-m-1)!} \\ &= \frac{-1}{z}z^{-n+1}\sum_{n-1\geq m\geq 0}\frac{(xnz)^{n-m-1}}{(n-m-1)!} \\ &= -z^{-n}\sum_{n-1\geq j\geq 0}\frac{(xnz)^j}{j!} = (-1)z^{-n}S_{n-1}(zxn).\end{aligned}$$

Part (2) follows from differentiating (1) $j-1$ times with respect to $z$. ∎

## 3. Asymptotics Outside the Disk $D(0;1/r_0)$

THEOREM 3.1. *Given the Appell family $\{p_n(x)\}$ with generating function $g(t)$ we have*

$$\frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} = \frac{1}{g(1/x)}\left(1+O(1/n)\right)$$

*uniformly for $x \in K$ where $K$ is a compact subset of the annulus $A(1/r_0;\infty)$.*

PROOF. We shall find an asymptotic approximation to $p_n(nx)$ in the region $A(1/r_0;\infty) = \left\{x : |x| > \frac{1}{r_0}\right\}$. Use the generating relation equation (1.1) to get

$$p_n(x) = \frac{1}{2\pi i}\oint_{|t|=\epsilon} \frac{e^{xt}}{g(t)t^{n+1}}\,dt,$$

where $\epsilon < r_0$. Since both sides of the above equation are entire functions of $x$, by analytic continuation this representation for $p_n(x)$ is valid for all $x \in C$. Hence we can replace $x$ by $nx$ to get

$$p_n(nx) = \frac{1}{2\pi i}\oint_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n \frac{dt}{tg(t)}. \tag{3.1}$$

The above expression is valid for $0 < \epsilon < r_0$ and is the starting point of the analysis in the sequel.

Let $K$ be an arbitrary compact subset $\subseteq \{x : |x| > \frac{1}{r_0}\}$ and let $x \in K$. We can certainly choose $\epsilon$ small enough so that for all $x \in K$, $|\epsilon x| < 1$. By a change of variables, we get

$$p_n(nx) = \frac{x^n}{2\pi i}\oint_{|t|=\epsilon|x|}\left(\frac{e^t}{t}\right)^n \frac{dt}{tg(t/x)}.$$

Observe that the zeros of $g(t/x)$ have the form $ax$ where $a \in Z(g)$. Moreover, they must lie outside the closed unit disk since $|x| > 1/r_0$, so we can deform the integration path from the circle with radius $\epsilon\,|x|$ to the unit circumference. Thus

$$\begin{aligned} p_n(nx) &= \frac{x^n}{2\pi i}\oint_{|t|=1}\left(\frac{e^t}{t}\right)^n \frac{dt}{tg(t/x)} \\ &= \frac{x^n}{2\pi i}\oint_{|t|=1} e^{n(t-\ln t)}\frac{dt}{tg(t/x)}. \end{aligned}$$

It can be easily seen that $t = 1$ is the saddle point of the integral and the classical saddle point method is applicable here [**3**]. Hence

$$p_n(nx) = \frac{(ex)^n}{\sqrt{2\pi n}\,g(\frac{1}{x})}\left(1+O(\frac{1}{n})\right),$$

where the implied $O$ constant holds uniformly for $x \in K$. □

The last equation can be written as

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(\frac{1}{x})}\left(1+O(\frac{1}{n})\right), \quad |x| > 1/r_0.$$

We have the:

COROLLARY 3.2. *(1) On the complement of the disk $D(0;1/r_0)$,*

$$\lim_{n\to\infty}\frac{1}{n}\ln\left|\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}}\right|=0$$

*where the limit holds uniformly on compact subsets.*
*(2) The zero attractor must be contained in the closed disk $\overline{D}(0;1/r_0)$.*

Note that part (2) follows easily from (1) since $g(t)$ never vanishes outside the disk $D(0;1/r_0)$.

## 4. Asymptotics on the Basic Regions $R_\ell$

Let $r_0, r_1, \ldots$ denote the distinct moduli of the zeros of the generating function $g$ for the Appell family $\{p_n(x)\}$. Fix a positive integer $\ell$. We fix a large $\rho > 0$ so it is not equal to any zero modulus $\{r_0, r_1, \ldots\}$. For each zero $a \in Z(g)$ with $|a| = r_\ell$, we consider the circle $|x| = 1/|a|$ and the disk $D(1/a;\delta_a)$.

Now the tangent line $T_a$ to the circle $|x| = 1/r_\ell$ at the point $1/a$ determines the half-plane $H_a$, which contains 0; that is, $\Re(ax) < 1$. We choose $\epsilon_\ell > 0$ to be less than the distance from the portion of the tangent line $T_a$ that lies outside the disk $D(1/a;\delta_a)$ to the circle $|x| = 1/|a|$ for any $|a| = r_\ell$; that is, $\epsilon_\ell < \sqrt{1/r_\ell^2+\delta_a^2}-1/r_\ell$. This has the effect that the circle $|x| = 1/|a| + \epsilon_\ell$ never intersects the portion of the tangent line $T_a$ outside the disk $D(1/a;\delta_a)$. Finally, we make the requirement the disks $D(1/a;\delta_a)$ be mutually disjoint for all $a \in Z(g)$ with $|a| < \rho$.

DEFINITION 4.1. With these conventions, the region $R_\ell$ is described in terms of the half-planes $H_a$ and disks as

$$R_\ell=\bigcap\left\{H_a\setminus D(\tfrac{1}{a};\delta_a) : |a|=r_\ell\right\}\cap D(0;1/r_\ell+\epsilon_\ell)\setminus D(0;\tfrac{1}{r_{\ell+1}}+\epsilon_{\ell+1}) \tag{4.1}$$

We note that the regions $R_\ell$ are not disjoint; in fact, by construction, its inner boundary which consists of the portion of the circle $|x| = \frac{1}{r_{\ell+1}}+\epsilon_{\ell+1}$ that lie outside the disks $D(1/a;\delta_a)$, $|a| = r_{\ell+1}$, actually lies inside the region $R_{\ell+1}$.

Note the order of dependence: first we have the cut-off modulus $\rho > 0$ for the moduli of the zeros; next, $\delta_a > 0$ for each $a \in Z(g)$ is given and is a function of $\rho$ (described later in this section), then finally, $\epsilon_\ell$ is determined relative to each zero moduli $r_\ell$ which is a function of $\delta_a$.

It is convenient to introduce a region that contains all of the $R_\ell$'s:

DEFINITION 4.2. Let $R_\rho$ be the domain given by

$$R_\rho=\bigcap\{H_a : a\in Z(g), |a|=r_0\}\setminus\left[\bigcup\{D(1/a;\delta_a : a\in Z(g), |a|<\rho\}\cup D(0;1\rho)\right] \tag{4.2}$$

For any $a \in Z(g)$ with $r_0 \le |a| < \rho$, let $s_a(t)$ be the singular part of

$$\frac{1}{tg(t)}$$

at its pole $a$. Next let $g_1(t)$ be a normalized version of the generating function $g(t)$ given as

$$g_1(t)=\frac{1}{tg(t)}-\sum\{s_a(t) : a\in Z(g), r_0\le|a|<\rho\} \tag{4.3}$$

so $g_1(t)$ is analytic in the disk: $|t| < \rho$.

We develop the asymptotics for $\{p_n(nx)\}$ on the regions $R_\ell$ where $r_0 \le r_\ell < \rho$. Now we saw already that we can write $p_n(nx)$ as

$$p_n(nx) = \frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n g_1(t)\,dt + \frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s(t)\,dt,$$

where $s(t) = \sum\{s_a(t) : a \in Z(g), r_0 \le |a| \le \rho\}$.

Lemma 4.3. *With $g_1(t)$ given above in equation (4.3), we have*

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n g_1(t)\,dt = \frac{x^{n-1}e^n}{\sqrt{2\pi n}} g_1(1/x)\;(1+O(1/n))$$

*uniformly on compact subsets of the annulus $A(1/\rho;\infty)$.*

Proof. Let $x \in K \subset A(1/\rho,\infty)$. By a change of variables, we write

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n g_1(t)\,dt = \frac{x^{n-1}}{2\pi i}\int_{|t|=\epsilon|x|}\left(\frac{e^{t}}{t}\right)^n g_1(t/x)\,dt.$$

By construction, $g_1(t/x)$ is analytic on a disk of radius greater than 1. So the contour in the last integral can be deformed to the unit circle $|t| = 1$ without changing its value. Finally, by an application of the saddle point method we find that

$$\frac{x^{n-1}}{2\pi i}\int_{|t|=1}\left(\frac{e^{t}}{t}\right)^n g_1(t/x)\,dt = \frac{x^{n-1}e^n}{\sqrt{2\pi n}} g_1(1/x)\left(1+O(\tfrac{1}{n})\right).$$

$\square$

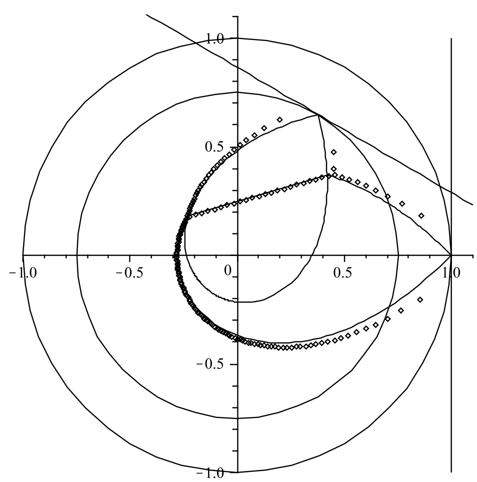


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Figure 2. Generic Plot of Polynomial Zeros and Zero Attractor When $g$ Has Two Roots; Tangent Lines and Circles Displayed

To state the next two lemmas, we need to introduce special polynomials $I_n(z)$ in $z^{-1}$ and $J(a; z)$ in $z$.

The polynomial $I_n(z)$ comes from expanding the derivative of $D_z^{m-1}(z^{-n}e^{nz})$. Consider

$$\begin{aligned} D_z^{m-1}(z^{-n}e^{nz}) &= \sum_{p=0}^{m-1}\binom{m-1}{p}(D_z^p z^{-n})(D_z^{m-1-p}e^{nz}) \\ &= \sum_{p=0}^{m-1}\binom{m-1}{p}(-n)(-n-1)\cdots(-n-p+1)z^{-n-p}(n^{m-1-p}e^{nz}) \\ &= z^{-n}e^{nz}n^{m-1}\sum_{p=0}^{m-1}\binom{m-1}{p}(-n)(-n-1)\cdots(-n-p+1)(nz)^{-p} \\ &= z^{-n}e^{nz}n^{m-1}\sum_{p=0}^{m-1}(-1)^p p!\binom{m-1}{p}\binom{n+p-1}{p}(nz)^{-p} \\ &= z^{-n}e^{nz}n^{m-1}I_{m-1}(nz), \end{aligned}$$

where $I_{m-1}(z)$ is given in

Definition 4.4.

$$I_{m-1}(z) = \sum_{p=0}^{m-1}(-1)^p p!\binom{m-1}{p}\binom{n+p-1}{p}z^{-p}. \tag{4.4}$$

For $a \in Z(g)$, we define $J(a;z)$ which are also polynomials in $z$. We write out the singular part $s_a(t)$ of the function $\frac{1}{tg(t)}$ at its nonzero pole $a$ by

$$s_a(t) := \sum_{m=1}^{\beta_a}\frac{b_{a,m}}{(t-a)^m}, \tag{4.5}$$

where $\beta_a$ is the order of $a$ as a zero of $g(t)$ so $b_{a,\beta_a} \neq 0$.

Definition 4.5. For $a \in Z(g)$, let $J(a;z)$ be the polynomial in $z$ given as

$$J(a;z) = \sum_{m=1}^{\beta_a}\frac{b_{a,m}}{(m-1)!}z^{m-1}I_{m-1}(az). \tag{4.6}$$

Lemma 4.6. *Let $a \in Z(g)$ and let $x \in K$, a compact subset of the half-plane $H_a$, $\Re(ax) < 1$. Then*

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt = -a^{-n}e^{nax}J(a;nx) + \frac{e^n x^{n-1}}{\sqrt{2\pi n}}s_a(1/x)\left(1+O(n^{1-3\alpha})\right)$$

*where $s_a(t)$ is the singular part of $1/(tg(t))$ at the zero $a$ of $g(t)$.*

Proof. We first write out the integral in terms of the singular part $s_a(t)$

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt = -\sum_{m=1}^{\beta_a}\frac{b_{a,m}}{(m-1)!}D_a^{m-1}\left(a^{-n}S_{n-1}(nax)\right)$$

where the coefficients $b_{a,m}$ are given in equation (4.5). We now study the asymptotics of the typical term $D_a^{m-1}(a^{-n}S_{n-1}(nax))$.

We may use the generalized half-plane Szegö asymptotics with $\frac{1}{3} < \alpha < \frac{1}{2}$ because of the restriction that $a \in Z(g)$ with $\Re(ax) < 1$ to obtain

$$D_a^{m-1}(a^{-n}S_{n-1}(nax)) = x^{n+m-1}D_{ax}^{m-1}((ax)^{-n}S_{n-1}(nax))$$
$$= x^{n+m-1}\left\{ D_z^{m-1}(z^{-n}e^{nz})\big|_{z=ax} - \frac{(m-1)!}{\sqrt{2\pi n}}\frac{e^n}{(1-ax)^m}\left(1+O(n^{1-3\alpha})\right)\right\}.$$

Combining these estimates we obtain

$$D_a^{m-1}(a^{-n}S_{n-1}(nax)) = x^{n+m-1}\{(ax)^{-n}e^{nax}n^{m-1}I_{m-1}(nax)$$
$$-\frac{(m-1)!}{\sqrt{2\pi n}}\frac{e^n}{(1-ax)^m}\left(1+O(n^{1-3\alpha})\right)\}$$
$$= a^{-n}e^{nax}(nx)^{m-1}I_{m-1}(nax) - \frac{(m-1)!}{\sqrt{2\pi n}}\frac{e^n x^{n+m-1}}{(1-ax)^m}\left(1+O(n^{1-3\alpha})\right). \tag{4.7}$$

Hence after summation we obtain

$$\frac{1}{2\pi i}\oint_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt = -\sum_{m=1}^{\beta_a}\frac{b_{a,m}}{(m-1)!}D_a^{m-1}(a^{-n}S_{n-1}(nax))$$
$$= -a^{-n}e^{nax}J(a;nx) + \frac{e^n x^{n-1}}{\sqrt{2\pi n}}s_a(\frac{1}{x})\left(1+O(n^{1-3\alpha})\right). \tag{4.8}$$

□

COROLLARY 4.7. *For $a \in Z(g)$, $|a| \le r_\ell$, we have*

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt = -a^{-n}e^{nax}J(a;nx) + \frac{e^n x^{n-1}}{\sqrt{2\pi n}}s_a(1/x)\left(1+O(n^{1-3\alpha})\right)$$

*uniformly on the compact subsets of $R_\ell$, where $s_a(t)$ is the singular part of $1/(tg(t))$ at the zero $a$ of $g(t)$.*

PROOF. By the definition of $R_\ell$, when $x \in R_\ell$ and $|a| = r_\ell$, we have $\Re(ax) < 1 - c(\delta)$. When $|a| < r_\ell$, we have $|xa| < 1 - c(\delta)$. So in both cases, the asymptotics stated in Proposition 2.1 applies. □

LEMMA 4.8. *Let $a \in Z(g)$ and let $x \in K$, where $K$ compact subset of the disk-complement $A(1/|a|;\infty)$. Then*

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt = \frac{e^n x^{n-1}}{\sqrt{2\pi n}}s_a(1/x)\left(1+O(n^{1-3\alpha})\right).$$

PROOF. We will use the disk-complement generalized Szegö asymptotics. For $z$ in the annulus $A(1+c,\infty)$, for any $c > 0$, we have

$$S_{n-1}(nz) = -\frac{z^n}{2\pi i}\oint_{|\zeta|=1}\frac{e^{n(\zeta-\ln\zeta)}}{\zeta - z}\,d\zeta$$

By Dividing $z^n$ and taking derivatives up to order $m-1$, we get

$$D_z^{m-1}(z^{-n}S_{n-1}(nz)) = -\frac{(m-1)!}{2\pi i}\oint_{|\zeta|=1}\frac{e^{n(\zeta-\ln\zeta)}}{(\zeta-z)^m}\,d\zeta$$
$$= -\frac{(m-1)!}{\sqrt{2\pi n}}\frac{e^n}{(1-z)^m}\left(1+O(n^{1-3\alpha})\right).$$

In the above, replace $z$ by $ax$ to obtain

$$\begin{aligned} D_a^{m-1}(a^{-n}S_{n-1}(nax)) &= x^{n+m-1}D_{ax}^{m-1}((ax)^{-n}S_{n-1}(nax)) \\ &= -\frac{(m-1)!}{\sqrt{2\pi n}}\frac{e^n x^{n+m-1}}{(1-ax)^m}\left(1+O(n^{1-3\alpha})\right). \end{aligned} \tag{4.9}$$

By summation, we obtain the asymptotics for the original integral:

$$\begin{aligned} \frac{1}{2\pi i}\oint_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt &= -\sum_{m=1}^{\beta_a}\frac{b_{a,m}}{(m-1)!}D_a^{m-1}(a^{-n}S_{n-1}(nx)) \\ &= \frac{e^n x^{n-1}}{\sqrt{2\pi n}}s_a(\frac{1}{x})\left(1+O(n^{1-3\alpha})\right). \end{aligned} \tag{4.10}$$

□

Corollary 4.9. *For $a \in Z(g)$ with $r_{\ell+1} \le |a| < \rho$, we have*

$$\frac{1}{2\pi i}\int_{|t|=\epsilon}\left(\frac{e^{xt}}{t}\right)^n s_a(t)\,dt = \frac{e^n x^{n-1}}{\sqrt{2\pi n}}s_a(1/x)\left(1+O(n^{1-3\alpha})\right),$$

*uniformly on the compact subsets of $R_\ell$.*

Proof. When $x \in R_\ell$, we have $r_{\ell+1} \le |a| < \rho$. By definition of $R_\ell$, we have $|xa| \ge 1 + c(\delta)$. Hence the asymptotics in Proposition 2.2 applies. □

The remaining case for the above integration involving $s_a(t)$ on the disk $D(1/a;\delta)$ will be handled in a later section.

Proposition 4.10. *For $x \in R_\ell$, we have*

$$\begin{aligned} \frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} = \frac{1}{x}\frac{1}{g_1(1/x)}&(1+O(1/n)) \\ &-\sqrt{2\pi n}\sum\left\{J(a;nx)\frac{1}{\phi(ax)^n} : a \in Z(g), |a| \le r_\ell\right\} \\ &+\sum\left\{\frac{1}{x}s_a\left(\frac{1}{x}\right) : a \in Z(g), |a| < \rho\right\}(1+O(n^{1-3\alpha})) \end{aligned}$$

*uniformly on the compact subsets of $R_\ell$, where $\phi(x) = xe^{1-x}$ and $1/3 < \alpha < 1/2$.*

Proof. Putting the last two corollaries into equation (3.1) and using Lemma 4.3 to simplify, we have the result. □

Proposition 4.11. *For $x \in R_\ell$, we have*

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(1/x)} - \sqrt{2\pi n}\sum\{\frac{J(a;nx)}{\phi^n(ax)} : a \in Z(g), |a| \le r_\ell\} + O(n^{1-3\alpha})$$

*uniformly on the compact subsets of $R_\ell$, where $1/3 < \alpha < 1/2$.*

Proof. By the definition of normalized version of the generating function $g_1(t)$ (see equation (4.3)), we see that

$$\frac{1}{x}g_1\left(\frac{1}{x}\right) = \frac{1}{g(1/x)} - \sum\left\{\frac{1}{x}s_a(\frac{1}{x}) : a \in Z(g), |a| < \rho\right\}. \tag{4.11}$$

We insert this into the expression for $\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}}$ in Proposition 4.11. Since the $s_a(1/x)$ term cancels, we have uniformly for $x \in R_\ell$:

$$\begin{aligned}(4.12)\quad \frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} &= \frac{1}{g(1/x)} \\ &-\sqrt{2\pi n}\sum\left\{(axe^{1-ax})^{-n}J(a;nx) : a \in Z(g), |a| < r_\ell\right\} + O\left(n^{1-3\alpha}\right).\end{aligned}$$

□

LEMMA 4.12. *If $a \in Z(g)$ with $|a| < \rho$ and $x \in R_\ell$, then*

$$J(a;nx) = \frac{b_{a,\beta_a}}{(\beta_a-1)!}(nx)^{\beta_a-1}\left(\frac{ax-1}{ax}\right)^{\beta_a-1}(1+o(1)).$$

PROOF. Recall that

$$J(a;nx) = \sum_{m=1}^{\beta_a}\frac{b_{a,m}I_{m-1}(nax)}{(m-1)!}, \quad I_{m-1}(nax) = \sum_{p=0}^{m-1}(-1)^p\binom{m-1}{p}\binom{n+p-1}{p}p!\,(nax)^{-p}.$$

It is easy to see that

$$\binom{n+p-1}{p}(nax)^{-p} = \frac{(ax)^{-p}}{p!}\left(1+o(1)\right),$$

that is, as $n \to \infty$

$$I_{m-1}(nax) \to \sum_{p=0}^{m-1}(-1)^p\binom{m-1}{p}(ax)^{-p} = \left(\frac{ax-1}{ax}\right)^{m-1}.$$

Hence

$$J(a;nx) = \frac{b_{a,\beta_a}}{(\beta_a-1)!}(nx)^{\beta_a-1}\left(\frac{ax-1}{ax}\right)^{\beta_a-1}(1+o(1)). \tag{4.13}$$

Since the coefficient $b_{a,\beta_a}$ in the definition of the singular part $s_a(t)$ is nonzero, we find for fixed $x$ that the precise order of $J(a;nx)$ as a polynomial in $n$ is $n^{\beta_a-1}$. □

We note the following

COROLLARY 4.13. $\lim_{n\to\infty}\frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} = \frac{1}{g(1/x)}$, *$x \in R_\ell$ provided $|\phi(ax)| > 1$ for all $a \in Z(g)$ with $|a| \le r_\ell$.*

We can summarize this section in the following

THEOREM 4.14. *On $R_\rho$, we have the following uniform asymptotics*

$$\begin{aligned}\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} &= \frac{1}{g(1/x)}(1+O(1/n)) \\ &-\sqrt{2\pi n}\sum\{\phi(ax)^{-n}J(a;nx) : a \in Z(g), |a| < \rho\} + O(n^{1-3\alpha})\end{aligned}$$

*where $1/3 < \alpha < 1/2$.*

It remains to develop the asymptotics in the disks $D(1/a;\delta_a)$ and well as determining domination among $a \in Z(g)$ of $|\phi(ax)|$.

## 5. Geometry of Szegö curves

Recall that $\phi(x) = xe^{1-x}$ is an entire function conformal on the open unit disk. The standard Szegö curve $\mathcal{S}$ is the portion of the level curve $|\phi(x)| = 1$ that lies inside the closed unit disk or equivalently, inside the closed left-hand plane $\Re(x) \leq 1$. $\mathcal{S}$ is a simple closed convex curve; in fact, it has the form $t = \pm\sqrt{e^{2(s-1)} - s^2}$ where $x = s + it$ and $s \in [-W(e^{-1}), 1]$ and $W$ is the principal branch of the Lambert $W-$function.

Definition 5.1. Let $a$ be a nonzero complex number. We call any curve of the form $\frac{1}{a}\mathcal{S}$ a Szegö curve.

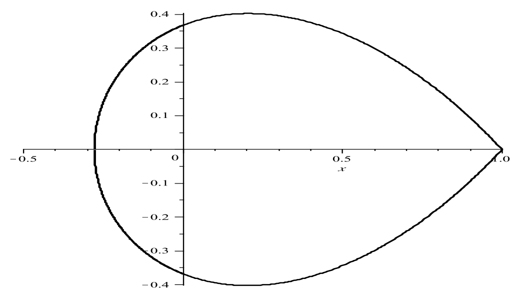


Figure 3. Szegö Curve: $|ze^{1-z}| = 1$ and $|z| \leq 1$

Remark 5.2. Note that the full curve $|\phi(x)| = 1$ divides the complex $x$-plane into three domains, one bounded (the interior of the standard Szegö curve $\mathcal{S}$) and two unbounded. The inequality $\left|\phi^{-1}(x)\right| > 1$consists of two domains: the interior of $\mathcal{S}$ and the unbounded domain that contains the real axis where $x > 1$. Furthermore, the deleted circumference $\{x : |x| = 1, x \neq 1\}$ lies in the domain where $\left|\phi^{-1}(x)\right| < 1$.

For brevity denote the interior of $\frac{1}{a}\mathcal{S}$ by $G_a$ so the interior of $\mathcal{S}$ is denoted by $G_1$. Of course, if $x \in G_1$, then $\left|\phi^{-1}(x)\right| > 1$. Let

$$G_1^+ := \left\{x : x \notin G_1, \left|\phi^{-1}(x)\right| > 1\right\}, \quad G_1^- := \left\{x : x \notin G_1, \left|\phi^{-1}(x)\right| < 1\right\}$$

$G_1^+$ is the unbounded domain that contains the real axis where $x > 1$ while $G_1^-$ is the remaining domain where $\left|\phi^{-1}(x)\right| < 1$. Since the difference between $G_1$ and a typical $G_a$ is a matter of rotation and stretching, the domains $G_a^+$ and $G_a^-$ are similarly defined. In terms of these notations, the above remarks can be equivalently phrased as

$$\{x : |x| \leq 1\} \setminus \overline{G}_1 \subset G_1^-. \tag{5.1}$$

In general, if $|a| > 0$, then

$$\left\{x : |x| \leq \frac{1}{|a|}\right\} \setminus \overline{G}_a \subset G_a^-. \tag{5.2}$$

Lemma 5.3. *The image of $\mathcal{S} \setminus \{x = 1\}$ under the inversion map $x \to \frac{1}{x}$ lies $G_1^-$.*

PROOF. We saw that the level curve $|\phi(x)| = 1$ divides the complex plane into three connected components whose boundaries are described in terms of

$$f(t) = \sqrt{e^{2(t-1)} - t^2}, \quad t \geq -W(e^{-1}) \simeq -0.2784645428.$$

For example, $\mathcal{S}$ is given by the two graphs of $\pm f(t)$, with $t \in [-W(e^{-1}), 1]$. We want to show that the inverted Szegö curve lies outside the standard Szegö curve $\mathcal{S}$ in the half plane $\Re(x) < 1$ and either above or below the the graph of $\pm f(t)$ when $t > 1$.

For convenience, let $\mathcal{G}$ denote the two domains $G_a$ and $G_a^+$ where $|\phi(x)| < 1$. Since $\mathcal{S}$ is symmetric about the real axis, it is enough to show that the portion of $\mathcal{S}$ with positive real part lies inside $\mathcal{G}$ under the map $T : w \mapsto 1/\overline{w}$.

Now $\mathcal{S} \setminus \{1\}$ lies inside the open unit disk. So the portion of the image of $\mathcal{S}$ that lies outside the unit disk with real part $< 1$ will lie inside the desired set $\mathcal{G}$.

Given the point $p(t) = (t, f(t))$ on the upper portion of the Szegö curve, its image under $T$ is given as

$$\left(\frac{t}{t^2 + f(t)^2}, \frac{f(t)}{t^2 + f(t)^2}\right) = \left(te^{-2(t-1)}, e^{-2(t-1)} f(t)\right), \; -W(e^{-1}) \leq t \leq 1$$

since $t^2 + f(t)^2 = e^{2(t-1)}$. Now $\Re(T(p(t))) = te^{-2(t-1)} < 1$ if $t < -W(-2e^{-2})/2 \simeq 0.2031878700$ and the modulus of $T(p(t))$ is $\sqrt{t^2 + f(t)^2}/(t^2+f(t)^2) = 1/\sqrt{t^2 + f(t)^2}$ which reduces to $1/\sqrt{e^{2(t-1)}} = e^{-(t-1)} > 1$ for $t < 1$. This shows that $T(p(t))$ lies inside the region $\mathcal{G}$ provided $t < -W(-2e^{-2})/2$.

It remains to examine the location of $T(p(t))$ for $-W(-2e^{-2})/2 \leq t < 1$. Of course, for such points, we know that their real part is greater than 1. Now the function $t/(t^2 + f(t)^2) = te^{-2(t-1)}$ is increasing on the interval $[-W(-2e^{-2})/2, 1/2]$ and decreasing on $[1/2, 1]$ and is $\geq 1$ on both intervals. It will be enough to show the following inequality:

$$\frac{f(t)}{t^2 + f(t)^2} = e^{-2(t-1)} f(t) > f\left(\frac{t}{t^2 + f(t)^2}\right) = f(te^{-2(t-1)}))$$

which is straightforward to verify. □

LEMMA 5.4. *Let $a, b$ be two distinct non-zero complex numbers. Then the intersection $\frac{1}{a}\mathcal{S} \cap \frac{1}{b}\mathcal{S}$ has at most two points.*

PROOF. The intersection of the two curves $\frac{1}{a}\mathcal{S} \cap \frac{1}{b}\mathcal{S}$ must satisfy $|\phi(ax)| = |\phi(bx)|$. This modulus condition determines a line; in fact, It is easy to give an explicit form for this line. Write $x = s + it$ and $b - a = \alpha + i\beta$. Then $|\phi(ax)| = |\phi(bx)|$ reduces to the line: $|ae^{-ax}| = \left|be^{-bx}\right| \cdot \left|e^{(b-a)x}\right| = |b/a|$; that is, $\Re((b-a)x) = \ln|b/a| \cdot \alpha s - \beta t = \ln|b/a|$. Since the Szegö curves are both convex, the number of intersection points is bounded above by 2. □

We need to determine exactly the size of this intersection.

LEMMA 5.5. *Choose $|a| > 1$ so that $1/a$ lies on the Szegö curve; that is, $|\phi(1/a)| = 1$. Then the equation $|\phi(ax)| = |\phi(x)|$ has a unique solution: $1/a$. In this case, $\frac{1}{a}\mathcal{S}$ is properly contained inside $\mathcal{S}$ except at the point $\frac{1}{a}$. Conversely, if $|a| > 1$ and $\frac{1}{a}\mathcal{S} \cap \mathcal{S}$ consists of just one point, then this common point must be $\frac{1}{a}$.*

Proof. We use the form of the equation for $|\phi(ax)| = |\phi(x)|$ from the above proof for Lemma 5.4 where we set $b = 1$. The slope of this line is $\alpha/\beta$. Recall that the upper portion of $\mathcal{S}$ is the graph of $y = \sqrt{e^{2(x-1)} - x^2}$ with derivative

$$y' = \frac{e^{2(x-1)} - x}{y} = \frac{x^2 + y^2 - x}{y}$$

We set $1/a = x_0 + iy_0$ where $y_0 = \sqrt{e^{2(x_0-1)} - x_0^2}$ so $1/a$ lies on $\mathcal{S}$. Write $a$ as

$$a = \frac{x_0}{x_0^2 + y_0^2} - \frac{y_0}{x_0^2 + y_0^2} i.$$

The slope of the line $|\phi(ax)| = |\phi(x)|$ is

$$\frac{\alpha}{\beta} = \frac{1 - x_0/(x_0^2 + y_0^2)}{y_0/(x_0^2 + y_0^2)} = \frac{x_0^2 + y_0^2 - x_0}{y_0}$$

Hence the slope of the tangent line at $1/a$ agrees with the slope of the line $|\phi(ax)| = |\phi(x)|$. Since $\mathcal{S}$ is convex, there is just one intersection point with the tangent line. □

The following two corollaries are immediate consequences of this lemma:

Corollary 5.6. *Let $|a| > |b| > 0$. Assume $\frac{1}{a} \notin \overline{G}_b$, the closure of $G_b$, then*

$$\left| \frac{1}{a}\mathcal{S} \cap \frac{1}{b}\mathcal{S} \right| = 2.$$

Corollary 5.7. *Let $|a| > |b| > 0$. If $\frac{1}{a} \in G_b$, then $\frac{1}{a}\mathcal{S}$ is properly contained in $G_b$.*

We now introduce the definition of *dominant zero* of the function $g$. Since $g$ is an entire function, the zeros of $g$ can be quite general. In fact, any discrete point set with a possible limit point at infinity is qualified as the zero set of $g$.

Definition 5.8. Let $a \in Z(g)$. The definition of dominant zero is inductive on the magnitude of $a$. First every zero $\alpha$ with $|\alpha| = r_0$ is dominant. Secondly, a zero $\alpha$ with $|\alpha| = r_1$ is dominant if

$$\frac{1}{\alpha} \notin \bigcup \left\{ \overline{G}_a : |a| = r_0 \right\}.$$

A zero $\alpha$, with $|\alpha| = r_2$, is dominant if

$$\frac{1}{\alpha} \notin \bigcup \left\{ \overline{G}_a : a \text{ dominant }, |a| \le r_1 \right\}$$

This procedure is carried out inductively.

Let $W$ denote the principal value of the Lambert $W$-function.

Lemma 5.9. *If $a' \in Z(g)$ such that $|a'| > r_0/W(e^{-1})$, then $a'$ must be a non-dominant zero. Hence there are at most finitely many dominant zeros.*

Proof. The proof follows from the fact that the radius of the largest open circular disk centered 0 that lies in the interior of the standard Szegö curve $\mathcal{S}$ is $W(e^{-1})$. □

Lemma 5.10. *Let $a$ and $b$ be two dominant zeros of $g$. Then $\frac{1}{a}\mathcal{S} \cap \frac{1}{b}\mathcal{S}$ consists of two points.*

Proof. If $|a| = |b|$, then equation of the line of intersection is reduced to the line $\arg x = -\frac{\arg a + \arg b}{2}$. It is easy to verify that, indeed, we have exactly two points of intersection. When $|a| > |b|$, by definition

$$\frac{1}{a} \notin \bigcup \{\overline{G}_\beta : \beta, \beta \text{ dominant}, |\beta| < |a|\}$$

Since $G_b \subset \bigcup \{G_\beta : \beta, \beta \text{ dominant}, |\beta| < |a|\}$, we have $\frac{1}{a} \notin \overline{G}_b$. Again by Lemma 5.6 we get the result. The case where $|a| < |b|$ is proved similarly. □

Definition 5.11. If $a$ and $b$ are two dominant zeros of $g$, then by Lemma 5.10 the intersection line $L_{a,b}$ $|\phi(ax)| = |\phi(bx)|$ always exists. Of the two half planes this line determines, let $E_{a^+,b}$ denote the one that contains $\frac{1}{a}$.

Lemma 5.12. *For two dominant zeros $a \neq b$ of the generating function $g(t)$, we have*

$$E_{a^+,b} = \{x : \left|\phi^{-1}(ax)\right| > \left|\phi^{-1}(bx)\right|\}$$

Proof. Let $x = \frac{1}{a}$ in the inequality to get $1 > \left|\phi^{-1}(b/a)\right|$. It is equivalent to showing that $1 > \left|\phi^{-1}(b/a)\right|$ is true for all distinct dominant zeros $a$ and $b$. We divide the situation into three cases:

Case 1: $|a| = |b|$. In this case we have $|b/a| = 1$ and $b/a \neq 1$, by Remark 5.2 the number $b/a$ lies in $G_1^-$. Hence we have $\left|\phi^{-1}(b/a)\right| < 1$.

Case 2: $|a| > |b|$. By definition of dominant zero we have $\frac{1}{a} \notin \overline{G}_b$. Since $\left|\frac{1}{a}\right| < \left|\frac{1}{b}\right|$, by equation (5.1) we see that $\frac{1}{a}$ lies $G_b^-$. Hence $\left|\phi^{-1}(b/a)\right| < 1$.

Case 3: $|a| < |b|$. To see $\left|\phi^{-1}(b/a)\right| < 1$, we invoke Lemma 5.3 to get $\frac{1}{a} \in G_b^-$. Hence $\left|\phi^{-1}(b/a)\right| < 1$. □

We introduce two key domains needed to describe the Appell polynomial asymptotics.

Definition 5.13. Let $D_0$ be the domain given as

$$D_0 := \bigcup \{G_a : a \text{ dominant zero of } g\}$$

so $D_0$ is a domain that contains 0.

Definition 5.14. For a dominant zero $a$, let

$$D_a := G_a \cap \bigcap \{E_{a^+,b} : b \text{ dominant }, b \neq a\}$$

Note that by Lemma 5.10 for all dominant $b \neq a$, $E_{a^+,b}$ is a non-empty domain that contains $\frac{1}{a}$. Hence $\bigcap \{E_{a^+,b} : b, b \text{ dominant }, b \neq a\}$ is a domain containing $\frac{1}{a}$. Therefore, $D_a$ is a non-empty domain.

Lemma 5.15. *(1) Let $a'$ be a non-dominant zero of the generating function $g$. We have*

$$\frac{1}{a'} \in \bigcup \{\overline{G}_a : a, |a| < |a'|, a \text{ dominant }\}$$

*(2) For all zeros $a$ of $g$, $\frac{1}{a} \in \overline{D}_0$.*
*(3) For all dominant zeros $a$ of $g$, we have $D_a \subset G_a \subset D_0$.*
*(4) $\bigcup \{D_a : a \text{ dominant}\} \subset D_0$*
*(5) For all dominant zero $a$, we have*

$$D_a = \{x : x \in G_a, \left|\phi^{-1}(ax)\right| > \left|\phi^{-1}(bx)\right| \text{ for all dominant } b \neq a\}$$

*(6) Let $a$ and $b$ are two distinct dominant zeros of $g$. We have*

$$D_a \cap D_b = \emptyset.$$

Proof. The proof of these statements follows mostly from definitions. We do not prove all of them.

Part (1) follows directly from definition. For (2), note that if $a$ is dominant, then, of course, we have $\frac{1}{a} \in \overline{G}_a$ and for $b$ dominant, $\frac{1}{a} \in E_{a^+,b}$. Hence by Definition 5.14, $\frac{1}{a} \in \overline{D}_a$. If $a$ is non-dominant, by (1) and Definition 5.13 we still have $\frac{1}{a} \in \overline{D}_a$. Hence (2) follows.

Part (5) follows from Definition 5.14 and Lemma 5.12. For (6), assume $x_0 \in D_a \cap D_b$. Since $x_0 \in D_a$, by (5) we have $\left|\phi^{-1}(ax_0)\right| > \left|\phi^{-1}(bx_0)\right|$. Similarly, we have $\left|\phi^{-1}(ax_0)\right| < \left|\phi^{-1}(bx_0)\right|$. A contradiction thus arises. Hence (6) follows. □

Lemma 5.16. $\bigcup\{D_a : a \text{ dominant zero of } g(t)\} \subset D_0$.

Proof. We prove a claim first.

Claim: If $x_0 \in D_0$ and $x_0 \notin \bigcup\{L_{a,b} : a, b \text{ dominant zeros, } a \neq b\}$, then $x_0 \in D_\alpha$ for some dominant $\alpha$. For the notation for the line segment $L_{a,b}$, see Definition 5.11.

Proof. Since $x_0 \notin \bigcup\{L_{a,b} : a, b \text{ dominant zero}\}$, the set $\{\left|\phi^{-1}(ax_0)\right| : a$ is dominant$\}$ consists of distinct numbers. Let $\left|\phi^{-1}(\alpha x_0)\right|$ be the unique maximum of the set. Hence for all dominant $b$, $b \neq \alpha$, we have $\left|\phi^{-1}(\alpha x_0)\right| > \left|\phi^{-1}(bx_0)\right|$. Next, since $x_0 \in D_0$, by Definition 5.13 there exists a dominant zero $\beta$ such that $x_0 \in G_\beta$. So $\left|\phi^{-1}(\alpha x_0)\right| > \left|\phi^{-1}(\beta x_0)\right| > 1$ and $|x_0| < \left|\frac{1}{\beta}\right|$. First, it is easy to see that $x_0 \notin \frac{1}{\alpha}S$, the boundary of $G_\alpha$ (otherwise it would contradict to $\left|\phi^{-1}(\alpha x_0)\right| > 1$ ).

Assume that $x_0 \notin G_\alpha$.

Case 1: $\left|\frac{1}{\alpha}\right| \geq \left|\frac{1}{\beta}\right|$. Since $|x_0| < \left|\frac{1}{\beta}\right|$, then $\left|\frac{1}{\alpha}\right| > |x_0|$. By equation (5.2) $x_0 \in G_\alpha^-$. Therefore, $\left|\phi^{-1}(\alpha x_0)\right| < 1$. This contradicts to $\left|\phi^{-1}(\alpha x_0)\right| > 1$.

Case 2: $\left|\frac{1}{\alpha}\right| < \left|\frac{1}{\beta}\right|$. Now both $\alpha$ and $\beta$ are dominant. We have $\frac{1}{\alpha} \notin \overline{G}_\beta$. By Lemma 5.3, $\overline{G}_\beta \setminus \overline{G}_\alpha \subset G_\alpha^-$. Because $x_0 \in \overline{G}_\beta \setminus \overline{G}_\alpha$, we have $x_0 \in G_\alpha^-$, which implies $\left|\phi^{-1}(\alpha x_0)\right| < 1$. This is still a contradiction. Thus $x_0 \in G_\alpha$. By (5) $x_0 \in D_\alpha$. □

For the proof of the lemma, we note that the set $D_0 \setminus \bigcup\left\{\overline{D}_a : a \text{ dominant}\right\}$ is an open set which we will assume is non-empty. Then there exists a disk $\Delta \subset D_0 \setminus \bigcup\left\{\overline{D}_a : a \text{ dominant}\right\}$. Observe that $\Delta \setminus \bigcup\{L_{a,b} : a, b \text{ dominant }\}$ is never empty. Thus exists $x_0 \in \Delta \setminus \bigcup\{L_{a,b} : a, b \text{ dominant }\}$. By the above claim $x_0 \in D_\beta$ for some dominant $\beta$. This is a contradiction since $x_0 \notin \left\{\overline{D}_a : a \text{ dominant}\right\}$. □

According to Lemma 5.16, the general picture for $D_a$ is now clear. Roughly, the set $\{D_\alpha : \alpha \text{ is a dominant zero of } g\}$ partitions $D_0$ so that the borders between two adjacent $D_a's$ are segments of the lines $L_{a,b}$.

Lemma 5.17. *Uniformly on the compact subsets of $\overline{D}(0; \frac{1}{r_0}) \setminus \overline{D}_0$, we have*

$$\lim_{n\to\infty} \frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(\frac{1}{x})}. \tag{5.3}$$

Proof. Let $K$ be a compact subset of $\overline{D}(0; \frac{1}{r_0}) \setminus \overline{D}_0$. By part 2 of Lemma 5.15, $K$ contains no zeros of $g$. Therefore, we can choose $\delta$ small enough so that $K$ does not intersect any disk $D(\frac{1}{a}; \delta)$, where $a \in Z(g)$. Recall the definition of the set $R_\ell, \ell \geq 0$. Let

$$K_\ell := K \cap R_\ell.$$

Note that by definition of $R_\ell$ we know, for all large $\ell$, $K_\ell = \emptyset$. Since $\bigcup_{l \geq 0} R_\ell \supset K$, we have

$$\bigcup_{\ell \geq 0} K_\ell = K.$$

There are at most finitely many $K_\ell$ in the above union. Consider a typical $K_\ell$. Let $x \in K_\ell$, so $x$ also lies in $R_\ell$. By the way $R_\ell$ is defined and a variant of equation (5.1) $x$ lies in $G_a^-$ for all $a \in Z(g)$, with $|a| \leq R_\ell$, we get $\left|\phi^{-1}(ax)\right| < 1$. Now we invoke Proposition 4.11 to obtain

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(\frac{1}{x})} + O(n^{1-3\alpha}).$$

Note in the above equation, the exponentially small terms corresponding to $\frac{J(a;nx)}{\phi^n(ax)}$ are absorbed in $O(n^{1-3\alpha})$. Hence $\lim_{n\to\infty} \frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(\frac{1}{x})}$ for $x \in K_\ell$. Since the number of $K_\ell$ is finite, proof of the lemma follows. □

We close this section with a strengthening of Theorem :

Theorem 5.18. *Let $\rho$ be chosen greater than $1/|a|$ where $a$ is any dominant zero of the generating function $g(t)$. Then on $R_\rho$, we have the following uniform asymptotics*

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(1/x)}\ (1 + O(1/n))$$
$$- \sqrt{2\pi n} \sum \left\{ \phi(ax)^{-n} J(a; nx) : a \in Z(g) \text{ and dominant } \right\} + O\left(n^{1-3\alpha}\right) + o(\Phi(x)),$$

*where $1/3 < \alpha < 1/2$ and $\Phi(x) = \max\{|\phi(ax)|^{-1} : a \in Z(g)$ and dominant $\}$.*

## 6. Asymptotics for Other Domains

### 6.1. Asymptotics Inside the Disk $D(1/a'; \delta)$ Where $a'$ Is a Non-Dominant Zero.

Proposition 6.1. *Let $a' \in Z(g)$. Then on the disk $D(1/a'; \delta_{a'})$, the normalized Appell polynomials have the asymptotics*

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \left(\frac{1}{g(1/x)} - \frac{1}{x} s_{a'}(\frac{1}{x})\right) \left(1 + O(\frac{1}{n})\right)$$
$$- \sqrt{2\pi n} \sum \left\{ \frac{J(a; nx)}{\phi^n(ax)} : a \in Z(g), |a| \leq |a'|, a \neq a' \right\} - \frac{\sqrt{2\pi n}}{(ex)^n} \sigma_{a'}(x) + O(n^{1-3\alpha}),$$

*where*

$$\sigma_{a'}(x) = \sum_{m=1}^{\beta_{a'}} \frac{b_{a',m}}{(m-1)!} D_{a'}^{m-1}((a')^{-n} S_{n-1}(na'x)).$$

Proof. The proof is very similar to that of Proposition 4.11. We shall not repeat it here. □

This proposition shows that we still need to estimate $\frac{\sqrt{2\pi n}}{(ex)^n}\sigma_{a'}(x)$. Since $x \in D(1/a';\delta)$, the approximations in Propositions 1 and 2 do not work. We handle this in the following proposition.

Proposition 6.2. *Let $a'$ be a non-dominant zero of $g$ with $|a'| < \rho$. Then there exists a choice of $\delta$ such that*

$$\frac{\sqrt{2\pi n}}{(ex)^n}\sigma_{a'}(x) = O(e^{6n\delta\rho})$$

Proof. To estimate $\sigma_a$, we make use of the elementary estimate: If $f(z)$ is analytic function of $z$, then for any $\epsilon > 0$, we have

$$\left|D_z^{j-1} f(z)\right| \leq \frac{(j-1)!}{\epsilon^{j-1}} \max_{|\zeta - z| = \epsilon} |f(\zeta)|.$$

By the definition of $\sigma_{a'}(x)$, we find

$$\begin{aligned}
|\sigma_{a'}| &\leq \left|\sum_{m=1}^{\beta_{a'}} \frac{b_{a',m}}{(m-1)!} D_{a'}^{m-1}(|a'|^{-n} S_{n-1}(na'x))\right| \\
&\leq \sum_{m=1}^{\beta_{a'}} \frac{|b_{a',m}|}{\delta_{a'}^{m-1}} \max_{|\zeta - a'| = \delta_{a'}} \left|\zeta^{-n} S_{n-1}(n\zeta x)\right| \\
&\leq K_{\delta_{a'}} \max_{|\zeta - a'| = \delta_{a'}} (|\zeta|^{-n} S_{n-1}(|\zeta x|\, n))
\end{aligned}$$

where $K_{\delta_{a'}} > 0$ is a constant that depends on the zero $a'$ and the radius $\delta_{a'}$.

To go further we observe for $x \in D(\frac{1}{a}, \delta_{a'})$ and $|\zeta - a'| = \delta_{a'}$:

$$\begin{aligned}
|\zeta x| \leq (|a'| + \delta_{a'})\,|x| &\leq |a'|\,|x| + |x|\,\delta_{a'} \\
&\leq 1 + |a'|\,\delta_{a'} + |x|\,\delta_{a'} = 1 + \delta_{a'}(|a'| + |x|).
\end{aligned}$$

Since $|a'| < \rho$ by assumption, $|\zeta x| \leq 1 + 2\rho\delta_{a'}$. But $|\zeta| \geq |a'| - \delta_{a'}$ and $|x| \geq \frac{1}{|a'|} - \delta_{a'}$, so we get

$$|\zeta x| \geq (|a'| - \delta_{a'})\left(\frac{1}{|a'|} - \delta_{a'}\right) \geq 1 - \delta_{a'}\left(\frac{1}{|a'|} + |a'|\right) \geq 1 - 2\delta_{a'}\rho.$$

Collecting these two inequalities, we get

$$1 - 2\delta_{a'}\rho \leq |\zeta x| \leq 1 + 2\delta_{a'}\rho.$$

Now use that $|S_{n-1}(nt)| \leq e^{nt}$:

$$\begin{aligned}
\max_{|\zeta - a'| = \delta_{a'}} |e\zeta x|^{-n} S_{n-1}(|\zeta x|\, n) &\leq e^{-n}\,|1 - 2\delta_{a'}\rho|^{-n}\, e^{n(1+2\delta_{a'}\rho)} \\
&= |1 - 2\delta_{a'}\rho|^{-n}\, e^{2n\delta_{a'}\rho}
\end{aligned}$$

For $0 \leq x \leq 1/2$, $1/(1-x) \leq e^{2x}$; if we choose $\delta_{a'}$ such that $2\delta_{a'}\rho \leq 1/2$, then we have $|1 - 2\delta_{a'}\rho|^{-n} \leq e^{4\delta_{a'}\rho}$. With this choice of $\delta$, we obtain the desired bound

$$\max_{|\zeta - a'| = \delta_{a'}} (|e\zeta x|^{-n} S_{n-1}(|\zeta x|\, n)) \leq e^{4\delta_{a'}\rho} e^{2n\delta_{a'}\rho} = e^{6n\delta_{a'}\rho}$$

□

**6.2. Asymptotics Inside the Domain $D_\beta$ Where $\beta$ Is a Dominant Zero.** Let $\beta_1, \beta_2, \cdots, \beta_k$ be the dominant zeros of $g$. For each $\beta_i, 1 \le i \le k$ define set $A_i$ as

$$A_i := \left\{ \frac{1}{\alpha} : \alpha \in Z(g), \frac{1}{\alpha} \in D_{\beta_i}, |a| < \rho \right\}$$

So $A_i$ consists of reciprocals of zeros that fall into $D_{\beta_i}$. Finally let the remaining part of reciprocals be denoted by $B$, namely,

$$B := \left\{ \frac{1}{\alpha} : \alpha \in Z(g), \alpha \text{ non-dominant}, \frac{1}{\alpha} \notin \bigcup_{j=1}^{k} A_j, |a| < \rho \right\}$$

The set $B$ consists of the reciprocals of those zeros lying on the border lines among $\{D_{\beta_j}\}$. Note that each $A_i \cup B$ is a finite set. If $\frac{1}{\alpha} \in A_i \cup B$, then $\alpha$ is non-dominant. We now investigate the asymptotics of $\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}}$ for $x \in D_{\beta_i}, 1 \le i \le k$. We remind the readers that there could be many zeros $a$ of $g$ such that $\frac{1}{a} \in D_{\beta_i}$. This fact prevents the situation given in equation (5.3) from occuring. We need a lemma for estimation.

Lemma 6.3. *If $0 < |a| < |b|$ and $\frac{1}{b} \in \overline{G}_a$, then for all $x \in \overline{G}_b$, we have*

$$\left|\phi^{-1}(bx)\right| \le \left|\phi^{-1}(ax)\right|.$$

*For $x \in G_b$, we have*

$$\left|\phi^{-1}(bx)\right| < \left|\phi^{-1}(ax)\right|.$$

Proof. Apply Lemma 5.7 and maximum modulus principle to the harmonic function $\ln\left|\phi^{-1}(ax)\right| - \ln\left|\phi^{-1}(bx)\right|$ for $x \in \frac{1}{b}\mathcal{S}$. □

We note that this result can be sharpen as: $\frac{1}{b} \in G_a$, then there exists $\delta > 0$ such that for all $x \in \overline{G}_b$, we have

$$e^{\delta}\left|\phi^{-1}(bx)\right| \le \left|\phi^{-1}(ax)\right|$$

Proposition 6.4. *For $x \in D_{\beta_i} \setminus \overline{D}(0; \frac{1}{\rho})$, there exists $\delta = \delta(\rho) > 0$ such that*

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = -\sqrt{2\pi n}\frac{J(\beta_i, nx)}{\phi^n(\beta_i x)} + o(\phi^{-n}(\beta_i x)) + O(e^{6n\rho\delta}).$$

Note that when $x$ lies in a compact subset of $D_{\beta_i} \setminus \overline{D}(0; \frac{1}{\rho})$, the term $O(e^{6n\rho\delta})$ can be absorbed in $o(\phi^{-n}(\beta_i x))$.

Proof. Let $K$ be a compact subset of $D_{\beta_i} \setminus \overline{D}(0, \frac{1}{\rho})$. We choose $\delta$ small enough so that for all $\frac{1}{a} \in B \setminus \overline{D}(0; \frac{1}{\rho})$ we have $D(\frac{1}{a}; \delta) \cap K = \phi$. Let

$$r_\ell = |\beta_i|.$$

Note that $R_{\ell-1} \cap D_{\beta_i} = \phi$. The first $R_j$ which possibly has a non-empty overlap with $K$ is $R_\ell$. Hence we define $K_j := R_{l+j} \cap K$, $j \ge 0$. Note that

$$\bigcup_{j \ge 0} K_j \subset K.$$

The left-hand side of the above is a finite union and equality of sets does not hold in general. What is missing in $\bigcup_{j\ge 0} K_j$ is that many small disks centered at some $\frac{1}{a}$ where $a \in Z(g)$ are not included in $\bigcup_{j\geq 0} K_j$. To see the pattern of estimation, we apply Proposition 4.11 to $K_0$ which is $R_\ell$. Thus for $x \in K_0$ we have

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \frac{1}{g(1/x)} - \sqrt{2\pi n}\sum\left\{\frac{J(a;nx)}{\phi^n(ax)} : a \in Z(g), |a| \leq r_\ell\right\} + O(n^{1-3\alpha})$$

Next, the summation is broken into three parts $\sigma_1 + \sigma_2 - \sqrt{2\pi n}\frac{J(\beta_i,nx)}{\phi^n(\beta_i x)}$, where

$$\sigma_1 := -\sqrt{2\pi n}\sum\left\{\frac{J(\alpha;nx)}{\phi^n(\alpha x)} : \alpha \text{ dominant}, \alpha \neq \beta_i, |\alpha| \leq r_\ell\right\}$$

and $\sigma_2$ is the summation over the remaining part of it. Thus for $x \in K_0$,

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \sigma_1 + \sigma_2 - \sqrt{2\pi n}\frac{J(\beta_i,nx)}{\phi^n(\beta_i x)} + O(n^{1-3\alpha}) \tag{6.1}$$

By part 5 of Lemma 5.15 each term in $\sigma_1$ is of $o(\phi^{-n}(\beta_i x))$. Hence

$$\sigma_1 = o(\phi^{-n}(\beta_i x)). \tag{6.2}$$

Let $\alpha$ be a zero that corresponds to a summand in $\sigma_2$. So $\alpha$ is non-dominant. By definition of $\sigma_2$, we get $|\alpha| \leq r_\ell$. Since $x \in D_{\beta_i}$, we have $|x| < \left|\frac{1}{\beta_i}\right|$ which equals $\frac{1}{r_\ell}$. Hence $|x| < \left|\frac{1}{\alpha}\right|$.

Case 1: $x \notin G_\alpha$. By equation (5.2) $x \in G_\alpha^-$. Hence $\left|\phi^{-1}(\alpha x)\right| \leq 1$.

Case 2: $x \in G_\alpha$. Since $\alpha$ is non-dominant, $\alpha$ lies in $\overline{G}_{\beta_j}$ for some dominant $\beta_j$. By Lemma 6.3 we get

$$\left|\phi^{-1}(\alpha x)\right| < \left|\phi^{-1}(\beta_j x)\right|$$

We know that when $x \in D_{\beta_i}$,

$$\left|\phi^{-1}(\beta_i x)\right| = \max\{\left|\phi^{-1}(\beta_m x)\right| : 1 \leq m \leq k\}$$

So we still get $\left|\phi^{-1}(\alpha x)\right| \leq \left|\phi^{-1}(\beta_i x)\right|$. Combining these two cases, we always have $\left|\phi^{-1}(\alpha x)\right| < \left|\phi^{-1}(\beta_i x)\right|$. Therefore,

$$\sigma_2 = o(\phi^{-n}(\beta_i x)). \tag{6.3}$$

Putting the results from equations (6.2) and (6.3) into equation (6.1) we get, for $x \in K_0$,

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = -\sqrt{2\pi n}\frac{J(\beta_i,nx)}{\phi^n(\beta_i x)} + o(\phi^{-n}(\beta_i x))$$

The argument works similarly for $x \in K_j, j \geq 1$. Hence for $x \in \bigcup_{j\geq 0} K_j$, we have

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = -\sqrt{2\pi n}\frac{J(\beta_i,nx)}{\phi^n(\beta_i x)} + o(\phi^{-n}(\beta_i x)).$$

Since $K \setminus \bigcup_{j\geq 0} K_j$ may possibly consists of small disks. It remains to study the behavior of $\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}}$ on any such disk. To this end we note that the number of zeros of $g$ contained in $D_{\beta_i} \setminus D(0;\frac{1}{\rho})$ is obviously finite. Let $D(1/a';\delta)$ be any such disk contained in $K \setminus \bigcup_{j\geq 0} K_j$. To apply Proposition 6.1 for $x \in D(1/a';\delta)$, we write $\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}}$ as

$$\frac{p_n(nx)}{(ex)^n/\sqrt{2\pi n}} = \tau_1(x) + \tau_2(x) - \frac{\sqrt{2\pi n}}{(ex)^n}\sigma_{a'}(x) + O(n^{1-3\alpha}),$$

where

$$\tau_1(x) = \frac{1}{x}(\frac{1}{(1/x)g(1/x)} - s_{a'}(\frac{1}{x}))(1 + O(\frac{1}{n}))$$

$$\tau_2(x) = -\sqrt{2\pi n}\sum\left\{\frac{J(a;nx)}{\phi^n(ax)} : a \in Z(g), |a| \le |a'|, a \ne a'\right\}.$$

Now $\tau_1(x)$ is obviously bounded in $D(1/a';\delta)$ since $s_{a'}(\frac{1}{x})$ is the singular part of $\frac{1}{(1/x)g(1/x)}$. Let $\alpha$ correspond to a summand in $\tau_2(x)$.

Case 1: $a' \notin \overline{G}_\alpha$. Since $|\alpha| \le |a'|$ and $\alpha \ne a'$, $\left|\frac{1}{\alpha}\right| \ge \left|\frac{1}{a'}\right|$. Using equation (5.2) we can obviously choose $\delta$ small enough so that for all $x \in D(1/a';\delta)$ we have $x \in G_\alpha^-$. So $\left|\phi^{-1}(\alpha x)\right| < 1$.

Case 2: $a' \in \overline{G}_\alpha$. Now $\alpha$ in non-dominant, there exists a dominant $\beta_j$ such that $\frac{1}{\alpha} \in \overline{G}_{\beta_j}$. Choosing $\delta$ small enough and carrying out a careful reasoning using Lemma 6.3 and the maximality of $\left|\phi^{-1}(x\beta_i)\right|$ we can show that for all $x \in D(1/a',\delta)$ , we have $\left|\phi^{-1}(\alpha x)\right| < \left|\phi^{-1}(x\beta_i)\right|$.

Combining these two cases we get $\left|\phi^{-1}(\alpha x)\right| < \left|\phi^{-1}(x\beta_i)\right|$. As a result, we obtain for $x \in D(1/a';\delta)$,

$$\tau_2(x) = o(\left|\phi^{-n}(x\beta_i)\right|).$$

Finally the term $\frac{\sqrt{2\pi n}}{(ex)^n}\sigma_{a'}(x)$ is $O(e^{6n\rho\delta})$ by Proposition 6.2. □

## 7. Zero Attractor and the Density of the Zeros

In our paper [**2**], we determined the limit points of the zeros of the Euler polynomials by means of the asymptotics and the zero density. Here, we separate out first the question of find the support of the zero density measure, which is, of course, the zero attractor. Then we determine the zero density by applying our general result in the appendix.

Proposition 7.1. *Let $f_n(x) = \sqrt{2\pi n}\, p_n(nx)/(xe)^n$. Then the following limits hold uniformly on compact subsets of the indicated domains:*

(1) *On the domain $A(1/r_0;\infty)$, $\lim_{n\to\infty}\frac{1}{n}\ln[f_n(x)] = 0$.*
(2) *On the domain $D_a \cap A(1/\rho;\infty)$ where $a$ is any dominant zero of $g$,* $\lim_{n\to\infty}\frac{1}{n}\ln[f_n(x)] = -\ln\phi(ax)$.

Proof. We use the asymptotic expansions for $p_n(nx)$ developed in the previous sections. □

To describe the zero attractor requires a closer examination of the boundary of each domain $D_a$ where $a$ is a dominant zero.

The boundary $\partial D_a$ where $a$ is a dominant zero of $g$ has several natural families: $\partial D_a \cap \partial D_0$ which is an "outer boundary" and a polygonal curve consisting of the line segments contained in $L_{a,b}$ where $b$ is another dominant zero of $g$. Note that $\partial D_a \cap \partial D_b$ is a subset of $\overline{D}_0$. It will be useful to subdivide $\partial D_a \cap \partial D_0$ into two connected components denoted by $\partial D_a^\pm$ that come from deleting $\{1/a\}$ from $[\partial D_a \cap \partial D_0]$.

LEMMA 7.2. *The zero attractor of the Appell polynomials* $\{p_n(nx)\}$ *must lie inside the compact set*

$$\bigcup \{\partial D_a : a \text{ is a dominant zero of } g\}.$$

PROOF. First, we let $x^*$ let in the infinite exterior of $D_0$. Recall that

$$\lim_{n\to\infty} \sqrt{2\pi n}\, p_n(nx)/(xe)^n = 1/g(1/x)$$

uniformly on compact subsets. If $x_{n_k}$ is a zero of $p_{n_k}(n_k x)$ and $x_{n_k} \to x^*$, then appealing to this limit we find that the limit must be 0 while the right-hand side is $1/g(1/x^*) \neq 0$. Secondly, suppose $x^*$ lies in the interior of $D_0$ but not on any boundary set $\partial D_a$, where $a$ is a dominant zero. By construction, $x$ will lie in the interior of one of the domains $D_b$, where $b$ is a dominant zero. Then $\lim_{n\to\infty} |\sqrt{2\pi n}\, p_n(nx)/(xe)^n|^{1/n} = |\phi(bx)|$ uniformly on compacta in the interior of $D_b$. By the same reasoning as before, $x^*$ cannot be a limit of zeros. □

The following Theorem is an immediate consequence of the above lemma together with the result of Sokal in section A.1 of the Appendix.

THEOREM 7.3. *Let* $\{p_n(x)\}$ *be an Appell family with generating function* $g(t)$. *Then the zero attractor of the normalized family* $\{p_n(nx)\}$ *is given by*

$$\bigcup \{\partial D_a : a \text{ is a dominant zero of } g\}.$$

*where* $D_a$ *is the domain given in Definition 5.8.*

PROOF. Let $a$ be any dominant zero of $g$ and let $x^* \in \partial D_a^{\pm}$. Let $\epsilon > 0$ be given. Then we find that

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} \right| = \begin{cases} 0, & x \in D(x^*;\epsilon) \setminus \overline{D}_0, \\ -\ln|\phi(ax)|, & x \in D(x^*;\epsilon) \cap D_0 \end{cases}$$

holds uniformly on compact subsets. Next suppose that $x^*$ lies on the line segment of the form $\partial D_a \cap \partial D_b$ where $D_b$ is a bordering domain of $D_a$. Again, we find that

$$\lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} \right| = \begin{cases} -\ln|\phi(ax)|, & x \in D(x^*;\epsilon) \cap D_a, \\ -\ln|\phi(bx)|, & x \in D(x^*;\epsilon) \cap D_b \end{cases}$$

which also holds uniformly on compact subsets. By Sokal's result [**5**] described in the appendix, we conclude that $x^*$ is in $\limsup Z(p_n)$ since there can be no harmonic function $v(x)$ on the disk $D(x^*;\epsilon)$ that satisfies the inequalities

$$\liminf_{n\to\infty} \ln \left| \frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} \right| \leq v(x) \leq \limsup_{n\to\infty} \ln \left| \frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} \right|.$$

This reasoning handles all $1/a$ where $a$ is a dominant zero of $g$. However, since the zero attractor must be a compact set and points in $D(1/a;\epsilon) \cap [\partial D_a \cap \partial D_0]$ lie in the zero attractor, we conclude that $1/a$ also lie in the attractor. □

THEOREM 7.4. *Let* $g(t)$ *be the generating function of the Appell family* $\{p_n(x)\}$. *Suppose* $a$ *and* $b$ *are distinct dominant zeros of* $g$.
*(1) The zero density measure on any proper subcurve of* $\partial D_a \cap \partial D_0$ *is the pull-back of the normalized Lebesgue measure on the unit circle under the conformal map* $\phi(ax)$ *where* $D_0$ *is the domain given in Definition 5.13.*
*(2) The zero density measure on any proper line segment of* $\partial D_a \cap \partial D_b$ *is a multiple of Lebesgue measure.*

Proof. For both parts, we can use the asymptotics given in Theorem 5.18.

For part (1), let $f_n(x) = \sqrt{2\pi n} g(1/x)p_n(nx)/(xe)^n$. Let $a$ be a dominant zero of $g$, and let $C$ be a proper subcurve of $\partial D_0 \cap \partial D_a^{\pm}$. Then there exists a neighborhood $U$ of $C$ such that $U \subset R_\rho \cap [(\mathbb{C} \setminus D_0) \cup D_a]$ so that the asymptotics in Theorem 5.18 can be written as

$$\frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} = \frac{1}{g(1/x)}\,(1 + O(1/n)) - \sqrt{2\pi n}\,\frac{J(a;nx)}{\phi(ax)^n} + O(n^{1-3\alpha}) + o(\Phi_{1,a}^n(x)),$$

where $\Phi_{1,a}(x) = \max\{1, |\phi^{-1}(ax)|\}$. Hence, by multiplying by $g(1/x)$), we find that $f_n(x)$ has the form:

$$f_n(x) = 1 + a_n(x)\phi(ax)^{-n} + e_n(x), \quad a_n(x) = -\sqrt{2\pi n}\, g(1/x)J(a;nx),$$

where

$$e_n(x) = \begin{cases} o(1), & x \in U \cap (\mathbb{C} \setminus D_0), \\ o(\phi(ax)^{-n}), & x \in U \cap (D_a \cap R_\rho). \end{cases}$$

Since $\phi(ax)$ is conformal in the disk $D(0;1/|a|)$, we may apply Theorem A.1 from the Appendix Section A.1 on the density of zeros.

Let $a$ and $b$ be two distinct dominant zeros of $g$ such that $\partial D_a \cap \partial D_b$ is nonempty. On $D_a \cap D_b \cap R_\rho$, the asymptotics in Theorem 5.18 can be written as

$$\begin{aligned}\frac{p_n(nx)}{(xe)^n/\sqrt{2\pi n}} &= \frac{1}{g(1/x)}\,(1 + O(1/n)) - \sqrt{2\pi n}\left(J(a;nx)\frac{1}{\phi(ax)^n} + J(b;nx)\frac{1}{\phi(bx)^n}\right. \\ &\quad \left. + \sum\{J(a';nx)\frac{1}{\phi(a'x)^n} : a' \text{ dominant zero}, a' \neq a, b\}\right) + O(n^{1-3\alpha}) + o(\Phi(x)^n) \\ &\quad \frac{1}{g(1/x)}\,(1 + O(1/n)) - \sqrt{2\pi n}\left(J(a;nx)\frac{1}{\phi(ax)^n} + J(b;nx)\frac{1}{\phi(bx)^n}\right) \\ &\qquad + O(n^{1-3\alpha}) + o(\Phi_{a,b}^n(x)),\end{aligned}$$

where $\Psi_{a,b}(x) = \max\{1/|\phi(ax)|, 1/|\phi(bx)|\}$.

Let $L$ be a proper line segment of the intersection $\partial D_a \cap \partial D_b$. Let $U$ be a neighborhood of $L$ so both $|\phi(ax)| < 1$ and $|\phi(bx)| < 1$ for $x \in U$. On the intersection $U \cap R_\rho$, we work with a different normalization than before:

$$T_n(x) = -\frac{\phi(ax)^n}{\sqrt{2\pi n}(xe)^n J(a;nx)}\, p_n(nx).$$

Note that in this normalization the term that contains $\phi(ax)^{-n}$ becomes the constant 1 for $T_n(x)$. Of course, this new normalization has exactly the same zeros as $p_n(nx)$ in $U$ so the zero density is unchanged. Then we find that

$$T_n(x) = 1 + a_n(x)\psi(x)^n + e_n(x),$$

where

$$\psi(x) = \frac{\phi(ax)}{\phi(bx)} = \frac{a}{b}e^{(b-a)x}, \quad a_n(x) = \frac{J(b;nx)}{J(a;nx)},$$

and

$$e_n(x) = -\frac{\phi(ax)^n}{\sqrt{2\pi n}J(a;nx)}\left(O(n^{1-3\alpha}) + o(\Phi_{a,b}^n(x))\right).$$

On $U$, we have that $|\phi(ax)|^n \le \Phi^n_{a,b}(x) = \max\{1, |\psi(x)|^n\}$; while on $D_a \cap U$, $|\psi(x)| < 1$ and on $D_b \cap U$, $|\psi(x)| > 1$. This allows us to write $e_n(x)$ as

$$e_n(x) = \begin{cases} o(\psi(x)^n), & x \in D_a \cap U, \\ \\ o(1), & x \in D_b \cap U. \end{cases}.$$

By construction, $\phi(ax)/\phi(bx) = \frac{a}{b} e^{(b-a)x}$ is a conformal map on $U \cap R_\rho$ that maps $L$ onto an arc of the unit circle. By Corollary A.3 in the Appendix section A.1, the result follows. □

We close with several examples that illustrate the main constructions in the paper.

Example 7.5. Let $g(t)$ be an entire function whose minimal modulus zero $a_1 = 1$ such that all its other zeros $a$ satisfy $1/|a| < W(e^{-1}) \simeq 0.27846$. Then the zero attractor for the associated Appell polynomials coincide with the classical Szegö curve in Figure 3.

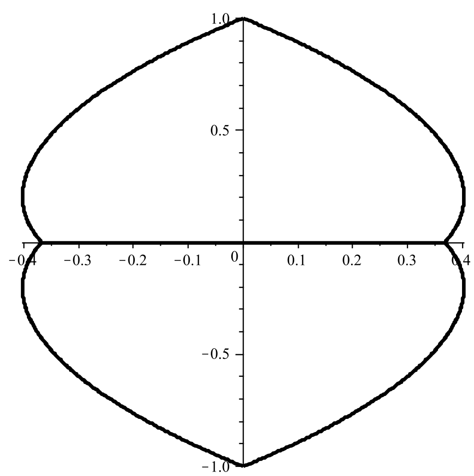


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Figure 4. Zero Attractor for Taylor polynomials of $\cos(x)$

Example 7.6. The higher order Euler polynomials $E_n^{(m)}(x)$, where $m \in \mathbb{Z}^+$, have generating function $g(t) = (e^t + 1)^m / 2^m$; while the higher order Bernoulli polynomials $B_n^{(m)}(t)$ have generating function $g(t) = (e^t - 1)^m / t^m$. Then their zero attractors are independent of $m$ and coincide with a scaled version of the zero attractor for the Taylor polynomials for $\cos(x)$, see Figure 4.

Example 7.7. The zero attractor for the Appell polynomials associated with generating function $g(t) = J_0(t)$, where $J_0(t)$ is the zero-th order Bessel function, is a scaled version of the zero attractor for the Taylor polynomials for $\cosh(x)$, see Figure 1. Here the minimal modulus zeros of $J_0(t)$, $a = \pm 2.404825558$, are the only dominant zeros and all the zeros of $J_0(t)$ lie on the real axis.

Example 7.8. Let $g(t) = (t-1)(t^2+2)$. See Figure 5 for its zero attractor and zeros for degree 400.

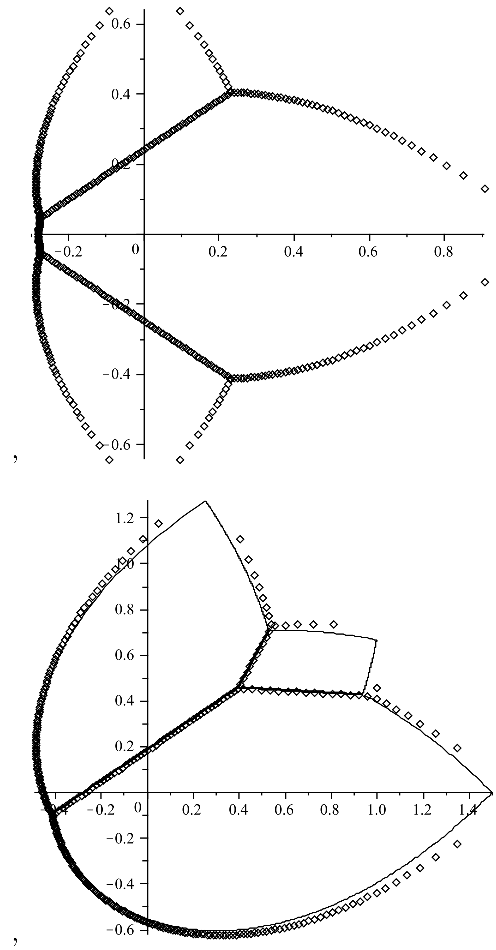


FIGURE 5. (a) Zeros for degree 400 polynomial with generating function $g(t) = (t-1)\,(t^2+2)$; (b) Zero Attractor with polynomial zeros

EXAMPLE 7.9. Consider the Appell polynomials with generating function

$$g(t) = (t - 1/a)(t - 1/b)(t - 1/c), \text{ with } a = 1.2e^{i3\pi/16}, b = 1.3e^{i7\pi/16}, c = 1.5.$$

In this case, all three roots of $g(t)$ are dominant. See Figures 6 and 7.

These last two examples both illustrate the following general fact. We assume that the generating function $g(t)$ has exactly three dominant zeros $a$, $b$, and $c$. Then the three lines determined by $|\phi(ax)| = |\phi(bx)|$, $|\phi(ax)| = |\phi(cx)|$, and $|\phi(bx)| = |\phi(cx)|$ have a common intersection point, a so-called "triple point." This follows by interpreting the lines as the boundary between the change of asymptotics of the Appell polynomial family; that is, the boundaries of the domains $D_a$, $D_b$, and $D_c$.

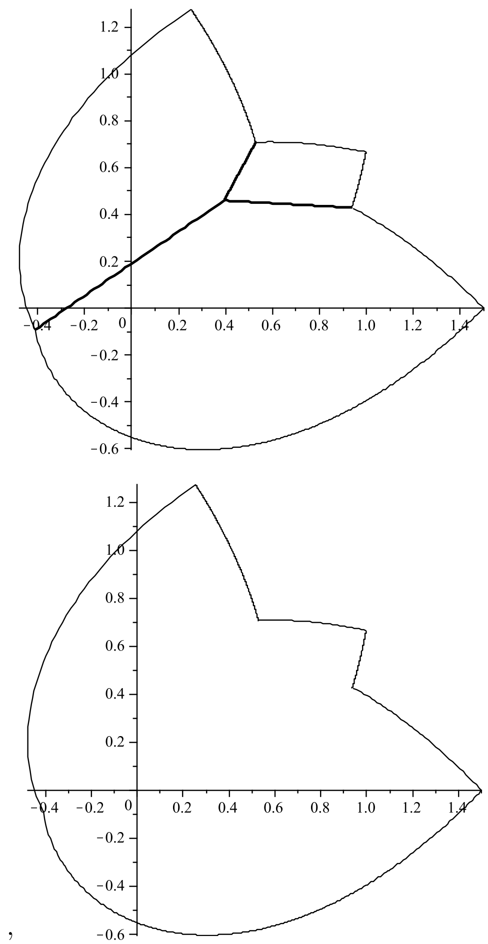


FIGURE 6. (a) Zero Attractor only, for generating function $g(t) = (t - 1/a)(t - 1/b)(t - 1/c)$, $a = 1.2e^{i3\pi/16}$, $b = 1.3e^{i7\pi/16}$, $c = 1.5$; (b) Boundary of the Domain $D_0$.

## Appendix A. Density of Zeros

**A.1. Introduction.** We generalize the density result for the zeros of the Euler polynomials in [**2**] to highlight how the asymptotic structure of the polynomial family may determine the density of its zeros.

Let $\psi(x)$ be an analytic function on a domain $D \subset \mathbb{C}$ that is conformal on $D$. We write $\zeta = \psi(x)$. We sometimes write $x(\zeta)$ for $x = \psi^{-1}(\zeta)$.

We assume that there exists $\epsilon_0 > 0$ and $0 \leq \alpha < \beta \leq 2\pi$ so that the annular sector

$$S = \{\rho e^{i\theta} : \rho \in [1 - \epsilon_0, 1 + \epsilon_0], \theta \in [\alpha, \beta]\} \tag{A.1}$$

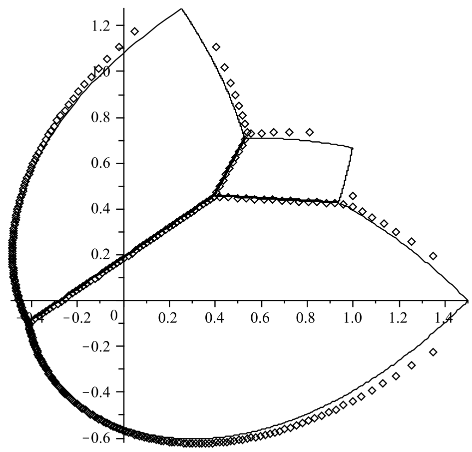

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FIGURE 7. Zeros for degree 400 polynomial together with the Zero Attractor, for generating function $g(t) = (t-1/a)(t-1/b)(t-1/c)$, $a = 1.2e^{i3\pi/16}$, $b = 1.3e^{i7\pi/16}$, $c = 1.5$

lies in the image $\psi(D)$. Next we define two subsectors of $S$ as

$$\begin{aligned} S_+ &= \{\rho e^{i\theta} : \rho \in [1-\epsilon_0, 1), \theta \in [\alpha, \beta]\} \\ S_- &= \{\rho e^{i\theta} : \rho \in (1, 1+\epsilon_0], \theta \in [\alpha, \beta]\}. \end{aligned}$$

Let $C$ be the unimodular curve $\psi^{-1}(\{e^{i\theta} : \theta \in [\alpha, \beta]\})$, so $|\phi(x)| = 1$ for $x \in C$. By construction, $C$ is smoothly parametrized as $x(e^{i\theta})$ for $\theta \in [\alpha, \beta]$. Of course, we have $\psi^{-1}(S) = \psi^{-1}(S_-) \cup C \cup \psi^{-1}(S_+)$ as a disjoint union.

Let $\{T_n(x)\}$ be a sequence of analytic functions on $\psi^{-1}(S)$ where we assume that the analytic functions satisfy the basic asymptotic relation:

$$T_n(x) = 1 + a_n(x)\psi(x)^{c_n} + e_n(x), \tag{A.2}$$

where $\{c_n\}$ is a increasing unbounded sequence of positive numbers, $\delta > 0$ is a constant so that $|a_n(x)| \geq \delta$, and $|a_n(x)| = \exp[o(c_n)]$, uniformly on $\psi^{-1}(S)$. The term $e_n(x)$ satisfy the following estimates uniformly:

$$e_n(x) = \begin{cases} o(\psi(x)^{c_n}), & x \in S_+, \\ o(1), & x \in S_-. \end{cases}$$

In the sequel, we may assume either form for $e_n(x)$ if $x$ lies on the common boundary $C$ of the two regions $S_\pm$ that is, $|\psi(x)| = 1$.

Let $Z_n$ be the set of all zeros of $T_n$ that lie in $\psi^{-1}(S)$, which we assume is finite for all $n$. For $[\gamma_1, \gamma_2] \subset (\alpha, \beta)$, let

$$N_n(\gamma_1, \gamma_2) = \#\{x \in Z_n : \arg x \in [\gamma_1, \gamma_2]\}. \tag{A.3}$$

Choose $\epsilon > 0$ so $3\epsilon < \epsilon_0$. By the Argument Principle, we find that

$$N_n(\gamma_1, \gamma_2) = \frac{1}{2\pi i} \int_\Gamma \frac{\frac{d}{d\zeta} T_n(x(\zeta))}{T_n(x(\zeta))} \, d\zeta$$

where $\Gamma$ is the boundary of the sector $\{\rho e^{i\theta} : \rho \in [1-\epsilon, 1+\epsilon], \theta \in [\gamma_1, \gamma_2]\}$. The closed contour $\Gamma$ naturally has four parts of the form $\Gamma_{1\pm\epsilon}$ and $\Gamma_{\gamma_j}$, $j = 1, 2$ where

$$\begin{aligned} \Gamma_{1\pm\epsilon} &= \{(1\pm\epsilon)e^{i\theta} : \theta \in [\gamma_1, \gamma_2]\}, \\ \Gamma_\gamma &= \{\rho e^{i\gamma} : \rho \in [1-\epsilon, 1+\epsilon]\}. \end{aligned}$$

THEOREM A.1. *Let $\alpha < \gamma_1 < \gamma_2 < \beta$, and let $N_n(\gamma_1, \gamma_2)$ denote the number of zeros of $T_n(x)$ whose arguments lie in $[\gamma_1, \gamma_2]$, given in equation (A.3). Then*

$$\lim_{n\to\infty} \frac{N_n(\gamma_1, \gamma_2)}{c_n} = \frac{\gamma_2 - \gamma_1}{2\pi};$$

*that is, the image of the zero density under $\psi$ is Lebesgue measure on an arc of the unit circle.*

We need to recall the notions of lim sup and lim inf of a sequence $\{X_n\}$ of compact sets in the complex plane. Now $x^* \in \limsup X_n$ if for every neighborhood $U$ of $x^*$, there exists a sequence $x_{n_k} \in X_{n_k} \cap U$ that converges to $x^*$ while $x^* \in \liminf X_n$ if for every neighborhood $U$ of $x^*$, there exists an index $n^*$ and a sequence $x_n \in X_n \cap U$, for $n \geq n^*$ that converges to $x^*$. It is known that if the $\liminf X_n$ and $\limsup X_n$ agree and are uniformly bounded, then the sequence $\{X_n\}$ converges in the Hausdorff metric. Consequently, when the density result holds, then the $\liminf Z(T_n)$ must agree with $\limsup Z(T_n)$. Hence, we have the following:

COROLLARY A.2. *As compact subsets of $\psi^{-1}(S)$, $Z(T_n)$ converges to the unimodular curve $C$ in the Hausdorff metric.*

Although we can determine the zero attractor and the zero density completely in the above framework, it is conceptually useful to have the result of Sokal that gives a description of the support of the zero density measure.

**[Sokal]** [**5**]: *Let $D$ be a domain in $\mathbb{C}$, and let $z_0 \in D$. Let $\{g_n\}$ be analytic functions on $D$, and let $\{a_n\}$ be positive real constants such that $\{|g_n|^{a_n}\}$ are uniformly bounded on the compact subsets of $D$. Suppose that there does* not *exist a neighborhood $V$ of $z_0$ and a function $v$ on $V$ that is either harmonic or else identically $-\infty$ such that $\liminf_{n\to\infty} a_n \ln|g_n(z)| \leq v(z) \leq \limsup_{n\to\infty} a_n \ln|g_n(z)|$ for all $z \in V$. Then $z_0 \in \liminf Z(g_n)$.*

We can state the asymptotic form for $T_n(x)$ in a more symmetric form:

$$T_n(x) = \psi_0(x) + \sum_{k=1}^{N} a_{n,k}(x)\psi_k(x)^{c_n} + e_n(x)$$

where $N$ is fixed and the error term has the form

$$e_n(x) = o(\max\{\psi_k(x)^{c_n}, 0 \leq k \leq n\})$$

This version explains the asymmetry in the above setup where we have $\psi_0(x) = 1$ and the zeros accumulate along the curve $|\psi_0(x)| = |\psi(x)|$.

When the analytic arc $C$ is a straight line segment and $\psi(x)$ has the form $e^{ax+b}$, where $a$ and $b$ are constants, the Density Theorem yields:

COROLLARY A.3. *If the analytic arc $C$ is a straight line segment and $\psi(x)$ is of the form $e^{ax+b}$, where $a$ and $b$ are constants, then the zero density along the line segment $C$ is a multiple of Lebesgue measure.*

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