

POLYNOMIALS ASSOCIATED WITH PARTITIONS: THEIR ASYMPTOTICS AND ZEROS

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ABSTRACT. Let p_n be the number of partitions of an integer n . For each of the partition statistics of counting their parts, ranks, or cranks, there is a natural family of integer polynomials. We investigate their asymptotics and the limiting behavior of their zeros as sets and densities.

1. INTRODUCTION

The purpose of this paper is to survey several natural polynomial families associated with integer partitions focusing on their asymptotics and the limiting behavior of their zeros. Our principal families are

- (1) Taylor polynomials of the analytic function $P(x) = \prod_{n \geq 1} (1 - x^n)^{-1}$, the generating function of the partition numbers (Section 2).
- (2) Polynomials $F_n(x)$ associated with counting partitions in parts (Section 3).
- (3) Polynomials associated with the rank or crank of a partition (Section 4).

We introduce several definitions used throughout the paper.

Definition 1. Let $Z(q_n)$ denote the finite set of zeros of the polynomial q_n . Then the zero attractor \mathcal{A} of the polynomial sequence $\{q_n\}$ whose degrees go to ∞ is the limit of $Z(q_n)$ in the Hausdorff metric Δ on the non-empty compact subsets \mathcal{K} of \mathbb{C} .

We recall the standard:

Definition 2. The asymptotic zero distribution for a sequence $\{q_n\}$ of polynomials whose degrees go to ∞ is the weak*-limit of the normalized counting measures of their zeros $\frac{1}{\deg(q_n)} \sum \{\delta_z : z \in Z(q_n)\}$.

We single out a useful compromise from the obtaining the full asymptotic zero distribution.

Definition 3. We say that the arguments of the zeros of a polynomial family $\{q_n(x)\}$ whose degrees go to ∞ are uniformly distributed on the unit circle as $n \rightarrow \infty$ if the normalized counting measures $\frac{1}{\deg(n)} \sum \{\delta_{\arg z} : z \in Z(q_n)\}$ converge in the weak*-topology to normalized Lebesgue measure on the unit circle.

The following result of Erdős and Turán ([10], Theorem 1) will be used repeatedly throughout the paper to determine that the arguments of zeros are uniformly distributed.

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Let $q(x)$ be the polynomial $\sum_{k=0}^n a_k x^k$ of degree n with non-zero constant term $a_0 \neq 0$. For $0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$(1) \quad \left| \#\{z : \arg z \in [\theta_1, \theta_2], q(z) = 0\} - \frac{\theta_2 - \theta_1}{2\pi} n \right| < 16 \sqrt{n \ln \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{a_0 a_n}} \right)}.$$

2. TAYLOR POLYNOMIALS OF $P(x)$

Let p_k be the number of partitions of a positive integer k with $p_0 = 1$ by convention. The ordinary generating function $P(x)$ for $\{p_k\}$ is

$$(2) \quad P(x) = \prod_{n \geq 1} \frac{1}{1 - x^n} = \sum_{k=0}^{\infty} p_k x^k.$$

A natural choice of polynomials associated with the partitions is simply the Taylor polynomials $s_n(x)$ of $P(x)$:

$$(3) \quad s_n(x) = \sum_{k=0}^n p_k x^k$$

since $P(x)$ is analytic in the open unit disk \mathbb{D} .

The asymptotics of these polynomials $s_n(x)$ depend on the classical result of the asymptotics of the partition numbers p_n :

$$(4) \quad p_n \Big/ \left[\frac{1}{4n\sqrt{3}} \exp \left(\pi \sqrt{\frac{2n}{3}} \right) \right] \rightarrow 1.$$

See either [1] or [4]. We first establish the limiting behavior of their zeros.

Theorem 1. (a) *The zero attractor of the Taylor polynomials $\{s_n(x)\}$ is the unit circle.*
(b) *The asymptotic zero density is Lebesgue measure on the unit circle.*

Proof. Recall the Eneström-Kakeya Theorem: If the coefficients of the polynomial $q(z) = \sum_{k=0}^n a_k z^k$ satisfy $a_n \geq a_{n-1} \geq \cdots \geq a_0 \geq 0$, then all the zeros of $p(z)$ lie in the closed unit disk (see [11], p. 136). Since the partition numbers are positive and increasing, the zeros of the Taylor polynomials $s_n(x)$ must lie in the closed unit disk $\overline{\mathbb{D}}$.

Next let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ be an analytic function with radius of convergence 1. To state the Jentzsch Theorem [10] concerning the zeros of the Taylor polynomials $t_n(x)$ of $f(x)$, recall that a is called a limit point of zeros of $t_n(x)$ if for every $\varepsilon > 0$ there are infinitely many indices n so $t_n(z_n) = 0$ with $|z_n - a| < \varepsilon$. Then the collection of all limit points of zeros of $t_n(x)$ must contain the unit circle.

Since $s_n(x)$ are the Taylor polynomials of the generating function $P(x)$ which is analytic and does not vanish in \mathbb{D} , no limit point of the polynomials $s_n(x)$ can lie inside \mathbb{D} since such a limit point must be a zero of $P(x)$. Since the radius of convergence of $P(x)$ is 1, we conclude that the limit points are exactly the unit circle. We conclude that the zero attractor is the unit circle since the all the zeros of $s_n(x)$ are bounded in modulus by 1.

For the polynomials $s_n(x)$, their constant terms are always 1 while their coefficients are all bounded above by p_n , so the right-hand side of inequality of Erdős-Turán (1) is dominated by $16 \sqrt{n \ln(n \sqrt{p_n})}$. Hence, we have the following limit by (4):

$$\left| \frac{1}{n} \#\{z : \arg z \in [\theta_1, \theta_2], q(z) = 0\} - \frac{\theta_2 - \theta_1}{2\pi} \right| < 16 \sqrt{\frac{1}{n} \ln(n \sqrt{p_n})} \rightarrow 0.$$

A compactness argument shows that the unit circle is the zero attractor. \square

Because of the non-negativity and monotonicity of the coefficients of $s_n(x)$ together with the subexponential growth of $s_n(1)$, both the zero attractor and the asymptotic zero distribution for $s_n(x)$ were quickly obtained. A more complete understanding of these polynomials, though, requires their asymptotics outside the unit disk. In general, it is very useful to have asymptotic expansions for a polynomial family throughout the complex plane. In [5], we obtained such expansions for the Euler and Bernoulli polynomials while in Section 3 we describe expansions for another partition polynomial family. Further, we note that the Euler and Bernoulli polynomial zeros are not uniformly distributed around the unit circle and that the zero distribution studied in Section 3 is more subtle than any of these examples.

Theorem 2. *Let $\delta > 0$ and $0 < \eta < 1/2$, then*

$$s_n(x) = \frac{x^{n+1}}{x-1} \frac{e^{a\lambda_n} \lambda_n^{-2}}{4\sqrt{3}} (1 + O_\delta(\lambda_n^{-\eta})),$$

where

$$(5) \quad a = \pi\sqrt{2/3}, \quad \lambda_n = \sqrt{n-1}/24$$

and the constant in the big oh term $O_\delta(\lambda_n^{-\eta})$ depends only on δ and holds uniformly for all x with $|x| \geq 1 + \delta$.

Proof. For any $0 < r < 1$, we have

$$p_n = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{P(\zeta)}{\zeta^{n+1}} d\zeta.$$

By summing over the above expression for the partition numbers p_n , we obtain an integral form for the Taylor polynomial:

$$s_n(x) = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{P(\zeta)}{\zeta} \left(\sum_{j=0}^n \left(\frac{x}{\zeta}\right)^j \right) d\zeta = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{P(\zeta)}{\zeta-x} \left(1 - \left(\frac{x}{\zeta}\right)^{n+1} \right) d\zeta.$$

Since $|x| \geq 1 + \delta$, we find, by using the Cauchy integral theorem, that

$$\frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{P(\zeta)}{\zeta-x} d\zeta = 0.$$

The integral for the Taylor polynomial $s_n(x)$ reduces to

$$(6) \quad s_n(x) = \frac{-x^{n+1}}{2\pi i} \oint_{|\zeta|=r} \frac{P(\zeta)}{\zeta-x} \zeta^{-n-1} d\zeta.$$

Next we define I_n as the integral:

$$(7) \quad I_n = \frac{1}{2\pi i} \oint_{|\zeta|=r} \frac{P(\zeta)}{\zeta-x} \zeta^{-n-1} d\zeta.$$

In particular, we must have:

$$(8) \quad s_n(x) = -x^{n+1} I_n.$$

Thus our goal is to find an asymptotic approximation for I_n . Our strategy follows very closely that of [4]. Consequently, we will adopt the same notation as Ayoub to avoid confusion. Not surprisingly, the methods come from a proof of the asymptotics of partition numbers originally by J. Uspensky and uses the functional equation of the modular function. Basically, the major contribution to the integral in (7) comes from the a small neighborhood of the strongest singularity $\zeta = 1$ of $P(\zeta)$.

We begin with the following well-known functional equation which is essential: For $\Re(\tau) > 0$,

$$(9) \quad P(e^{-2\pi\tau}) = \Psi(\tau)P(e^{-2\pi/\tau}),$$

where

$$\Psi(\tau) = \sqrt{\tau} \exp \left[\frac{\pi}{12} \left(\frac{1}{\tau} - \tau \right) \right].$$

Now we put $\zeta = e^{-2\pi\tau}$ with $d\zeta = e^{-2\pi\tau} 2\pi i d\phi$ where $\tau = \alpha - i\phi$, with $\alpha = \alpha(n) > 0$. Note that we shall choose α so that $\alpha \rightarrow 0$ as $n \rightarrow \infty$. The specific form of α will be made clear below. Using the functional equation, we write I_n as:

$$(10) \quad I_n = \int_{-1/2}^{1/2} \frac{P(e^{-2\pi\tau})}{e^{-2\pi\tau} - x} e^{2\pi n\tau} d\phi = J + \tilde{I}_n,$$

where

$$J = \int_{-1/2}^{1/2} \frac{\Psi(\tau)}{e^{-2\pi\tau} - x} e^{2\pi n\tau} d\phi, \quad \tilde{I}_n = \int_{-1/2}^{1/2} \frac{P(e^{-2\pi\tau}) - \Psi(\tau)}{e^{-2\pi\tau} - x} e^{2\pi n\tau} d\phi.$$

To estimate \tilde{I}_n , we break the interval into three parts; a neighborhood of origin, say $-\phi_0 \leq \phi \leq \phi_0$, and the remaining two segments from $-1/2$ to $-\phi_0$ and ϕ_0 to $1/2$. Choose $\phi_0 = \lambda\alpha$ and λ that satisfies $2\pi = \alpha(1 + \lambda^2)$, that is, $\phi_0 = (2\pi\alpha - \alpha^2)^{1/2}$.

We proceed as in [4] to get the estimates:

Lemma 1. (a) For $|\phi| \leq \phi_0$ we have

$$(11) \quad P(e^{-2\pi\tau}) - \Psi(\tau) = O(1).$$

(b) For $\phi_0 \leq \phi \leq \frac{1}{2}$ or $-\frac{1}{2} \leq \phi \leq -\phi_0$ we have

$$(12) \quad P(e^{-2\pi\tau}) - \Psi(\tau) = O(e^{\pi/(48\alpha)}).$$

Proof. Equation (9) is essential for the proof here. For details see equations (14) and (19) on page 150 of [4]. \square

We use this lemma to estimate \tilde{I}_n . Define $\tilde{I}_{n,1}$, $\tilde{I}_{n,2}$, and $\tilde{I}_{n,3}$ as:

$$\begin{aligned} \tilde{I}_n &= \left(\int_{-1/2}^{-\phi_0} + \int_{-\phi_0}^{\phi_0} + \int_{\phi_0}^{1/2} \right) \left(\frac{P(e^{-2\pi\tau}) - \Psi(\tau)}{e^{-2\pi\tau} - x} e^{2\pi n\tau} \right) d\phi \\ &= \tilde{I}_{n,1} + \tilde{I}_{n,2} + \tilde{I}_{n,3}. \end{aligned}$$

From equation (11),

$$\tilde{I}_{n,2} = O \left(\int_{-\phi_0}^{\phi_0} \left| \frac{1}{e^{-2\pi\tau} - x} \right| |e^{2\pi n\tau}| d\phi \right).$$

For $|x| \geq 1 + \delta$

$$\left| \frac{1}{e^{-2\pi\tau} - x} \right| \leq \frac{1}{|x| - |e^{-2\pi\tau}|} = \frac{1}{|x| - e^{-2\pi\alpha}} \leq \frac{1}{|x| - 1} \leq \frac{1}{\delta}.$$

Hence $\tilde{I}_{n,2} = O_\delta(e^{2\pi n\alpha})$, whereas from equation (12) $\tilde{I}_{n,3} = O_\delta(e^{2\pi n\alpha + \pi/(48\alpha)})$; and exactly the same estimate holds for $\tilde{I}_{n,1}$.

From equation (10), we have now shown the following:

Lemma 2.

$$(13) \quad I_n = J + O_\delta \left(e^{2\pi n\alpha + \pi/(48\alpha)} \right),$$

We use the functional equation (9) to obtain

$$J = \int_{-1/2}^{1/2} \frac{(\alpha - i\phi)^{1/2}}{e^{-2\pi\tau} - x} \exp\left(\frac{\pi}{12(\alpha - i\phi)} + 2\pi\left(n - \frac{1}{24}\right)(\alpha - i\phi)\right) d\phi.$$

For convenience, we put

$$(14) \quad m = 2\pi\left(n - \frac{1}{24}\right) = 2\pi\lambda_n^2.$$

We change variables $\phi = \alpha u$ to get

$$J = \alpha^{3/2} \int_{-1/(2\alpha)}^{1/(2\alpha)} \frac{(1 - iu)^{1/2}}{e^{-2\pi\alpha(1-iu)} - x} \exp\left(\frac{\pi}{12\alpha(1-iu)} + m\alpha(1-iu)\right) du.$$

To obtain an asymptotic approximation for J , we set the coefficients of $\frac{1}{1-iu}$ and $1-iu$ to be equal. Thus $\frac{\pi}{12\alpha} = m\alpha = \sigma$, and so

$$(15) \quad \alpha = \sqrt{\frac{\pi}{12m}}, \quad \sigma = \sqrt{\frac{\pi m}{12}},$$

where m was defined in equation (14). This is how α is made explicit. Consequently,

$$(16) \quad \begin{aligned} J &= \alpha^{3/2} \int_{-1/(2\alpha)}^{1/(2\alpha)} \frac{(1-iu)^{1/2}}{e^{-2\pi\alpha(1-iu)} - x} \exp\left[\sigma\left(\frac{1}{1-iu} + (1-iu)\right)\right] du \\ &= \alpha^{3/2} e^{2\sigma} \int_{-u_0}^{u_0} \frac{(1-iu)^{1/2}}{e^{-2\pi\alpha(1-iu)} - x} \exp[-\sigma g(u)] du, \end{aligned}$$

where

$$g(u) = \frac{u^2}{1-iu}, \quad u_0 = \frac{1}{2\alpha}.$$

Note that from equation (15) $\sigma = \sigma(n) \rightarrow \infty$ and $\alpha \rightarrow 0$ as $n \rightarrow \infty$.

To approximate J we follow ([9], page 91). We choose ε to lie in the interval $(1/3, 1/2)$. Write

$$(17) \quad \begin{aligned} J &= \alpha^{3/2} e^{2\sigma} \left[\int_{-u_0}^{-\sigma^{-\varepsilon}} + \int_{-\sigma^{-\varepsilon}}^{\sigma^{-\varepsilon}} + \int_{\sigma^{-\varepsilon}}^{u_0} \right] \frac{(1-iu)^{1/2} \exp[-\sigma g(u)]}{e^{-2\pi\alpha(1-iu)} - x} du \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Lemma 3. (a) Both J_1 and J_3 equal $\frac{\alpha^{3/2} e^{2\sigma} \sqrt{\pi}}{\sigma} O_{\delta}(\sigma^{1-3\varepsilon})$.

(b) $J_2 = \frac{\alpha^{3/2} e^{2\sigma} \sqrt{\pi}}{1-x} (1 + O_{\delta}(\sigma^{1-3\varepsilon}))$.

Proof. We estimate J_2 first. Note for $-\sigma^{-\varepsilon} \leq u \leq \sigma^{-\varepsilon}$ we have

$$\begin{aligned} (1-iu)^{1/2} &= 1 + O(\sigma^{-\varepsilon}) = 1 + O(\sigma^{1-3\varepsilon}) \\ \frac{1}{e^{-2\pi\alpha(1-iu)} - x} &= \frac{1}{1-x + O(\alpha)} = \frac{1}{1-x} \frac{1}{1 + O_{\delta}(\alpha)} \\ &= \frac{1}{1-x} (1 + O_{\delta}(\alpha)) = \frac{1}{1-x} (1 + O_{\delta}(\sigma^{-1})) = \frac{1}{1-x} (1 + O_{\delta}(\sigma^{1-3\varepsilon})), \\ g(u) &= \frac{u^2}{1-iu} = u^2 + O(\sigma^{-3\varepsilon}) \end{aligned}$$

so that

$$\exp[-\sigma g(u)] = \exp[-\sigma u^2](1 + O(\sigma^{1-3\varepsilon})).$$

Making the above substitutions, we find

$$(18) \quad J_2 = \alpha^{3/2} e^{2\sigma} \int_{-\sigma^{-\varepsilon}}^{\sigma^{-\varepsilon}} \frac{(1-iu)^{1/2} \exp[-\sigma g(u)]}{e^{-2\pi\alpha(1-iu)} - x} du$$

$$(19) \quad = \alpha^{3/2} e^{2\sigma} \int_{-\sigma^{-\varepsilon}}^{\sigma^{-\varepsilon}} \frac{\exp[-\sigma u^2]}{1-x} (1 + O_\delta(\sigma^{1-3\varepsilon})) du$$

$$(20) \quad = \frac{\alpha^{3/2} e^{2\sigma}}{1-x} \left(\int_{-\sigma^{-\varepsilon}}^{\sigma^{-\varepsilon}} e^{-\sigma u^2} du \right) (1 + O_\delta(\sigma^{1-3\varepsilon}))$$

Now

$$\int_{-\sigma^{-\varepsilon}}^{\sigma^{-\varepsilon}} e^{-\sigma u^2} du = \int_{-\infty}^{\infty} e^{-\sigma u^2} du - \left[\int_{-\infty}^{-\sigma^{-\varepsilon}} + \int_{\sigma^{-\varepsilon}}^{\infty} \right] e^{-\sigma u^2} du.$$

It is not hard to see that since $\int_{-\infty}^{\infty} e^{-\sigma u^2} du = \sqrt{\pi}/\sqrt{\sigma}$

$$\left[\int_{-\infty}^{-\sigma^{-\varepsilon}} + \int_{\sigma^{-\varepsilon}}^{\infty} \right] e^{-\sigma u^2} du = o(\sigma^{1-3\varepsilon})$$

so that

$$\int_{-\sigma^{-\varepsilon}}^{\sigma^{-\varepsilon}} e^{-\sigma u^2} du = \frac{\sqrt{\pi}}{\sqrt{\sigma}} (1 + o(\sigma^{1-3\varepsilon})).$$

Hence from equation (20) we get

$$\begin{aligned} J_2 &= \frac{\alpha^{3/2} e^{2\sigma}}{1-x} \frac{\sqrt{\pi}}{\sqrt{\sigma}} (1 + o(\sigma^{1-3\varepsilon})) (1 + O_\delta(\sigma^{1-3\varepsilon})) \\ &= \frac{\alpha^{3/2} e^{2\sigma}}{1-x} \frac{\sqrt{\pi}}{\sqrt{\sigma}} (1 + O_\delta(\sigma^{1-3\varepsilon})). \end{aligned}$$

Recall

$$J_3 = \alpha^{3/2} e^{2\sigma} \int_{\sigma^{-\varepsilon}}^{u_0} \frac{(1-iu)^{1/2} \exp[-\sigma g(u)]}{e^{-2\pi\alpha(1-iu)} - x} du.$$

We have the estimates

$$\begin{aligned} |J_3| &\leq \alpha^{3/2} e^{2\sigma} \int_{\sigma^{-\varepsilon}}^{u_0} \left| \frac{(1-iu)^{1/2} \exp[-\sigma g(u)]}{e^{-2\pi\alpha(1-iu)} - x} \right| du \\ &= \alpha^{3/2} e^{2\sigma} \int_{\sigma^{-\varepsilon}}^{u_0} \frac{(1+u^2)^{1/4} \exp[-\sigma \Re g(u)]}{|e^{-2\pi\alpha(1-iu)} - x|} du \\ (21) \quad &\leq \frac{\alpha^{3/2} e^{2\sigma}}{\delta} \int_{\sigma^{-\varepsilon}}^{u_0} (1+u^2)^{1/4} \exp\left[-\sigma \frac{u^2}{1+u^2}\right] du. \end{aligned}$$

Since $u^2/(1+u^2)$ is an increasing function of u , we have, for $u_0 \geq u \geq \sigma^{-\varepsilon}$, $u^2/(1+u^2) \geq \sigma^{-2\varepsilon}/(1+\sigma^{-2\varepsilon})$. This implies $\exp(-\sigma u^2/(1+u^2)) \leq \exp(-\sigma^{1-2\varepsilon}/(1+\sigma^{-2\varepsilon})) \leq \exp(-\sigma^{1-2\varepsilon}/2)$.

By assumption $1/3 < \varepsilon < 1/2$, so we find that $\exp(-\frac{\sigma^{1-2\varepsilon}}{2})$ is much smaller than $\sigma^{1-3\varepsilon}$.

Hence by the inequality (21) we get

$$J_3 = \frac{\alpha^{3/2} e^{2\sigma} \sqrt{\pi}}{\sqrt{\sigma}} o_\delta(\sigma^{1-3\varepsilon}).$$

Exactly the same estimate holds for J_1 . □

We now return to the proof of the Theorem. By the definition of J_1, J_2, J_3 (see equation (17)), we see that

$$J = \frac{\alpha^{3/2} e^{2\sigma} \sqrt{\pi}}{1-x \sqrt{\sigma}} (1 + O_\delta(\sigma^{1-3\epsilon})).$$

From equation (13)

$$I_n = \frac{\alpha^{3/2} e^{2\sigma} \sqrt{\pi}}{1-x \sqrt{\sigma}} (1 + O_\delta(\sigma^{1-3\epsilon})) + O_\delta(e^{2\pi n \alpha + \pi/(48\alpha)}).$$

To see the final result, we recall the equations (14), (5), and (15). It is convenient that we express everything in terms of λ_n which equals $\sqrt{n-1/24}$. Thus, with $a = \pi\sqrt{2/3}$,

$$\begin{aligned} \alpha &= \frac{1}{2\sqrt{6}\lambda_n}, & \sigma &= \frac{\pi\lambda_n}{\sqrt{6}}, \\ 2\sigma &= a\lambda_n. \\ 2\pi n \alpha + \pi/(48\alpha) &= \frac{\pi n}{\sqrt{6}\lambda_n} + \frac{\pi\lambda_n}{4\sqrt{6}} \\ &= \frac{5\pi\lambda_n}{4\sqrt{6}} + o(1) = \frac{5a\lambda_n}{8} + o(1). \end{aligned}$$

Since $e^{5a\lambda_n/8}$ is dominated by $e^{a\lambda_n}$, we have

$$I_n = \frac{e^{a\lambda_n} \lambda_n^{-2}}{(1-x)4\sqrt{3}} (1 + O_\delta(\lambda_n^{1-3\epsilon})).$$

By comparing with equation (8) and setting $\eta = 1 - 3\epsilon$, we find that the proof is complete. \square

3. POLYNOMIALS FOR PARTITIONS WITH PARTS

Let $p_k(n)$ denote the number of partitions of n with exactly k parts. Define the polynomials $F_n(x) = \sum_{k=1}^n p_k(n)x^k$, the *partition with parts polynomials*. They have generating function:

$$P(x, u) = \prod_{k \geq 1} \frac{1}{1-xu^k} = \sum_{n=1}^{\infty} F_n(x)u^n.$$

With $x = 1$, $P(1, u)$ reduces to the generating function $P(x)$ for the partition numbers. To calculate these polynomials, we make use of the recurrence $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$. and the fact that about half their coefficients are actually given by the partition numbers: $p_{n-k}(n) = p(k)$, $2k < n-1$.

It is also known that the coefficients of $F_n(x)$ are unimodal for n sufficiently large ([1], page 100). These polynomials are mentioned in [7] where it is pointed out that they have complex zeros. Unfortunately, these facts do not give a hint to the complexity of their zeros (see Figure 2b). In fact, Richard Stanley plotted the zeros of $F_{200}(x)$ and asked what happens at $n \rightarrow \infty$. The proofs of the following results are found in [6].

The asymptotics for $F_n(x)$ outside the unit disk can be found using the method of Darboux. We state:

Theorem 3. *On compact subsets K that lie in the open set $\{z : |z| > 1\}$, the polynomials $F_n(x)$ have the asymptotic form*

$$F_n(x) = x^n P\left(1, \frac{1}{x}\right) + O(|x|^{Cn}),$$

where $1/2 < C < 1$ and the big O term holds uniformly in the compact set K .

From these asymptotics, we can give a simple argument that there is no limit point of zeros outside the closed unit disk. Let $\delta > 0$ be given. Suppose $\{x_n\}$ is a sequence of zeros; that is, $F_n(x_n) = 0$, that converges to x^* , say, and that $|x_n| \geq 1 + \delta$ for all n . Then by Theorem 3,

$$0 = \frac{F_n(x_n)}{x_n^n} = P(1, 1/x_n) + O(|x_n|^{(C-1)n}).$$

Since $P(1, 1/x^*) \neq 0$, we obtain a contradiction since $C < 1$. Hence, the zero attractor must lie inside the closed unit disk \mathbb{D} .

We find that the arguments of the zeros of $F_n(x)$ are uniformly distributed around the unit circle by writing $F_n(x)$ as $xg_n(x)$ and applying the result of Erdős-Turán [10] (see equation (1)). Note that $g_n(1) = p_n$ and $g_n(x)$ is monic and $g_n(0) = 1$. We state this result formally as:

Theorem 4. *The arguments of the zeros of $\{F_n(x)\}$ are uniformly distributed on the unit circle as $n \rightarrow \infty$.*

Understanding the behavior of zeros inside the unit disk \mathbb{D} requires a more detailed analysis using the Hardy-Ramanujan circle method. A difficulty to overcome is that the functional equation of the modular function is unavailable for the generating function $P(x, u)$. An important first step in applying the circle method is to rewrite the generating function $P(x, u)$, for $|x| < 1$ fixed, in a neighborhood of a rational point $e^{2\pi i h/k}$ inside the unit disk \mathbb{D} where h and k are relatively prime integers. Write u as $e^{2\pi i(h/k + iz)}$ with $\Re(z) > 0$ small. The factorization below required careful estimates with L -functions:

$$\ln[P(x, e^{2\pi i(h/k + iz)})] = e^{w_{h,k}} e^{\Psi(z)} e^{j_{h,k}},$$

where

$$\begin{aligned} w_{h,k} &= \frac{1}{2k} \ln(1 - x^k) + \sum_{\ell, \ell \nmid k} \frac{x^\ell}{\ell} \frac{1}{e^{-2\pi i \ell h/k} - 1}, \quad (h, k) = 1, \\ \Psi(z) &= \frac{\text{Li}_2(x^k)}{2\pi k^2} \frac{1}{z}, \\ j_{h,k}(z) &= \frac{1}{2\pi i} \int_{-3/4 - i\infty}^{-3/4 + i\infty} Q_{h,k}(s) \Gamma(s) (2\pi z)^{-s} ds, \end{aligned}$$

where $\text{Li}_2(x)$ is the dilogarithm function given on \mathbb{D} as $\sum_{n=1}^{\infty} x^n/n^2$ (see [2], p. 102). and $Q_{h,k}(s)$ is defined by means of a series expansion for $\Re(s) \geq \sigma_0 > 1$:

$$Q_{h,k}(s) = \sum_{m \geq 1} \sum_{\ell \geq 1} \frac{x^\ell}{\ell} e^{2\pi i \ell m h/k} (\ell m)^{-s}$$

which admits an analytic continuation to \mathbb{C} with a unique singularity, a simple pole at $z = 1$.

Next we introduce the quantities needed for the asymptotic expansion for the polynomials $F_n(x)$:

$$I_k = \frac{1}{\sqrt{\pi}} \frac{1}{n^{3/4}} \left[\frac{\sqrt{\text{Li}_2(x^k)}}{k} \right]^{1/2} \exp \left(2\sqrt{n} \frac{\sqrt{\text{Li}_2(x^k)}}{k} \right).$$

Theorem 5. *Let K be a compact subset of the open upper unit disk. Then the partition polynomials $F_n(x)$ have the asymptotic form*

$$\begin{aligned} F_n(x) &= e^{w_{0,1}} I_1 + (-1)^n e^{w_{1,2}} I_{1,2} + e^{-2\pi i n/3} e^{w_{1,3}} I_3 \\ &\quad + e^{-4\pi i n/3} e^{w_{2,3}} I_3 + o(I_1 + I_2 + I_3). \end{aligned}$$

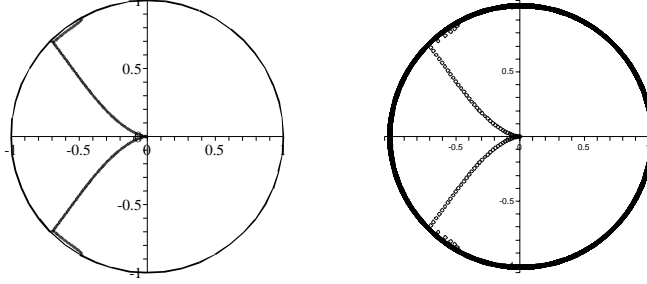


FIGURE 1. (a) Zero Attractor for Partitions with Parts Polynomial; (b) All Zeros of $F_n(x)$, $n = 10,000$

uniformly on K .

For simplicity, it is enough to give the zero attractor in the upper unit disk \mathbb{D}^+ since the coefficients of $F_n(x)$ are all real. Introduce the non-negative subharmonic functions $f_k(x) = \frac{1}{k} \Re[\sqrt{\text{Li}_2(x^k)}]$ for $|x| \leq 1$. Let $\mathcal{R}(k)$ be the subset of \mathbb{D}^+ given by:

$$\mathcal{R}(k) = \{z \in \mathbb{D}^+ : |z| \leq 1, f_k(z) > f_j(z), j \in \{1, 2, 3\}, j \neq k\}.$$

Using the same argument as above, we can easily show that there are no limit point of zeros that lies in any of the three regions $\mathcal{R}(1)$, $\mathcal{R}(2)$, or $\mathcal{R}(3)$. In fact, the zero attractor consists of the boundaries of these regions. See Figure 1a. To describe them, let $C_{k,\ell}$ be the curves given by $f_k(x) = f_\ell(x)$, $x \in \mathbb{D}^+$ and their subcurves $\gamma_{k,\ell} = C_{k,\ell} \cap \overline{\mathcal{R}(k)}$, where $1 \leq k < \ell \leq 3$.

Theorem 6. *The zero attractor of $F_n(x)$ in the upper half-plane consists of the unit semi-circle together with the three curves $\gamma_{k,\ell}$, $1 \leq k < \ell \leq 3$.*

A basic estimation of the number of zeros of $F_n(x)$ relative to the unit disk \mathbb{D} demonstrates a striking dichotomy.

Theorem 7. (a) *Let $\varepsilon > 0$. Then $\#\{z : F_n(x) = 0, 1 \leq |z| \leq 1 + \varepsilon\} = O(n)$.*
 (b) *Let K be any compact subset of the open unit disk \mathbb{D} . Then*

$$\#(Z(F_n) \cap K) = O(\sqrt{n}).$$

Corollary 1. *The asymptotic zero distribution for $\{F_n(x)\}$ is Lebesgue measure on the unit circle.*

Clearly, the standard definition for the asymptotic zero distribution (Definition 2) ignores any contribution from the zeros inside the unit disk \mathbb{D} . As a consequence, it is necessary to extend that definition:

Definition 4. *The asymptotic zero distribution of order α on a domain D for a sequence $\{q_n\}$ of polynomials whose degrees go to ∞ is the weak*-limit μ of the normalized counting measures of their zeros $\frac{1}{\deg(q_n)^\alpha} \sum \{\delta_z : z \in Z(q_n) \text{ and } z \in D\}$.*

For the sake of exposition, we will restrict our discussion of the order $1/2$ asymptotic zero distribution μ for $\{F_n(x)\}$ to the upper unit disk \mathbb{D}^+ . Since the zero attractor consists of three analytic curves, the support of the measure μ is supported exactly on those curves; in particular, it will be enough to describe μ in a neighborhood of each of them.

Theorem 8. *Let $1 \leq k < \ell \leq 3$. For each curve $\gamma_{k,\ell}$ in the zero attractor there exists a neighborhood $U_{k,\ell}$ of $\gamma_{k,\ell}$ and a conformal map $G_{k,\ell}$ on $U_{k,\ell}$ that maps $\gamma_{k,\ell}$ into the unit circle such that the asymptotic zero distribution of $\{F_n(x)\}$ in $U_{k,\ell}$ of order $1/2$ is the pull-back of Lebesgue measure on the unit circle.*

For the specifics of these mappings, see [6]. Their construction comes from the explicit asymptotic expansion of the polynomials $F_n(x)$.

4. RANK AND CRANK POLYNOMIALS

We now emphasize another way to look at the partition in parts polynomials $F_n(x)$ relative to their generating function to show its similarities with other generating functions that appear in partition theory. It is well known (see [2], p. 568) that

$$P(z, q) = 1 + \sum_{n=1}^{\infty} F_n(z)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}z^n}{(q; q)_n(zq; q)_n} = \frac{1}{(zq; q)_{\infty}};$$

namely, $F_n(x)$ is the coefficient of q^n of the generating function. With this viewpoint, there are several other natural polynomial families defined in terms of either Durfee squares (see [7]), ranks, or cranks. Here we are using the standard notations $(q)_n = (1-q)(1-q^2)\cdots(1-q^{n-1})$ and, more generally, $(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$; next, when $|q| < 1$, let $(a; q)_{\infty} = \lim_{n \rightarrow \infty} (a; q)_n$ and $(q)_{\infty}$ for $(1; q)_{\infty}$.

For a partition λ of n , its Durfee square is the largest square that lies inside its Ferrers graph (see [1], Chapter 2). The polynomials $d_n(z)$ for Durfee squares were introduced in [7] and are given in terms of their generating function:

$$D(z, q) = \sum_{n=1, k=1}^{\infty} d(n, k)q^n z^k = \sum_{n=1}^{\infty} d_n(z)q^n = \sum_{n=1}^{\infty} \frac{q^{n^2}z^n}{(q; q)_n^2}$$

where $d(n, k)$ is the number of partitions of n with a Durfee square of size k . Further, in [7] and [8], they conjecture that the associated polynomials $\{d_n(z)\}$ have only negative real zeros. Note that the Erdős-Turán result does not apply here since the degree of $d_n(z)$ is $\lfloor \sqrt{n} \rfloor$.

F. Dyson introduced the statistic of rank for a partition λ of n as the difference between its largest part and the number of its parts (see [1], p. 142). We introduce the rank polynomials $r_n(z)$ as follows. Consider their generating function:

$$R(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n)z^m q^n = \sum_{n=0}^{\infty} \sum_{m=-(n-1)}^{n-1} N_n(z)q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq; q)_n(z^{-1}q; q)_n}$$

where $N(m, n)$ is the number of partitions of n with rank m and $\{N_n(z)\}$ are symmetric Laurent polynomials. Set $r_n(z)$ to be the principal part of $N_n(z)$ and call it the *rank polynomial*:

$$r_n(z) = \sum_{m=0}^{n-1} N(m, n)z^m.$$

Let λ be the partition of n given as $\lambda_1 + \cdots + \lambda_s + 1 + \cdots + 1$, where there are exactly r 1's. Let $o(\lambda)$ be the number of parts $> r$. Then the crank of λ is λ_1 if $r = 0$ and $o(\lambda) - r$ if $r > 0$ [3]. Let $M(m, n)$ be the number of partitions of n whose crank is exactly m . Then

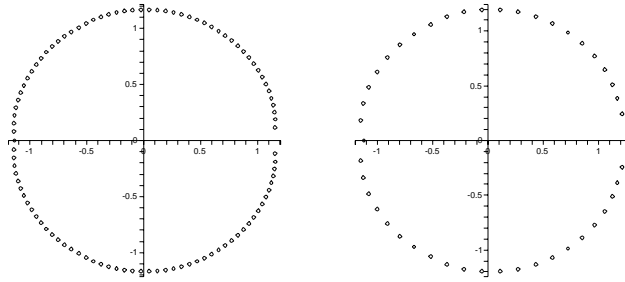


FIGURE 2. Zeros of the rank polynomial degree 100; crank polynomial degree 50

$C(z, q)$ is their generating function for $n > 1$ where

$$\begin{aligned} C(z, q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M_n(z) q^n \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - zq^n)(1 - z^{-1}q^n)} = \frac{(q)_{\infty}}{(z, q)_{\infty}(z^{-1}, q)_{\infty}}. \end{aligned}$$

Let $c_n(z)$ be the principal part of $M_n(z)$ and call it the *crank polynomial*:

$$c_n(z) = \sum_{m=0}^n M(m, n) z^m.$$

We can apply the Erdős-Turán result on the asymptotic distribution of the arguments of the zeros to the two families for the rank and crank polynomials since their coefficients are all non-negative, they are monic, and both quotients $r_n(1)/\sqrt{r_n(0)}$, $c_n(1)/\sqrt{c_n(0)}$ are bounded above by $\sqrt{p_n}$. We record this as a theorem:

Theorem 9. *The arguments of the zeros of both the rank and crank polynomials are uniformly distributed on the unit circle as $n \rightarrow \infty$.*

From explicit computation, we find that their zero attractor appears to be the unit circle (see Figure 2). It is very natural to attempt to extend the work for the partition in parts polynomials $F_n(x)$ to establish this conjecture.

Furthermore, it would be interesting to see how any partition polynomials in this paper fit into the statistical mechanics framework described in Vershik's paper [12].

5. SUMMARY

For all but one of the partition polynomial families, the unit circle has a dominant role. Their zero attractor is either equal or contains the unit circle while their asymptotic zero distribution involves Lebesgue measure on the unit circle. All this makes it even more intriguing to understand the meaning of the subtle two-scale asymptotics of the partition in parts polynomials $F_n(x)$ in Section 3.

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