Non-Commutative Harmonic Analysis
Semidirect Product Groups

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Problems and Goals

- Harmonic analysis on non-commutative Lie groups requires a computationally effective means to find their operator-valued Fourier transform and its inverse.
- We present a general scheme for a class of semidirect product Lie groups that include the euclidean motion group and other 3-dimensional Lie groups.
- We emphasize induced representations and integral operators.
Goals continued

• Tutorial approach to semidirect products and induced representations
• Reduce non-commutative analysis to well-known 2D Fourier Transform questions
Semidirect Product Examples

These groups are given by a one-parameter family of linear transformations on $\mathbb{R}^2$.

- Euclidean Motion Group: $\mathbb{R}^2 \rtimes S^1$
- Hyperbolic Motion Group: $\mathbb{R}^2 \rtimes \mathbb{R}$
- Scaling Group: $\mathbb{R}^2 \rtimes \mathbb{R}^*$
- Spiraling Group: $\mathbb{R}^2 \rtimes \mathbb{R}$
- Heisenberg Group: $\mathbb{R}^2 \rtimes \mathbb{R}$
Euclidean Motion Group: $\mathbb{R}^2 \rtimes S^1$

The group of rotations $\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ acts on $\mathbb{R}^2$.

These transformations preserve the form $x^2 + y^2$.

The orbit of a point is a circle.
Hyperbolic Motion Group: \( \mathbb{R}^2 \rtimes \mathbb{R} \)

The group of hyperbolic “rotations” \[
\begin{bmatrix}
\cosh t & \sinh t \\
\sinh t & \cosh t
\end{bmatrix}
\]
acts on \( \mathbb{R}^2 \).

These transformations preserve the form \( x^2 - y^2 \).

The orbit of point is a branch of a hyperbola.
Scaling Group: $\mathbb{R}^2 \times \mathbb{R}^*$

The group of scaling factors $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ acts on $\mathbb{R}^2$.

It is a non-unimodular group.

The orbit of a point is a ray.
Spiraling Group: \( \mathbb{R}^2 \times \mathbb{R} \)

The group

\[
e^t \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t \sin t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

acts on \( \mathbb{R}^2 \).

It is a non-unimodular group.

The orbit of point in \( \mathbb{R}^2 \) is a spiral.
## Summary of Groups

<table>
<thead>
<tr>
<th>Group</th>
<th>Geometric Transformation</th>
<th>Property</th>
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</table>
| \( \text{SE}(2) \) | \[
\begin{bmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{bmatrix}
\] | unimodular |
| \( \text{SE}(1, 1) \) | \[
\begin{bmatrix}
\cosh t & \sinh t \\
\sinh t & \cosh t
\end{bmatrix}
\] | unimodular |
| Scale(2) | \[
\begin{bmatrix}
r & 0 \\
0 & r
\end{bmatrix}
\] | non-unimodular |
| Spiral(2) | \( e^t \cos t \begin{bmatrix}1 & 0 \\0 & 1\end{bmatrix} + e^t \sin t \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix} \) | non-unimodular |
Commutative Fourier Transform

- \( \phi \in L^1(\mathbb{R}) \)

- \( \pi : \mathbb{R} \to S^1 \) – irreducible representation; \( \pi(t) = e^{i\lambda t} \)
  \[ \pi(gh) = \pi(g)\pi(h) \]

- **Fourier Transform** \( \hat{\phi}(\lambda) = \int_{\mathbb{R}} \phi(t) e^{i\lambda t} \, dt \)

- **Inversion Formula** \( \phi(t_0) = \int_{\mathbb{R}} e^{i\lambda t_0} \hat{\phi}(\lambda) \, d\lambda. \)
Non-commutative Fourier Transform

- $\phi \in L^1(G)$
- $\pi : G \rightarrow U(\mathcal{H})$ – irreducible representation
- Fourier Transform $\pi(\phi) = \int_G \phi(g)\pi(g) \, dg$
- Inversion Formula $\phi(g_0) = \int_{\hat{G}} \text{Tr}[\pi(g_0)^*\pi(\phi)] \, d\mu_G(\pi)$. 
Basic Elements

- **Dual of the Group:** $\hat{G}$ – set of the irreducible representations of $G$.

- **Character of Representation:** $\phi \mapsto \text{Tr}[\pi(\phi)]$

- $\text{Tr}[\pi(g_0) \ast \pi(\phi)]$

- **Plancherel Measure:** $\mu_G(\pi)$ on the dual space $\hat{G}$.

- **Reduced Dual Space:** $\hat{G}_\rho$ – the support of the Plancherel measure in $\hat{G}$.
Assumptions

- $N$ is a normal separable locally compact abelian subgroup which is a normal subgroup of $G$
  \[ N = \mathbb{R}^2 \]  in our examples

- $H$ is a separable locally compact subgroup of $G$ acting $N$.
  \[ H = \mathbb{R} \]  in our examples

- $G$ is a regular semidirect product group (Mackey theory)

- The action of $H$ on $\hat{N}$ is essentially free.

- Haar measure $m_{\hat{N}}$ is $H$-invariant.
  Relax this condition for non-unimodular examples
Induced Representation

- $\omega$: a multiplication character of $N$ of the semidirect product $N \rtimes H$.
  $\omega: N \rightarrow S^1$

- $\pi_\omega = \text{Ind}_N^G \omega$ acts on $L^2(H)$.

- $f \in L^2(H), \quad g_0 = (n_0, h_0), \quad h \in H$

- $[\pi_\omega(g_0)f](h) = \omega(h^{-1}n_0)f(h_0^{-1}h)$

- $\pi(\phi) = \int_G \phi(g)\pi(g)\,dg \quad \text{General Notation}$

- Integral Operator $[\pi(\phi)f](h_1) = \int_H K_\phi(h_1, h_2)f(h_2)\,dh_2$
Notation for Kernel Function

- **Partial Fourier Transform relative to** $N$ \( [\mathcal{F}_N \phi](\omega| h) \)

- $\phi \in L^1(N \rtimes H)$, write $\phi(n, h)$; $\omega \in \hat{N}$

- take the Fourier transform relative to $N$ and hold $h \in H$ fixed

- $N = \mathbb{R}^2$ in our examples

- $h \cdot \omega$ means $\omega(h^{-1}v)$ where $v \in \mathbb{R}^2$, $h \in H$

- \( [\mathcal{F}_N \phi](h \cdot \omega| h) \)
Kernel Function for $\pi_\omega$

- $K_\phi(h_1, h_2) = \int_N \phi(n, h_1 h_2^{-1}) \omega(\theta(h_1)^{-1} n) \, dn$

Fourier Transform relative to $N$

- $K_\phi(h_1, h_2) = [\mathcal{F}_N \phi](h_1 \cdot \omega \mid h_1 h_2^{-1})$

- $K_\phi(h, h) = [\mathcal{F}_N \phi](h \cdot \omega \mid 1_H)$

- $\text{Tr} [\pi(\phi)] = \int_H K_\phi(h, h) \, dh = \int_H [\mathcal{F}_N \phi](h \cdot \omega \mid 1_H) \, dh$
Plancherel Formula

\[ \hat{G}_\rho = \hat{N}/H = \hat{\mathbb{R}^2}/H \]

\[ m\hat{N} = m\hat{\mathbb{R}^2} = m\hat{\mathbb{R}^2}/H m_H \]

\[ \phi(1_G) = \phi(1_N, 1_H) = \int_{\hat{N}} [\mathcal{F}_N \phi](\lambda \mid 1_H) d\lambda \]

\[ = \int_{\hat{N}/H} \int_H [\mathcal{F}_N \phi](h \cdot \omega \mid 1_H) dh d\omega \]
Convolution

\[ K_{\phi_1 \ast \phi_2}(h_1, h_2) = \int_H K_{\phi_1}(h_1, h) K_{\phi_2}(h, h_2) \, dh \]
SE(2) Example

- $\hat{G}_\rho = \hat{N}/H$; $N = \mathbb{R}^2$, $H = S^1$;  
  $\mathbb{R}^2/S^1$ – space of concentric circles with radius $r$ 
  write $\pi_r$, $r > 0$ for corresponding irreducible

- $H$ acts essentially freely on $\hat{N}$

- $\hat{N} = \hat{N}/H \times H$

- $(r, e^{i\theta}) \leftrightarrow re^{i\theta}$

- $m_{\hat{N}} = m_{\hat{N}/H} m_H$
• Plancherel measure $\mu_G$ is a measure on $\hat{N}/H$

• $m_{\mathbb{R}^2} = \mu_G m_{S^1} = rd\theta \Rightarrow \mu_G = rdr$

• $\text{Tr}[\pi_r(\phi)] = \int_{-\pi}^{\pi} [\mathcal{F}_{\mathbb{R}^2}\phi](re^{i\theta} | 1) d\theta$

• $\phi(1_G) = \int_0^{\infty} \text{Tr}[\pi_r(\phi)] rdr = \int_0^{\infty} \int_{-\pi}^{\pi} [\mathcal{F}_{\mathbb{R}^2}\phi](re^{i\theta} | 1) d\theta rdr$
Hyperbolic Rotations

- The orbits of $\mathbb{R}$ in $\mathbb{R}^2$ are the two families of hyperbolas
  $x^2 - y^2 = r^2$ [$x > 0$ or $x < 0$; $r > 0$] and
  $y^2 - x^2 = r^2$ [$y > 0$ or $y < 0$; $r > 0$].

- Fix $r \neq 0$, corresponding orbit $O_{+,r}$
  $x^2 - y^2 = r^2 \leftrightarrow r \begin{bmatrix} \cosh t \\ \sinh t \end{bmatrix}$

- Fix $r \neq 0$, corresponding orbit $O_{-,r}$
  $y^2 - x^2 = r^2 \leftrightarrow r \begin{bmatrix} \sinh t \\ \cosh t \end{bmatrix}$

- $m_{\mathbb{R}^2} = r_+ dr_+ dt + r_- dr_- dt$

- $t \cdot \omega = t \cdot \omega \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \omega \left( \begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$
Hyperbolic Rotations Continued

\[ \text{Tr}[\omega_{+r}(\phi)] \]

\[ = \int_{\mathbb{R}} [\mathcal{F}_{\mathbb{R}^2} \phi] \left( \begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1_H \end{bmatrix} \right) dt \]

\[ \text{Tr}[\omega_{-r}(\phi)] \]

\[ = \int_{\mathbb{R}} [\mathcal{F}_{\mathbb{R}^2} \phi] \left( \begin{bmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix} \right) \begin{bmatrix} 1_H \end{bmatrix} \right) dt \]
Hyperbolic Rotations Continued

\[
\text{Tr}[\pi_{\pm,r}(\phi)] = \int_{\mathbb{R}} [\mathcal{F}_{\mathbb{R}^2}\phi](t\omega_{\pm,r}|1_H) \, dt
\]

Plancherel measure: \( \mu_G = r \, dr \)

\[
\phi(1_G) = \sum_{\pm} \int_{\mathbb{R}} \int_{\mathbb{R}} [\mathcal{F}_{\mathbb{R}^2}\phi](t\omega_{\pm,r}|1_H) \, dt \, r \, dr
\]
Scale Group

- Left Haar measure \( \frac{dx dy dr}{r^3} \)
- Right Haar measure \( \frac{dx dy dr}{r} \)
- Modular Function \( \Delta(x, y, r) = r^2 \)
- Orbit Space: \( \mathbb{R}^2 / \mathbb{R} \) identified with \( S^1 \)
- \( \pi_\omega \) acts on \( L^2(\mathbb{R}^+, dr/r) \) for \( \omega \in \widehat{\mathbb{R}^2} \)
- \( [\pi_\omega(n_0, r_0)f](r) = \omega(r^{-1}n_0)f(r_0^{-1}r) \)
- Kernel Function

\[
K(h_1, h_2) = \int_N \phi(n, h_1h_2^{-1}) \omega(h_1^{-1}n) \, dn \\
= [\mathcal{F}_{\mathbb{R}^2} \phi](h_1 \cdot \omega | h_1h_2^{-1})
\]
For $\theta \in \mathbb{S}^1$

$$\text{Tr}[\pi_\theta(\phi)] = \int_{\mathbb{R}^+} [\mathcal{F}_{\mathbb{R}^2}\phi](r \cdot \omega_\theta | 1_H) \frac{dr}{r}$$

$$= \int_{\mathbb{R}^+} [\mathcal{F}_{\mathbb{R}^2}\phi](re^{i\theta} | 1_H) \frac{dr}{r}$$

**Note:** $m_{\mathbb{R}^2} = r^2 d\theta \frac{dr}{r}$

$$\phi(1_G) \neq \int_{-\pi}^{\pi} \text{Tr}[\pi_\theta(\phi)] \, d\theta$$
\[
\phi(1_G) = \int_{-\pi}^{\pi} \int_{0}^{\infty} r^2 \mathcal{F} \Phi(re^{i\theta} | 1_H) \frac{dr}{r} d\theta
\]

\[
= \int_{-\pi}^{\pi} \int_{0}^{\infty} \mathcal{F} [\nabla^2 \Phi](re^{i\theta} | 1_H) \frac{dr}{r} d\theta
\]

\[
= \int_{-\pi}^{\pi} \text{Tr}[D\Phi] d\theta
\]

Non-unimodular Plancherel Formula

Big overhead to study scaling this way
Direct Inversion of Kernel Function

- $\pi_\omega(\phi)$ is an integral operator on $L^2(H)$

- **Kernel Function** $F(\phi; \omega)(h_1, h_2) = (\mathcal{F}_N\phi)(h_1 \cdot \omega \mid h_1 h_2^{-1})$

- **Proposed Fourier Transform** $F(\phi; \omega)(h_1, h_2)$

- $F(\phi; \omega)(h_1, h_2^{-1} h_1) = (\mathcal{F}_N\phi)(h_1 \cdot \omega \mid h_2)$

- Do not use trace of $\pi(\phi)$ in general

- $\phi(g_0) = \phi(n_0, h_0)$

  $$= \int_{\hat{N}/H} \int_H F(\phi, \omega)(h, h_0^{-1} h) (h \cdot \omega)(n_0) dh d\omega$$
Examples

• For $\text{SE}(2)$

\[
F(\phi, \omega_r)(\theta_1, \theta_2) = [\mathcal{F}_{\mathbb{R}^2}\phi](e^{i\theta_1}r \mid \theta_1 - \theta_2)
\]

\[
\phi(n_0, \theta_0) = \int_0^\infty \int_{-\pi}^{\pi} [\mathcal{F}_{\mathbb{R}^2}\phi](e^{i\theta}\omega_r \mid \theta_0)(e^{i\theta} \cdot \omega_r)(n_0) \, d\theta 
\]

• For hyperbolic rotations

\[
F(\phi, \omega_{\pm,r})(t_1, t_2) = [\mathcal{F}_{\mathbb{R}^2}\phi](t \cdot \omega_{\pm,r} \mid t_1 - t_2)
\]

\[
\phi(n_0, t_0) = \sum_{\pm} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\mathcal{F}_{\mathbb{R}^2}\phi](t\omega_{\pm,r} \mid t_0)(t\omega_{\pm,r})(n_0) \, dt 
\]
Examples

- For the scale group

\[ F(\phi, \omega_\theta)(r_1, r_2) = \left[ F_{\mathbb{R}^2_\phi} \right] (r_1 \cdot \omega_\theta | r_1 r_2^{-1}) \]

\[ \phi(n_0, r_0) = \int_{-\pi}^{\pi} \int_0^\infty \left[ F_{\mathbb{R}^2_\phi} \right] (r \omega_\theta | r_0) (r \omega_\theta)(n_0) \, d\theta \, rdr \]

* uniform method for semidirect products whether or not they are unimodular or non-unimodular
Computational Issues

- Kernel function approach coincides with Chirikjian-Kyatkin method for $\text{SE}(2)$
  More flexible – extends to other semidirect products

- Need to sample $\mathcal{F}_{\mathbb{R}^2}(\phi)(\omega | h)$ on $H$-orbits in $\mathbb{R}^2$ – non-uniform sampling problem

- For $\text{SE}(2)$ and $\text{Scale}(2)$, reduces to polar form of FT
  For hyperbolic rotations, sampling along family of hyperbolas
  Spiraling group and Heisenberg group – next to investigate