Lecture 2: Stochastic calculus

David Nualart

Department of Mathematics
Kansas University

Gene Golub SIAM Summer School 2016
Drexel University

The Wiener integral

- Let $B = \{B_t, t \geq 0\}$ be a Brownian motion.
- The integral of a step function $\varphi_t = \sum_{j=0}^{m-1} a_j \mathbf{1}_{(t_j, t_{j+1}]}(t) \in \mathcal{E}$ is defined by

$$\int_0^\infty \varphi_t dB_t = \sum_{j=0}^{m-1} a_j (B_{t_{j+1}} - B_{t_j})$$

- The mapping $\varphi \to \int_0^\infty \varphi_t dB_t$ from $\mathcal{E} \subset L^2(\mathbb{R}_+)$ to $L^2(\Omega)$ is linear and isometric :

$$E\left[\left(\int_0^\infty \varphi_t dB_t\right)^2\right] = \sum_{j=0}^{m-1} a_j^2 (t_{j+1} - t_j) = \int_0^\infty \varphi_t^2 dt = \|\varphi\|_{L^2(\mathbb{R}_+)}^2.$$

- $\mathcal{E}$ is a dense subspace of $L^2(\mathbb{R}_+)$. Therefore, the mapping

$$\varphi \to \int_0^\infty \varphi_t dB_t$$

can be extended to a linear isometry between $L^2(\mathbb{R}_+)$ and the Gaussian subspace of $L^2(\Omega)$ spanned by the Brownian motion.

White noise

- A white noise on $\mathbb{R}^m$ is a Gaussian centered family of random variables

$$\{W(A), A \in \mathcal{B}(\mathbb{R}^m), |A| < \infty\}$$

such that

$$E[W(A)W(B)] = |A \cap B|.$$

- The mapping $\mathbf{1}_A \to W(A)$ can be extended to a linear isometry from $L^2(\mathbb{R}^m)$ to the Gaussian space spanned by W :

$$\varphi \to \int_{\mathbb{R}^m} \varphi(x) W(dx).$$

- The Brownian motion B defines a white noise on $\mathbb{R}_+$ by setting

$$B(A) = \int_0^\infty \mathbf{1}_A(t) dB_t, \quad A \in \mathcal{B}(\mathbb{R}_+), \ |A| < \infty.$$

Progressively measurable processes

Let $\mathcal{F}_t$ be the filtration generated by the Brownian motion and the sets of probability sero.

Definition

We say that $u = \{u_t, t \geq 0\}$ is *progressively measurable* if for any $t \geq 0$, the restriction of u to $\Omega \times [0, t]$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable.

- Let $\mathcal{P}$ be the σ-field of sets $A \subset \Omega \times \mathbb{R}_+$ such that $\mathbf{1}_A$ is progressively measurable.
- We denote by $L^2(\mathcal{P})$ the Hilbert space $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, P \times \ell)$, where ℓ is the Lebesgue measure, equipped with the norm

$$\|u\|^2 = E\left(\int_0^\infty u_s^2 ds\right).$$

Stochastic integrals

- $u = \{u_t, t \geq 0$ is a *simple process* if

$$u_t = \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(t_j, t_{j+1}]}(t),$$

where $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ and ϕ_j are $\mathcal{F}_{t_j}$-measurable random variables such that $E(\phi_j^2) < \infty$.

- We define the stochastic integral of u as

$$I(u) := \int_0^\infty u_t dB_t = \sum_{j=0}^{n-1} \phi_j \left(B_{t_{j+1}} - B_{t_j} \right).$$

Proposition

The space $\mathcal{E}$ of simple processes is dense in $L^2(\mathcal{P})$.

Proof :

(i) If u belongs to $L^2(\mathcal{P})$, we define

$$u_t^{(n)} = n \int_{(t-\frac{1}{n})\vee 0}^{t} u_s ds = n \left(\int_0^t u_s ds - \int_0^t u_{(s-\frac{1}{n})\vee 0} ds \right),$$

The processes $u_t^{(n)}$ are continuous in $L^2(\Omega)$ and satisfy

$$\lim_{n\to\infty} E\left(\int_0^\infty \left| u_t - u_t^{(n)} \right|^2 dt \right) = 0.$$

Indeed, for each ω we have

$$\int_0^\infty \left| u(t,\omega) - u^{(n)}(t,\omega) \right|^2 dt \overset{n\to\infty}{\longrightarrow} 0$$

and we can apply the dominated convergence theorem because

$$\int_0^\infty \left| u^{(n)}(t,\omega) \right|^2 dt \le \int_0^\infty |u(t,\omega)|^2 dt.$$

(iii) Suppose $u \in L^2(\mathcal{P})$ is continuous in $L^2(\Omega)$. In this case, we can choose the approximating processes

$$u_t^{(n,N)} = \sum_{j=0}^{n-1} u_{t_j} \mathbf{1}_{(t_j,t_{j+1}]}(t),$$

where $t_j = \frac{jN}{n}$. The continuity in mean square of u implies that

$$\begin{aligned} E\left(\int_0^\infty \left|u_t - u_t^{(n)}\right|^2 dt\right) &\leq E\left(\int_N^\infty u_t^2 dt\right) \\ &\quad + N \sup_{|t-s|\leq N/n} E\left(|u_t - u_s|^2\right). \end{aligned}$$

This converges to zero if we first let $n \to \infty$ and then $N \to \infty$.

(i) *Linearity :*

$$\int_0^\infty (au_t + bv_t)\, dB_t = a \int_0^\infty u_t dB_t + b \int_0^\infty v_t dB_t.$$

(ii) *Zero mean :*

$$E\left(\int_0^\infty u_t dB_t\right) = 0.$$

In fact,

$$\begin{aligned} E\left(\int_0^\infty u_t dB_t\right) &= \sum_{j=0}^{n-1} E\left[\phi_j \left(B_{t_{j+1}} - B_{t_j}\right)\right] \\ &= \sum_{j=0}^{n-1} E[\phi_j]E[B_{t_{j+1}} - B_{t_j}] = 0. \end{aligned}$$

(iii) *Isometry property* :

$$E\left[\left(\int_0^\infty u_t dB_t\right)^2\right] = E\left(\int_0^\infty u_t^2 dt\right).$$

Proof : Set $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$E\left(\phi_i\phi_j\Delta B_i\Delta B_j\right) = \begin{cases} 0 & \text{if} \quad i \neq j \\ E\left(\phi_j^2\right)(t_{j+1} - t_j) & \text{if} \quad i = j \end{cases}$$

because if $i < j$ the random variables $\phi_i\phi_j\Delta B_i$ and ΔB_j are independent and if $i = j$ the random variables ϕ_i^2 and $(\Delta B_i)^2$ are independent. So, we obtain

$$\begin{aligned} E\left[\left(\int_0^\infty u_t dB_t\right)^2\right] &= \sum_{i,j=0}^{n-1} E\left(\phi_i\phi_j\Delta B_i\Delta B_j\right) = \sum_{i=0}^{n-1} E\left(\phi_i^2\right)(t_{i+1} - t_i) \\ &= E\left(\int_0^\infty u_t^2 dt\right). \end{aligned}$$

□

Proposition

The stochastic integral can be extended to a linear isometry :

$$I : L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, P \times \ell) \to L^2(\Omega).$$

Proof : This follows form the fact that $\mathcal{E}$ is dense in $L^2(\mathcal{P})$. $\square$.

- The stochastic integral has the following properties :

$$E\left[I(u)\right] = 0$$

and

$$E\left[I(u)I(v)\right] = E\left(\int_0^\infty u_s v_s ds\right).$$

Example

$$\boxed{\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T}$$

Proof : The process B_t being continuous in mean square, we can choose as approximating sequence

$$u_t^{(n)} = \sum_{j=1}^n B_{t_{j-1}} \mathbf{1}_{(t_{j-1},t_j]}(t),$$

where $t_j = \frac{jT}{n}$, and we obtain

$$\begin{aligned}\int_0^T B_t dB_t &= \lim_{n\to\infty} \sum_{j=1}^n B_{t_{j-1}} \left(B_{t_j} - B_{t_{j-1}}\right) \\ &= \frac{1}{2}\lim_{n\to\infty} \sum_{j=1}^n \left(B_{t_j}^2 - B_{t_{j-1}}^2\right) - \frac{1}{2}\lim_{n\to\infty} \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)^2 \\ &= \frac{1}{2}B_T^2 - \frac{1}{2}T.\end{aligned}$$

Indefinite stochastic integrals

For $u \in L^2(\mathcal{P})$, we define the stochastic process

$$\boxed{\int_0^t u_s dB_s := \int_0^\infty u_s \mathbf{1}_{[0,t]}(s) dB_s}$$

Properties :

1. *Additivity* : For any $a \leq b \leq c$ we have

$$\int_a^b u_s dB_s + \int_b^c u_s dB_s = \int_a^c u_s dB_s.$$

2. *Factorization* : If $a < b$ and F is a bounded and $\mathcal{F}_a$-measurable random variable, then

$$\int_a^b F u_s dB_s = F \int_a^b u_s dB_s.$$

3. *Martingale property :*

Proposition

Let $u \in L^2(\mathcal{P})$. The indefinite stochastic integral

$$M_t = \int_0^t u_s dB_s$$

is a square integrable martingale with respect to the filtration $\mathcal{F}_t$ and admits a continuous version.

Proof :

(i) We first prove the martingale property. Suppose that $u \in \mathcal{E}$ has the form

$$u_t = \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(t_j, t_{j+1}]}(t).$$

Then, for any $s \le t$,

$$\begin{aligned}
E\left(\int_0^t u_v dB_v | \mathcal{F}_s\right) &= \sum_{j=0}^{n-1} E\left(\phi_j (B_{t_{j+1}\wedge t} - B_{t_j \wedge t}) | \mathcal{F}_s\right) \\
&= \sum_{j=0}^{n-1} E\left(E\left(\phi_j (B_{t_{j+1}\wedge t} - B_{t_j \wedge t}) | \mathcal{F}_{t_j \vee s}\right) \mathcal{F}_s\right) \\
&= \sum_{j=0}^{n-1} E\left(\phi_j E\left(B_{t_{j+1}\wedge t} - B_{t_j \wedge t} | \mathcal{F}_{t_j \vee s}\right) \mathcal{F}_s\right) \\
&= \sum_{j=0}^{n-1} \phi_j (B_{t_{j+1}\wedge s} - B_{t_j \wedge s}) = \int_0^s u_v dB_v.
\end{aligned}$$

(ii) So, $M_t = \int_0^t u_s dB_s$ is an $\mathcal{F}_t$-martingale if $u \in \mathcal{E}$.

(iii) If $u^{(n)}$ is a sequence of simple processes that converge to u in $L^2(\mathcal{P})$, then for each $t \geq 0$,

$$\int_0^t u_s^{(n)} dB_s \xrightarrow{L^2(\Omega)} \int_0^t u_s dB_s.$$

Taking into account that the convergence in $L^2(\Omega)$ implies the convergence in $L^2(\Omega)$ of the conditional expectations, we deduce that $\int_0^t u_s dB_s$ is a martingale.

(iv) Let us prove the continuity. Let $u \in L^2(\mathcal{P})$ and consider a sequence of simple processes $u^{(n)}$ which converges to u in $L^2(\mathcal{P})$. By the continuity of the paths of the Brownian motion, the stochastic integral $M_t^{(n)} = \int_0^t u_s^{(n)} dB_s$ has a continuous trajectories. Then, Doob's maximal inequality yields for any $T \geq 0$

$$P\left(\sup_{0\leq t\leq T} |M_t^{(n)} - M_t^{(m)}| > \lambda\right) \leq \frac{1}{\lambda^2} E\left(|M_T^{(n)} - M_T^{(m)}|^2\right)$$
$$= \frac{1}{\lambda^2} E\left(\int_0^T \left|u_t^{(n)} - u_t^{(m)}\right|^2 dt\right) \overset{n,m\to\infty}{\longrightarrow} 0.$$

We can choose an increasing sequence of natural numbers $n_k, k = 1, 2, \ldots$ such that

$$P\left(\sup_{0\leq t\leq T} |M_t^{(n_{k+1})} - M_t^{(n_k)}| > 2^{-k}\right) \leq 2^{-k}.$$

The events $A_k := \left\{ \sup_{0 \le t \le T} |M_t^{(n_{k+1})} - M_t^{(n_k)}| > 2^{-k} \right\}$ verify

$$\sum_{k=1}^{\infty} P(A_k) < \infty.$$

Hence, Borel-Cantelli lemma implies that $P(\limsup_k A_k) = 0$. Set $N = \limsup_k A_k$. Then for any $\omega \notin N$ there exists $k_1(\omega)$ such that for all $k \ge k_1(\omega)$

$$\sup_{0 \le t \le T} |M_t^{(n_{k+1})}(\omega) - M_t^{(n_k)}(\omega)| \le 2^{-k}.$$

As a consequence, if $\omega \notin N$, the sequence $M_t^{(n_k)}(\omega)$ is uniformly convergent on $[0, T]$ to a continuous function $J_t(\omega)$. On the other hand, we know that for any $t \in [0, T]$, $M_t^{(n_k)}$ converges in L^2 to $\int_0^t u_s dB_s$. So, $J_t(\omega) = \int_0^t u_s dB_s$ almost surely, for all $t \in [0, T]$.

Since $T > 0$ is arbitrary, this implies the existence of a continuous version for M_t. $\square$

5. *Maximal inequality :*

$$P\left(\sup_{t\geq 0}|M_t| > \lambda\right) \leq \frac{1}{\lambda^2} E\left(\int_0^\infty u_t^2 dt\right).$$

and

$$E\left(\sup_{t\geq 0}|M_t|^2\right) \leq 4E\left(\int_0^\infty u_t^2 dt\right).$$

Quadratic variation of a martingale

Theorem

Let M_t be a continuous and square integrable martingale such that $M_0 = 0$. Then, there is a unique continuous and increasing process $\langle M \rangle_t$ such that $\langle M \rangle_0 = 0$ and the process

$$M_t^2 - \langle M \rangle_t$$

is a martingale.

- For each sequence of partitions $\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = t\}$ such that $|\pi^n| \to 0$, we have

$$\sum_{j=0}^{k_n - 1} (M_{t_{j+1}^n} - M_{t_j^n})^2 \xrightarrow{P} \langle M \rangle_t .$$

6. *Quadratic variation of the integral process*

Proposition

If u_t is a progressively measurable process such that $E\left(\int_0^t u_s^2 ds\right) < \infty$ for each $t > 0$, then

$$\left\langle \int_0^{\cdot} u_s dB_s \right\rangle_t = \int_0^t u_s^2 ds.$$

Proof : Since $\int_0^t u_s^2 ds$ is an increasing continuous process that vanishes at 0, it suffices to show that

$$\left(\int_0^t u_s dB_s \right)^2 - \int_0^t u_s^2 ds$$

is a martingale. This is easy if $u \in \mathcal{E}$, and in the general case follows form the density of $\mathcal{E}$ in $L^2(\mathcal{P})$. $\square$

6. *Stochastic integration up to a stopping time* :

Proposition

Suppose that $u \in L^2(\mathcal{P})$ and let τ is be a finite stopping time. Then the process $u\mathbf{1}_{[0,\tau]}$ also belongs to $L^2(\mathcal{P})$ and we have :

$$\int_0^\infty u_t \mathbf{1}_{[0,\tau]}(t) dB_t = \int_0^\tau u_t \, dB_t.$$

Proof :

(i) Suppose first that $u_t = F1_{(a,b]}(t)$, where $0 \leq a < b$, $F \in L^2(\Omega, \mathcal{F}_a, P)$ and τ takes values in a finite set $\{0 \leq t_1 \leq \cdots \leq t_n\}$. One one hand, we have

$$\int_0^\tau u_t \, dB_t = F(B_{b\wedge\tau} - B_{a\wedge\tau}).$$

On the other hand, the process $\mathbf{1}_{[0,\tau]}$ si simple because

$$\mathbf{1}_{(0,\tau]}(t) = \sum_{j=1}^n \mathbf{1}_{\{\tau \geq t_j\}} \mathbf{1}_{(t_{j-1},t_j]}(t)$$

and $\mathbf{1}_{\{\tau \geq t_j\}} = \mathbf{1}_{\{\tau \leq t_{j-1}\}^c} \in \mathcal{F}_{t_{j-1}}$. Therefore,

$$\begin{aligned}
\int_0^\infty u_t \mathbf{1}_{[0,\tau]}(t) dB_t &= F \sum_{j=1}^n \mathbf{1}_{\{\tau \geq t_j\}} \int_0^\infty \mathbf{1}_{(a,b]\cap(t_{j-1},t_j]}(t) dB_t \\
&= F \sum_{i=1}^n \mathbf{1}_{\{\tau = t_i\}} \int_0^\infty \mathbf{1}_{(a,b]\cap[0,t_i]}(t) dB_t \\
&= F(B_{b\wedge\tau} - B_{a\wedge\tau}).
\end{aligned}$$

(ii) For a finite stopping time τ, we approximate τ by the sequence of stopping times $\tau_n = \sum_{i=1}^{n2^n} \frac{i}{2^n} \mathbf{1}_{\{\frac{i-1}{2^n} \le \tau < \frac{i}{2^n}\}}$, that satisfy $\tau_n \downarrow \tau$. Taking the limit as n tends to infinity we deduce the equality in the case of a simple process.

(iii) In the general case, we approximate u by simple processes $u^{(n)}$ in the norm of $L^2(\mathcal{P})$.

The convergence

$$\int_0^\tau u_t^{(n)} dB_t \xrightarrow{L^2(\Omega)} \int_0^\tau u_t dB_t$$

follows from Doob's maximal inequality.

Integral of general processes

- Let $L^2_{loc}(\mathcal{P})$ the set of progressively measurable processes $u = \{u_t, t \geq 0\}$, such that for all $t \geq 0$

$$P\left(\int_0^t u_s^2 ds < \infty\right) = 1.$$

- Suppose that $u \in L^2_{loc}(\mathcal{P})$. For each $n \geq 1$ we define the stopping time

$$T_n = \inf\left\{t \geq 0 : \int_0^t u_s^2 ds = n\right\}$$

and the sequence of processes $u_t^{(n)} = u_t \mathbf{1}_{[0,T_n]}(t)$, which belong to $L^2(\mathcal{P})$.

Proposition

There exists an adapted and continuous process $\int_0^t u_s dB_s$ *such that for any* $n \geq 1$,

$$\int_0^t u_s^{(n)} dB_s = \int_0^t u_s dB_s, \quad \text{on} \quad t \leq T_n.$$

Proof : If $n \leq m$, on the set $\{t \leq T_n\}$ we have

$$\int_0^t u_s^{(n)} dB_s = \int_0^t u_s^{(m)} dB_s,$$

and $T_n \uparrow \infty$. $\square$

- The process $M_t := \int_0^t u_s dB_s$ is a continuous *local martingale*, that is, there exist a sequence of stopping times $T_n \uparrow \infty$, such that for each $n \geq 1$, $M_{t \wedge T_n}$ is a uniformly integrable martingale.

- Instead of the isometry property, the stochastic integral of processes in $L^2_{loc}(\mathcal{P})$ has the following continuity property in probability :

Proposition

Suppose that $u \in L^2_{loc}(\mathcal{P})$. *For all* $K, \delta > 0$, $T > 0$ *we have :*

$$P\left(\left|\int_0^T u_s dB_s\right| \geq K\right) \leq P\left(\int_0^T u_s^2 ds \geq \delta\right) + \frac{\delta}{K^2}.$$

Proof :

- Consider the stopping time defined by

$$\tau = \inf \left\{ t \geq 0 : \int_0^t u_s^2 ds = \delta \right\},$$

with the convention that $\tau = T$ if $\int_0^T u_s^2 ds < \delta$.

- We have

$$\begin{aligned} P\left(\left| \int_0^T u_s dB_s \right| \geq K \right) &\leq P\left(\int_0^T u_s^2 ds \geq \delta \right) \\ &\quad + P\left(\left| \int_0^T u_s dB_s \right| \geq K, \int_0^T u_s^2 ds \leq \delta \right). \end{aligned}$$

- On the other hand,

$$
\begin{aligned}
P\left(\left|\int_0^T u_s dB_s\right| \geq K, \int_0^T u_s^2 ds \leq \delta\right) &= P\left(\left|\int_0^T u_s dB_s\right| \geq K, \tau = T\right) \\
&\leq P\left(\left|\int_0^\tau u_s dB_s\right| \geq K\right) \\
&\leq \frac{1}{K^2} E\left(\left|\int_0^\tau u_s dB_s\right|^2\right) \\
&= \frac{1}{K^2} E\left(\int_0^\tau u_s^2 ds\right) \leq \frac{\delta}{K^2}.
\end{aligned}
$$

Stochastic integrals with respect to local martingales

- Let M_t be a continuous local martingale with respect to a filtration $\mathcal{F}_t$ satisfying conditions (i) and (ii), such that $M_0 = 0$.
- We can defined the integral $I_M(u) = \int_0^\infty u_t dM_t$ for progressively measurable processes u_t such that

$$E\left(\int_0^\infty u_s^2 d\langle M\rangle_s\right) < \infty.$$

Basic properties :

1. $E(I_M(u)^2) = E\left(\int_0^\infty u_s^2 d\langle M\rangle_s\right)$.
2. If $M_t = \int_0^t \theta_s dB_s$ and $u\theta \in L^2(\mathcal{P})$, we have

$$\int_0^t u_s dM_s = \int_0^t u_s \theta_s dB_s.$$

3. We have

$$\left\langle \int_0^\cdot u_s dM_s \right\rangle_t = \int_0^t u_s^2 d\langle M\rangle_s.$$

Itô's formula

- Itô's stochastic integral does not follow the chain rule of classical calculus.
- *Example :*

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2 - \frac{t}{2},$$

  whereas if x_t is a differentiable function such that $x_0 = 0$,

$$\int_0^t x_s dx_s = \int_0^t x_s x_s' ds = \frac{1}{2} x_t^2.$$

- In differential form

$$d(B_t^2) = 2B_t dB_t + dt,$$

  and dt comes from $(dB_t)^2 \sim dt$ and the Taylor expansion up to the second order.

- Let $\mathcal{F}_t$ be the filtration generated by the Brownian motion B and the null sets.
- Denote by $L^1_{loc}(\mathcal{P})$ the space of progressively measurable processes $v = \{v_t, t \geq 0\}$ such that for all $t > 0$,

$$P\left(\int_0^t |v_s|\, ds < \infty\right) = 1.$$

Definition

A continuous and adapted stochastic process $\{X_t, t \geq 0\}$ is called an *Itô process* if

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds,$$

where $u \in L^2_{loc}(\mathcal{P})$ and $v \in L^1_{loc}(\mathcal{P})$.

- In differential notation we will write

$$\boxed{dX_t = u_t dB_t + v_t dt}$$

- We say that a function $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is of class $C^{1,2}$ if $f(t,x)$ is twice differentiable with respect to x and once differentiable with respect to t, with continuous partial derivatives.

Theorem (Itô's formula)

Suppose that X is an Itô process. Let $f \in C^{1,2}$. Then, the process $Y_t = f(t, X_t)$ is again an Itô process with the representation

$$
\begin{aligned}
Y_t &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) u_s dB_s \\
&\quad + \int_0^t \frac{\partial f}{\partial x}(s, X_s) v_s ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) u_s^2 ds.
\end{aligned}
$$

- In differential notation Itô's formula can be written as

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,X_s)\,(dX_t)^2\,,$$

where $(dX_t)^2$ is computed using the product rule

$\times$	dB_t	dt
dB_t	dt	0
dt	0	0

- In the particular case $u_t = 1$, $v_t = 0$, $X_0 = 0$, the process X_t is the Brownian motion B_t, and Itô's formula has the following simple version

$$\begin{aligned} f(t,B_t) &= f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \int_0^t \frac{\partial f}{\partial t}(s,B_s)ds \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_s)ds. \end{aligned}$$

Proof :

- Suppose $v_t = 0$ and f does not depend on t, that is,

$$X_t = X_0 + \int_0^t u_s dB_s.$$

We claim that

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) u_s dB_s + \frac{1}{2} \int_0^t f''(X_s) u_s^2 ds.$$

(i) By a localization argument we may assume that $f \in C_b^2(\mathbb{R})$, $\int_0^\infty u_s^2 ds \le N$ and $\sup_{t\ge 0}|X_t| \le N$.
In fact, consider the sequence of stopping times

$$T_N = \inf\left\{ t \ge 0 : \int_0^t u_s^2 ds \ge N, \text{or } |X_t| \ge N \right\}.$$

Let f_N be a function in $C_0^2(\mathbb{R})$ such that $f(x) = f_N(x)$ for $|x| \le N$. Then, if $u_t^{(N)} = u_t \mathbf{1}_{[0,T_N]}(t)$ and

$$X_t^{(N)} = X_0 + \int_0^t u_s^{(N)} dB_s = X_0 + \int_0^{T_N \wedge t} u_s dB_s,$$

we have

$$f_N(X_t^{(N)}) = f_N(X_0) + \int_0^t f_N'(X_s^{(N)}) u_s^{(N)} dB_s + \frac{1}{2}\int_0^t f_N''(X_s^{(N)})(u_s^{(N)})^2 ds.$$

which implies

$$f(X_{T_N \wedge t}) = f_N(X_0) + \int_0^{T_N \wedge t} f'(X_s) u_s dB_s + \frac{1}{2}\int_0^{T_N \wedge t} f''(X_s) u_s^2 ds.$$

Then we let $N \to \infty$ to get the result.

(ii) Consider the uniform partition $0 = t_0 < t_1 < \cdots < t_n = t$, where $t_i = \frac{it}{n}$. We can write, using Taylor's formula

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i=0}^{n-1} \left[f(X_{t_{i+1}}) - f(X_{t_i}) \right] \\ &= \sum_{i=0}^{n-1} f'(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(\widetilde{X}_i)(X_{t_{i+1}} - X_{t_i})^2, \end{aligned}$$

where $\widetilde{X}_i$ is a random point between X_{t_i} and $X_{t_{i+1}}$.

(iii) It is an easy exercise to show that

$$\sum_{i=0}^{n-1} f'(X_{t_i})(X_{t_{i+1}} - X_{t_i}) \xrightarrow{L^2(\Omega)} \int_0^t f'(X_s) u_s dB_s.$$

(iv) For the second term we write

$$
\begin{aligned}
&\int_0^t f''(X_s)u_s^2 ds - \sum_{i=0}^{n-1} f''(\widetilde{X}_i)(X_{t_{i+1}} - X_{t_i})^2 \\
&\quad = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[f''(X_s) - f''(X_{t_i})\right] u_s^2 ds \\
&\qquad + \sum_{i=0}^{n-1} f''(X_{t_i}) \left(\int_{t_i}^{t_{i+1}} u_s^2 ds - \left(\int_{t_i}^{t_{i+1}} u_s dB_s \right)^2 \right) \\
&\qquad + \sum_{i=0}^{n-1} [f''(X_{t_i}) - f''(\widetilde{X}_i)] \left(\int_{t_i}^{t_{i+1}} u_s dB_s \right)^2 \\
&=: A_1^n + A_2^n + A_3^n.
\end{aligned}
$$

Then, it suffices to show that each term A_i^n converges to zero in probability.

(v) We have

$$A_1^n \le \sup_{|s-r|\le t/n} |f''(X_s) - f''(X_r)| \int_0^t u_s^2 ds$$

and

$$A_3^n \le \sup_{0\le i\le n-1} |f''(X_{t_i}) - f''(\widetilde{X}_i)| \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} u_s dB_s \right)^2 .$$

Clearly both expressions converge to zero in probability as n tends to infinity.

(vi) Using that the sequence $\xi_i = \int_{t_i}^{t_{i+1}} u_s^2 ds - \left(\int_{t_i}^{t_{i+1}} u_s dB_s\right)^2$ is bounded and satisfies

$$E[\xi_i | \mathcal{F}_{t_i}] = 0,$$

we obtain

$$\begin{aligned}
E[(A_n^2)^2] &= \sum_{i=0}^{n-1} E[f''(X_{t_i})^2 \xi_i^2] \le \|f''\|_\infty^2 \sum_{i=0}^{n-1} E[\xi_i^2] \\
&= 2\|f''\|_\infty^2 \sum_{i=0}^{n-1} \left[\left(\int_{t_i}^{t_{i+1}} u_s^2 ds \right)^2 + \left(\int_{t_i}^{t_{i+1}} u_s dB_s \right)^4 \right] \\
&\le 2\|f''\|_\infty^2 \Big(N \sup_i \int_{t_i}^{t_{i+1}} u_s^2 ds \\
&\quad + \sup_i |X_{t_{i+1}} - X_{t_i}|^2 \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} u_s dB_s \right)^2 \Big),
\end{aligned}$$

which converges to zero in probability as $n \to \infty$. □

Examples :

1. $f(x) = x^2$ and $X_t = B_t$. We obtain

$$B_t^2 = 2 \int_0^t B_s dB_s + t,$$

because $f'(x) = 2x$ and $f''(x) = 2$.

2. $f(x) = x^3$ and $X_t = B_t$, we obtain

$$B_t^3 = 3 \int_0^t B_s^2 dB_s + 3 \int_0^t B_s ds,$$

because $f'(x) = 3x^2$ and $f''(x) = 6x$. More generally, if $n \geq 2$ is a natural number,

$$B_t^n = n \int_0^t B_s^{n-1} dB_s + \frac{n(n-1)}{2} \int_0^t B_s^{n-2} ds.$$

3. $f(t,x) = e^{ax - \frac{a^2}{2}t}$, $X_t = B_t$, and $Y_t = e^{aB_t - \frac{a^2}{2}t}$, we obtain

$$Y_t = 1 + a \int_0^t Y_s dB_s$$

because

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0. \tag{1}$$

In differential form

$$dY_t = aY_t dB_t.$$

4. If a function $f \in C^{1,2}$ satisfies the equality (1), then,

$$f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s, B_s) dB_s.$$

This implies that $f(t, B_t)$ is a continuous local martingale.
It is a square integrable martingale if :

$$E\left[\int_0^t \left(\frac{\partial f}{\partial x}(s, B_s)\right)^2 ds\right] < \infty$$

for all $t \geq 0$.

The stochastic calculus can be extended to continuous semimartingales of the form

$$X_t = X_0 + A_t + M_t,$$

where M_t is a continuous local martingale with $M_0 = 0$ and A_t is a continuous process with trajectories of bounded variation on any finite interval with $A_0 = 0$. In this case,

$$\boxed{df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t,}$$

where $\langle M\rangle_t$ is the quadratic variation of M.

Stratonovich integral

Proposition

Consider two continuous semimartingales X_t and Y_t. Then, for every sequence $\pi^{(n)} = \{0 = t_0^n \le t_1^n \le \cdots \le t_n^n = t\}$ of partitions of $[0, t]$ such that $|\pi^{(n)}| \to 0$, the following limit in probability exists :

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2}(Y_{t_i^n} + Y_{t_{i+1}^n})(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t Y_s \circ dX_s,$$

and it is called the Stratonovich integral of Y with respect to X.

- We have
$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s dX_s + \frac{1}{2}\langle X, Y\rangle_t,$$
where $\langle X, Y\rangle_t$ is the covariation between the local martingale components of X and Y.

- The Stratonovich integral follows the rules of the classical calculus. That is, if $f \in C^3(\mathbb{R})$,

$$\boxed{f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s.}$$

Proof :

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_t \\ &= f(X_0) + \int_0^t f'(X_s) \circ dX_s - \frac{1}{2} \langle f'(X), X \rangle_t + \frac{1}{2} \int_0^t f''(X_s) d\langle X \rangle_t. \end{aligned}$$

Finally, taking into account that the local martingale part of $f'(X)$ is $\int_0^t f''(X_s) dM_s$, we obtain

$$\langle f'(X), X \rangle_t = \int_0^t f''(X_s) d\langle M \rangle_s. \quad \square$$

Multidimensional Itô formula

- Suppose that $B_t = (B_t^1, B_t^2, \ldots, B_t^m)$ is an m-dimensional Brownian motion. Consider an n-dimensional Itô process of the form

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds,$$

  where v_t is an n-dimensional process and u_t is a process with values in the set of $n \times m$ matrices and we assume that the components of u belong to $L^2_{loc}(\mathcal{P})$ and those of v belong to $L^1_{loc}(\mathcal{P})$.

- Then, if $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^{1,2}$, the process $Y_t = f(t, X_t)$ is again an Itô process with the representation

$$
\begin{aligned}
dY_t &= \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)dX_t^i \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)dX_t^i dX_t^j.
\end{aligned}
$$

- The product of differentials $dX_t^i dX_t^j$ is computed by means of the product rules : $dB_t^i dt = 0$, $(dt)^2$ and

$$
dB_t^i dB_t^j = \begin{cases} 0 & \text{if} \quad i \neq j \\ dt & \text{if} \quad i = j \end{cases}
$$

Exercise : Show that if $i \neq j$, $\lim_{|\pi|\to 0} \sum_{k=1}^{n} \Delta B_k^i \Delta B_k^j = 0$ in L^2.

- In this way we obtain

$$dX_t^i dX_t^j = \left(\sum_{k=1}^m u_t^{ik} u_t^{jk} \right) dt = (u_t u_t')_{ij}\, dt,$$

  which implies

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \nabla f(t, X_t)dX_t + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)(u_t u_t')_{ij} dt.$$

- As a consequence we can deduce the following *integration by parts formula* : Suppose that X_t and Y_t are Itô processes. Then,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t dX_s dY_s.$$

Recurrence and transience of the Brownian motion

Proposition

Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and let B_t be a d-dimensional Brownian motion. Then, the process

$$X_t = f(t, B_t) - \int_0^t \left(\frac{1}{2} \Delta f(s, B_s) + \frac{\partial f}{\partial t}(s, B_s) \right) ds$$

is a local martingale. If moreover,

$$\sum_{i=1}^{d} \left(\frac{\partial f}{\partial x_i}(t, x) \right)^2 \leq \phi(t) e^{K\|x\|},$$

for some continuous function ϕ, then X_t is a martingale.

- For $a > 0$ and $x \in \mathbb{R}^d$, consider the stopping time

$$T_a^x = \inf\{t \geq 0 : \|B_t + a\| = x\}.$$

Proposition

For $a < \|x\| < b$,

$$P(T_a^x < T_b^x) = \begin{cases} \frac{\log b - \log \|x\|}{\log b - \log a}, & d = 2 \\ \\ \frac{\|x\|^{2-d} - b^{2-d}}{a^{2-d} - b^{2-d}}, & d \geq 3 \end{cases}$$

Proof :

Consider the function

$$f(x) = \Psi(\|x\|) = \begin{cases} \log \|x\|, & d = 2 \\ \|x\|^{2-d}, & d \geq 3 \end{cases}$$

Because $\Delta f = 0$, $f(B_{t \wedge T_a^x \wedge T_b^x})$ is a martingale, which implies

$$E(f(B_{T_a^x \wedge T_b^x})) = f(x).$$

This yields

$$\Psi(a)P(T_a^x < T_b^x) + \Psi(b)P(T_b^x < T_a^x) = f(x),$$

and together with

$$P(T_a^x < T_b^x) + P(T_b^x < T_a^x) = 1$$

we obtain the result. □

Letting $b \to \infty$ we get :

Corollary

For $0 < a < \|x\|$

$$P(T_a^x < \infty) = \begin{cases} 1, & d = 2 \\ \frac{\|x\|^{2-d}}{a^{2-d}}, & d \geq 3. \end{cases}$$

As a consequence, for $d = 2$ *the Brownian motion is recurrent, that is, for every non-empty set* $\mathcal{O} \subset \mathbb{R}^2$,

$$P(B_t \in \mathcal{O}, \quad \text{for some} \quad t \geq 0) = 1.$$

Martingale representation theorem

Suppose that $\mathcal{F}_t$ is the filtration generated by the Brownian motion B_t and the null sets.

Theorem

Let $F \in L^2(\Omega, \mathcal{F}_\infty, P)$. Then, there exists a unique process u in the space $L^2(\mathcal{P})$ such that

$$F = E(F) + \int_0^\infty u_s dB_s.$$

Example : $F = B_T^3$. By Itô's formula and integrating by parts

$$\begin{aligned} B_T^3 &= \int_0^T 3B_t^2 dB_t + 3\int_0^T B_t dt = \int_0^T 3B_t^2 dB_t + 3\left(TB_T - \int_0^T t dB_t\right) \\ &= \int_0^T 3B_t^2 dB_t + 3\int_0^T (T-t) dB_t \\ &= \int_0^T 3\left[B_t^2 + (T-t)\right] dB_t. \end{aligned}$$

Proof :

(i) Suppose first that

$$F = \exp\left(\int_0^\infty h_s dB_s - \frac{1}{2}\int_0^\infty h_s^2 ds\right), \qquad (2)$$

where $h \in L^2(\mathbb{R}_+)$. Define

$$Y_t = \exp\left(\int_0^t h_s dB_s - \frac{1}{2}\int_0^t h_s^2 ds\right).$$

By Itô's formula applied to the function $f(x) = e^x$ and the process $X_t = \int_0^t h_s dB_s - \frac{1}{2}\int_0^t h_s^2 ds$, we obtain

$$Y_t = 1 + \int_0^t Y_s h(s) dB_s.$$

Hence,

$$F = 1 + \int_0^\infty Y_s h_s dB_s$$

and we get the desired representation because $E(F) = 1$.

(ii) By linearity, the representation holds for linear combinations of exponentials of the form (2).

(iii) In the general case, any random variable $F \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be approximated in L^2 by a sequence F_n of linear combinations of exponentials of the form (2). Then, we have

$$F_n = E(F_n) + \int_0^\infty u_s^{(n)} dB_s.$$

By the isometry of the stochastic integral

$$\begin{aligned} E\left[(F_n - F_m)^2\right] &\geq \mathrm{Var}(F_n - F_m) \\ &= E\left[\left(\int_0^\infty \left(u_s^{(n)} - u_s^{(m)}\right) dB_s\right)^2\right] \\ &= E\left[\int_0^\infty \left(u_s^{(n)} - u_s^{(m)}\right)^2 ds\right]. \end{aligned}$$

(iv) Hence, $u^{(n)}$ is a Cauchy sequence in $L^2(\mathcal{P})$ and it converges to a process u in $L^2(\mathcal{P})$.

(v) Applying again the isometry property, and taking into account that $E(F_n)$ converges to $E(F)$, we obtain

$$\begin{aligned} F &= \lim_{n\to\infty} F_n = \lim_{n\to\infty} \left(E(F_n) + \int_0^\infty u_s^{(n)} dB_s \right) \\ &= E(F) + \int_0^\infty u_s dB_s. \end{aligned}$$

(vi) Uniqueness : Suppose that $u^{(1)}$ and $u^{(2)}$ are processes in $L^2(\mathcal{P})$ such that

$$F = E(F) + \int_0^\infty u_s^{(1)} dB_s = E(F) + \int_0^\infty u_s^{(2)} dB_s.$$

Then

$$0 = E\left[\left(\int_0^\infty \left(u_s^{(1)} - u_s^{(2)} \right) dB_s \right)^2 \right] = E\left[\int_0^\infty \left(u_s^{(1)} - u_s^{(2)} \right)^2 ds \right]$$

and, hence, $u_s^{(1)}(\omega) = u_s^{(2)}(\omega)$ for almost all (s, ω).

Corollary

Suppose that $\{M_t, t \geq 0\}$ *is a square integrable martingale with respect to* $\mathcal{F}_t$*. Then there exists a unique progressively measurable process u such that* $E\left(\int_0^t u_s^2 ds\right)^2 < \infty$ *for all t and*

$$M_t = E(M_0) + \int_0^t u_s dB_s.$$

- In particular, M_t has a continuous version.

Theorem (Burkholder-Davis-Gundy)

For any $p > 0$ we have

$$c_p E\left[\left|\int_0^T u_s^2 ds\right|^{\frac{p}{2}}\right] \le E\left[\sup_{t\in[0,T]}\left|\int_0^t u_s dB_s\right|^p\right] \le C_p E\left[\left|\int_0^T u_s^2 ds\right|^{\frac{p}{2}}\right].$$

- More generally, for any continuous local martingale,

$$c_p E\left[\langle M\rangle_t^{\frac{p}{2}}\right] \le E\left[\sup_{t\in[0,T]} |M_t|^p\right] \le C_p E\left[\langle M\rangle_t^{\frac{p}{2}}\right].$$

Change of measures

- Let $L \geq 0$ be a nonnegative random variable such that $E(L) = 1$. Then,

$$\boxed{Q(A) = E\left(\mathbf{1}_A L\right)}$$

defines a new probability and we say that L is the *density* of Q with respect to P, that is, $\frac{dQ}{dP} = L$.

- The expectation of a random variable X in the probability space $(\Omega, \mathcal{F}, Q)$ is computed by the formula

$$E_Q(X) = E(XL).$$

- The probability Q is absolutely continuous with respect to P, that means,

$$P(A) = 0 \Longrightarrow Q(A) = 0.$$

- If L is strictly positive, then the probabilities P and Q are *equivalent* (that is, mutually absolutely continuous), that means,

$$P(A) = 0 \iff Q(A) = 0.$$

Girsanov theorem

- Let $\{B_t, t \geq 0\}$ be a Brownian motion. Given a process $\theta \in L^2(\mathcal{P})$, consider the local martingale

$$\boxed{L_t = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)} \tag{3}$$

which satisfies the linear stochastic differential equation

$$L_t = 1 + \int_0^t \theta_s L_s dB_s.$$

Lemma (Novikov condition)

If

$$E\left(\exp\left(\frac{1}{2}\int_0^\infty \theta_s^2 ds\right)\right) < \infty,$$

then $\{L_t, t \geq 0\}$ is a uniformly integrable martingale.

Consequences :

1. The random variable L_∞ is a density in the probability space $(\Omega, \mathcal{F}_\infty, P)$ and defines a probability Q such that $L_\infty = \frac{dQ}{dP}$.
2. For any $t \geq 0$, $L_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$. In fact, if $A \in \mathcal{F}_t$ we have

$$\begin{aligned} Q(A) &= E(\mathbf{1}_A L_\infty) = E(E(\mathbf{1}_A L_\infty | \mathcal{F}_t)) \\ &= E(\mathbf{1}_A E(L_\infty | \mathcal{F}_t)) \\ &= E(\mathbf{1}_A L_t). \end{aligned}$$

Theorem (Girsanov theorem)

Suppose θ satisfies Novikov condition. In the probability space $(\Omega, \mathcal{F}_\infty, Q)$ the stochastic process

$$\boxed{W_t = B_t - \int_0^t \theta_s ds,}$$

is a Brownian motion.

Proof :

- It is enough to show that in the probability space $(\Omega, \mathcal{F}_\infty, Q)$, for all $s < t \leq T$ the increment $W_t - W_s$ is independent of $\mathcal{F}_s$ and has the normal distribution $N(0, t-s)$.

  These properties follow from the following relation, for all $s < t$, $A \in \mathcal{F}_s$, $\lambda \in \mathbb{R}$,

$$E_Q\left(\mathbf{1}_A e^{i\lambda(W_t - W_s)}\right) = Q(A) e^{-\frac{\lambda^2}{2}(t-s)}. \tag{4}$$

- In order to show (4) we write

$$\begin{aligned} E_Q\left(\mathbf{1}_A e^{i\lambda(W_t - W_s)}\right) &= E\left(\mathbf{1}_A e^{i\lambda(W_t - W_s)} L_t\right) \\ &= E\left(\mathbf{1}_A L_s \Psi_{s,t}\right) e^{-\frac{\lambda^2}{2}(t-s)}, \end{aligned}$$

  where

$$\Psi_{s,t} = \exp\left(\int_s^t (i\lambda + \theta_v) dB_v - \frac{1}{2}\int_s^t (i\lambda + \theta_v)^2 dv\right).$$

  Then the desired result follows form

$$E\left[\Psi_{s,t} | \mathcal{F}_s\right] = 1.$$

Application

- Let $\{B_t, t \geq 0\}$ be a Brownian motion. Fix a real number θ, and define

$$L_t = \exp\left(\theta B_t - \frac{\theta^2}{2} t\right).$$

- Let Q be the probability on each σ-field $\mathcal{F}_t$ such that for all $t > 0$

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = L_t.$$

By Girsanov theorem, for all $T > 0$, in the probability space $(\Omega, \mathcal{F}_T, Q)$ the process $B_t - \theta t := \widetilde{B}_t$ is a Brownian motion in the time interval $[0, T]$. That is, in this space B_t is a Brownian motion with drift θt.

- Set
$$\tau_a = \inf\{t \geq 0, B_t = a\},$$
where $a \neq 0$. For any $t \geq 0$ the event $\{\tau_a \leq t\}$ belongs to the σ-field $\mathcal{F}_{\tau_a \wedge t}$ because for any $s \geq 0$
$$\begin{aligned}\{\tau_a &\leq t\} \cap \{\tau_a \wedge t \leq s\} = \{\tau_a \leq t\} \cap \{\tau_a \leq s\} \\ &= \{\tau_a \leq t \wedge s\} \in \mathcal{F}_{s \wedge t} \subset \mathcal{F}_s.\end{aligned}$$

- Consequently, using the Optional Stopping Theorem we obtain
$$\begin{aligned}Q\{\tau_a \leq t\} &= E\left(\mathbf{1}_{\{\tau_a \leq t\}} L_t\right) = E\left(\mathbf{1}_{\{\tau_a \leq t\}} E(L_t | \mathcal{F}_{\tau_a \wedge t})\right) \\ &= E\left(\mathbf{1}_{\{\tau_a \leq t\}} L_{\tau_a \wedge t}\right) = E\left(\mathbf{1}_{\{\tau_a \leq t\}} L_{\tau_a}\right) \\ &= E\left(\mathbf{1}_{\{\tau_a \leq t\}} e^{\theta a - \frac{1}{2}\theta^2 \tau_a}\right) \\ &= \int_0^t e^{\theta a - \frac{1}{2}\theta^2 s} f(s) ds,\end{aligned}$$
where f is the density of the random variable τ_a.

- We know that

$$f(s) = \frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}}.$$

Hence, with respect to Q the random variable τ_a has the probability density

$$\frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{(a-\theta s)^2}{2s}}, \; s > 0.$$

- Letting, $t \uparrow \infty$ we obtain

$$Q\{\tau_a < \infty\} = e^{\theta a} E\left(e^{-\frac{1}{2}\theta^2 \tau_a}\right) = e^{\theta a - |\theta a|}.$$

If $\theta = 0$ (Brownian motion without drift), the probability to reach the level is one. If $\theta a > 0$ (the drift θ and the level a have the same sign) this probability is also one. If $\theta a < 0$ (the drift θ and the level a have opposite sign) this probability is $e^{2\theta a}$.