Theoretical Tutorial Session 1

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Basic properties for Brownian motion

1. Selfsimilarity:
For any $a > 0$, the process $\{a^{-\frac{1}{2}} B_{at}, t \geq 0\}$ is also a Brownian motion.

Proof: Let $Y_t = a^{-\frac{1}{2}} B_{at}$. It is obvious that the process $\{Y_t = a^{-\frac{1}{2}} B_{at}, t \geq 0\}$ is an Gaussian process with zero mean and continuous trajectories. It is sufficient to show that for any $0 \leq s \leq t < +\infty$ the covariance function $E(Y_t Y_s) = s$ which is proved as follows:

$$E(Y_t Y_s) = E(a^{-\frac{1}{2}} B_{at} a^{-\frac{1}{2}} B_{as}) = a^{-1} E(B_{at} B_{as}) = a^{-1}(as) = s.$$

2. For any $h > 0$, the process $\{B_{t+h} - B_h, t \geq 0\}$ is a Brownian motion.

Proof: Let $Y_t = B_{t+h} - B_h$. We know that the process $\{Y_t = B_{t+h} - B_h, t \geq 0\}$ is an Gaussian process with zero mean and continuous trajectories. Next, we calculate its covariance function: for any $0 \leq s \leq t < +\infty$,

$$\begin{aligned} E(Y_t Y_s) &= E((B_{t+h} - B_h)(B_{s+h} - B_h)) \\ &= E(B_{t+h}B_{s+h}) - E(B_h B_{s+h}) - E(B_{t+h}B_h) + E(B_h B_h) \\ &= (s+h) - h - h + h \\ &= s. \end{aligned}$$

Therefore, the process $\{B_{t+h} - B_h, t \geq 0\}$ is a Brownian motion.

3. The process $\{-B_t, t \geq 0\}$ is a Brownian motion.

Proof: This process is a Gaussian process with zero mean and continuous trajectories, and for any $0 \leq s \leq t < +\infty$,

$$E((-B_t)(-B_s)) = E(B_t B_s) = s.$$

Thus, The process $\{-B_t, t \geq 0\}$ is a Brownian motion.

Chebyshev's inequality

If ξ is a random variable with $E(|\xi|^p) < \infty$, then for any $\lambda > 0$

$$P(|\xi| > \lambda) \leq \frac{E(|\xi|^p)}{\lambda^p}.$$

Borel-Cantelli lemma

Lemma

Let $A_1, A_2, A_3, \dots$ be a sequence of events in a probability space. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then the probability that infinitely many of them occur is 0, that is,

$$P(\limsup_{n\to\infty} A_n) = 0.$$

Note that

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

Strong law of large numbers

Theorem

If $\xi_1, \xi_2, \xi_3, \ldots$ is a sequence of iid random variables and $\mu = E(\xi_1) < \infty$, then

$$\frac{\sum_{i=1}^{n} \xi_i}{n} = \frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n} \to \mu$$

almost surely as $n \to \infty$.

4. Almost surely $\lim_{t\to\infty} \frac{B_t}{t} = 0$ and the process

$$X_t = \begin{cases} tB_{1/t}, & t > 0; \\ 0, & t = 0 \end{cases}$$

is a Brownian motion.

<u>Proof:</u> We first show that $\lim_{n\to\infty} \frac{B_n}{n} = 0$ which is true by using the strong law of large numbers since

$$\frac{B_n}{n} = \frac{\sum_{i=1}^{n}(B_i - B_{i-1})}{n}$$

and $B_1 - B_0, B_2 - B_1, \ldots, B_n - B_{n-1}, \ldots$ are iid with law $N(0,1)$.

Then for any $t > 0$ and $t \in [n, n+1)$ for some n, we have

$$\left|\frac{B_t}{t} - \frac{B_n}{n}\right| = \left|\frac{B_t}{t} - \frac{B_n}{t} + \frac{B_n}{t} - \frac{B_n}{n}\right|$$
$$\leq \frac{\sup_{s\in[n,n+1]} |B_s - B_n|}{t} + |B_n|\left(\frac{1}{n} - \frac{1}{t}\right)$$
$$\leq \frac{\sup_{s\in[n,n+1]} |B_s - B_n|}{n} + \frac{|B_n|}{n}. \qquad (1)$$

Note that $\lim_{n\to\infty} \frac{B_n}{n} = 0$ implies

$$\lim_{n\to\infty} \frac{|B_n|}{n} = 0. \qquad (2)$$

Since the sequence of random variables

$$\{ \sup_{s\in[n-1,n]} |B_s - B_n|, n \geq 1\}$$

are iid with the same law as the integrable random variable $\sup_{s\in[0,1]} |B_s|$, using the strong law of large numbers again we can show that

$$\frac{\sum_{i=1}^{n} \sup_{s\in[i-1,i]} |B_s - B_i|}{n} \to E(\sup_{s\in[0,1]} |B_s|) < \infty$$

almost surely, as $n \to \infty$.

Then, the above results yields

$$
\begin{aligned}
&\frac{\sup_{s\in[n,n+1]}|B_s - B_n|}{n} \\
=& \frac{\sum_{i=1}^{n}\sup_{s\in[i-1,i]}|B_s - B_i|}{n} - \frac{\sum_{i=1}^{n-1}\sup_{s\in[i-1,i]}|B_s - B_i|}{n} \\
=& \frac{\sum_{i=1}^{n}\sup_{s\in[i-1,i]}|B_s - B_i|}{n} - \frac{\sum_{i=1}^{n-1}\sup_{s\in[i-1,i]}|B_s - B_i|}{n-1}\left(\frac{n-1}{n}\right) \\
\to& E(\sup_{s\in[0,1]}|B_s|) - E(\sup_{s\in[0,1]}|B_s|)(1) = 0, \qquad (3)
\end{aligned}
$$

as $n \to \infty$.

The fact of $\frac{|B_t|}{t} \le \left|\frac{B_t}{t} - \frac{B_n}{n}\right| + \frac{|B_n|}{n}$ together with (1), (2) and (3) imply

$$\lim_{t\to\infty}\frac{B_t}{t} = 0.$$

For the process X, using $\lim_{t\to\infty} \frac{B_t}{t} = 0$ almost surely we can prove that the trajectories are continuous almost surely.

The process X is a Gaussian process with zero mean. For any $0 \leq s \leq t < \infty$, we have

$$E(X_t X_s) = E((tB_{1/t})(sB_{1/s})) = (t)(s)E(B_{1/t}B_{1/s}) = (t)(s)(1/t) = s.$$

Therefore, X is also a Brownian motion.

Fatou's lemma

Lemma

- *Let* $\xi_1, \xi_2, \xi_3, \ldots$ *be a sequence of non-negative random variables on a probability space* $(\Omega, \mathcal{F}, P)$, *then*

$$E(\liminf_{n\to\infty} \xi_n) \leq \liminf_{n\to\infty} E(\xi_n).$$

- *Let* $\xi_1, \xi_2, \xi_3, \ldots$ *be a sequence of non-negative random variables on a probability space* $(\Omega, \mathcal{F}, P)$. *If there exists a non-negative integrable random variable* η *such that* $\xi_n \leq \eta$ *for all* n, *then*

$$\limsup_{n\to\infty} E(\xi_n) \leq E(\limsup_{n\to\infty} \xi_n).$$

Hewitt-Savage 0-1 law

Theorem

Let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of independent and identically-distributed (iid) random variables taking values in a set $\mathcal{E}$. Then, any event whose occurrence or non-occurrence is determined by the values of these random variables and whose occurrence or non-occurrence is unchanged by finite permutations of the indices, has probability either 0 or 1.

5. $\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty$ and $\liminf_{t\to\infty} \frac{B_t}{\sqrt{t}} = -\infty$.

Proof: It suffices to show that $\limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} = \infty$ and $\liminf_{n\to\infty} \frac{B_n}{\sqrt{n}} = -\infty$. For any positive constant C, we have

$$
\begin{aligned}
P\left(\limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} > C\right) &= P\left(\limsup_{n\to\infty} \left\{\frac{B_n}{\sqrt{n}} > C\right\}\right) \\
&\geq \limsup_{n\to\infty} P\left(\left\{\frac{B_n}{\sqrt{n}} > C\right\}\right) \text{ (Fatou's lemma)} \\
&= \limsup_{n\to\infty} P\left(B_1 > C\right) \text{ (Scaling property)} \\
&> 0.
\end{aligned}
$$

Let $\xi_n = B_n - B_{n-1}$, $n \geq 1$. Then $\xi_1, \xi_2, \xi_3, \ldots$ is a sequence of iid random variables. Note that the event

$$\limsup_{n\to\infty} \left\{ \frac{B_n}{\sqrt{n}} > C \right\} = \limsup_{n\to\infty} \left\{ \frac{\sum_{k=1}^{n} \xi_k}{\sqrt{n}} > C \right\}$$

is unchanged by finite permutations of the indices. Hence, by Hewitt-Savage $0-1$ law together with the last inequality on previous page, we have

$$P\left(\limsup_{n\to\infty} \left\{ \frac{B_n}{\sqrt{n}} > C \right\} \right) = 1.$$

Since C is arbitrary, we can show that $\limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} = \infty$.

Since $\{-B_t, t \geq 0\}$ is also a Brownian motion, we can show that

$$\liminf_{n\to\infty} \frac{B_n}{\sqrt{n}} = \liminf_{n\to\infty} \frac{-B_n}{\sqrt{n}} = -\limsup_{n\to\infty} \frac{B_n}{\sqrt{n}} = -\infty.$$

Remark: From the Exercise 5, we obtain the following result

$$P(\limsup_{t\to\infty} B_t = +\infty) = 1, \tag{4}$$

and

$$P(\liminf_{t\to\infty} B_t = -\infty) = 1. \tag{5}$$

Furthermore,

$$P(\limsup_{t\to\infty} B_t = +\infty,\ \liminf_{t\to\infty} B_t = -\infty) = 1. \tag{6}$$

6. Show that for each fixed $t \geq 0$

$$P\left(\omega \in \Omega : \limsup_{h \to 0^+} \frac{B_{t+h} - B_t}{h} = \infty\right) = 1.$$

Proof: Given this Brownian motion B, we construct another Brownian motion X defined in Exercise 4:

$$X_t = \begin{cases} tB_{1/t}, & t > 0; \\ 0, & t = 0. \end{cases}$$

Then for any $t \geq 0$, we obtain

$$\begin{aligned} \limsup_{n \to \infty} \frac{B_{t+\frac{1}{n}} - B_t}{\frac{1}{n}} &= \limsup_{n \to \infty} \frac{B_{\frac{1}{n}} - B_0}{\frac{1}{n}} \text{ (Exercise 2)} \\ &\geq \limsup_{n \to \infty} \sqrt{n} B_{\frac{1}{n}} \\ &= \limsup_{n \to \infty} \frac{X_n}{\sqrt{n}} = \infty \text{ (Previous exercise).} \end{aligned}$$

7. Almost surely the paths of B are not differentiable at any point $t \geq 0$, more precisely, the set

$$\left\{\omega \in \Omega : \text{for each } t \in [0, \infty), \text{ either } \limsup_{h \to 0^+} \frac{B_{t+h} - B_t}{h} = \infty \right.$$
$$\left. \text{or } \liminf_{h \to 0^+} \frac{B_{t+h} - B_t}{h} = -\infty \right\}$$

contains an event $A \in \mathcal{F}$ with $P(A) = 1$.

Proof: Denote

$$D^+ B_t = \limsup_{h \to 0^+} \frac{B_{t+h} - B_t}{h}$$

and

$$D_+ B_t = \liminf_{h \to 0^+} \frac{B_{t+h} - B_t}{h}.$$

It suffices to show the result on the interval $[0,1]$. For any fixed integers $j \geq 1$ and $k \geq 1$, define

$$A_{jk} = \bigcup_{t\in[0,1]} \bigcap_{h\in[0,1/k]} \{\omega \in \Omega : |B_{t+h}(\omega) - B_t(\omega)| \leq jh\}.$$

Note that A_{jk} is not an event, that is $A_{jk} \notin \mathcal{F}$. Note also that

$$\{\omega \in \Omega : -\infty < D_+ B_t(\omega) \leq D^+ B_t(\omega) < \infty, \text{ for some } t \in [0,1]\}$$
$$= \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_{jk}.$$

Then the proof of the result will be complete if we can find an event $C \in \mathcal{F}$ with $P(C) = 0$ such that $A_{jk} \subseteq C$ for all j and k.

For any fixed j and k, we consider a sample path $\omega \in A_{jk}$, that is, there exists $t \in [0,1]$ with

$$|B_{t+h}(\omega) - B_t(\omega)| \leq jh$$

for all $h \in [0, 1/k]$.

Let n be an integer with $n \geq 4k$. Then there exists some i, $1 \leq i \leq n$ such that $\frac{i-1}{n} \leq t < \frac{i}{n}$. Then it is also easy to verify that, for $l = 1, 2, 3$,

$$\frac{i+l}{n} - t \leq \frac{l+1}{n} \leq \frac{1}{k}.$$

Thus,

$$\begin{aligned}|B_{\frac{i+1}{n}}(\omega) - B_{\frac{i}{n}}(\omega)| &\leq |B_{\frac{i+1}{n}}(\omega) - B_t(\omega)| + |B_{\frac{i}{n}}(\omega) - B_t(\omega)| \\ &\leq \frac{2j}{n} + \frac{j}{n} = \frac{3j}{n},\end{aligned}$$

$$\begin{aligned}\left|B_{\frac{i+2}{n}}(\omega) - B_{\frac{i+1}{n}}(\omega)\right| &\leq \left|B_{\frac{i+2}{n}}(\omega) - B_t(\omega)\right| + \left|B_{\frac{i+1}{n}}(\omega) - B_t(\omega)\right| \\ &\leq \frac{3j}{n} + \frac{2j}{n} = \frac{5j}{n},\end{aligned}$$

and

$$\begin{aligned}\left|B_{\frac{i+3}{n}}(\omega) - B_{\frac{i+2}{n}}(\omega)\right| &\leq \left|B_{\frac{i+3}{n}}(\omega) - B_t(\omega)\right| + \left|B_{\frac{i+2}{n}}(\omega) - B_t(\omega)\right| \\ &\leq \frac{4j}{n} + \frac{3j}{n} = \frac{7j}{n}.\end{aligned}$$

Now denote

$$C_i^{(n)} = \bigcap_{l=1}^{3} \left\{\omega \in \Omega : \left|B_{\frac{i+l}{n}}(\omega) - B_{\frac{i+l-1}{n}}(\omega)\right| \leq \frac{(2l+1)j}{n}\right\}.$$

Then, we can see that $C_i^{(n)}$ is an event in $\mathcal{F}$ and $A_{jk} \subseteq \bigcup_{i=1}^{n} C_i^{(n)}$ for all $n \geq 4k$.

Note that the random variables

$$\sqrt{n}\left(B_{\frac{i+l}{n}} - B_{\frac{i+l-1}{n}}\right),\ l = 1,\ 2, 3,$$

are iid with law $N(0, 1)$. Note also that if a random variable ξ has law $N(0, 1)$ then

$$P(|\xi| \le x) \le x,\ \text{for any } x \ge 0.$$

Therefore, we have

$$P(C_i^{(n)}) \le \left(\frac{3j}{\sqrt{n}}\right)\left(\frac{5j}{\sqrt{n}}\right)\left(\frac{7j}{\sqrt{n}}\right) = \frac{105j^3}{n^{\frac{3}{2}}},\ i = 1, 2, \cdots, n.$$

Then

$$P(\bigcup_{i=1}^{n} C_i^{(n)}) \le \frac{105j^3}{\sqrt{n}}.$$

Take

$$C = \bigcap_{n=4k}^{\infty} \bigcup_{i=1}^{n} C_i^{(n)} \in \mathcal{F}.$$

Then $A_{jk} \subseteq C$ and it also holds that

$$P(C) = P(\bigcap_{n=4k}^{\infty} \bigcup_{i=1}^{n} C_i^{(n)}) \leq \inf_{n \geq 4k} P(\bigcup_{i=1}^{n} C_i^{(n)}) = 0,$$

which completes the proof.

Properties of the conditional expectation

8. Linearity:

$$E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G}).$$

<u>Proof:</u> $aE(X|\mathcal{G}) + bE(Y|\mathcal{G})$ is $\mathcal{G}$-measurable. For any $A \in \mathcal{G}$,

$$\begin{aligned}\int_A E(aX + bY|\mathcal{G})dP = & \int_A (aX + bY)dP \\ & = a\int_A XdP + b\int_A YdP \\ & = a\int_A E(X|\mathcal{G})dP + b\int_A E(Y|\mathcal{G})dP \\ & = \int_A (aE(X|\mathcal{G}) + bE(Y|\mathcal{G}))dP,\end{aligned}$$

which implies

$$E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G}).$$

9. $E(E(X|\mathcal{G})) = E(X)$.

Proof: In the definition of conditional expectation, let $A = \Omega$, then

$$E(E(X|\mathcal{G})) = \int_{\Omega} E(X|\mathcal{G})dP = \int_{\Omega} XdP = E(X).$$

10. If X and $\mathcal{G}$ are independent, then $E(X|\mathcal{G}) = E(X)$.

Proof: It is obvious that $E(X)$ is $\mathcal{G}$-measurable. For any $A \in \mathcal{G}$,

$$\begin{aligned}\int_A E(X|\mathcal{G})dP = \int_A XdP &= E(1_A X) \\ &= E(1_A)E(X) \text{ (X and 1_A are independent)} \\ &= P(A)E(X) \\ &= \int_A E(X)dP.\end{aligned}$$

Thus, it implies that $E(X|\mathcal{G}) = E(X)$.

11. If X is $\mathcal{G}$-measurable, then $E(X|\mathcal{G}) = X$.

Proof: The proof follows from the fact that X is $\mathcal{G}$-measurable and the definition of conditional expectation: for any $A \in \mathcal{G}$

$$\int_A E(X|\mathcal{G}) dP = \int_A X dP.$$

Dominated convergence theorem

Theorem

Let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of random variables. If the sequence $\{\xi_n\}$ converges almost surely to a random variable ξ and there exists a positive and integrable random variable η such that $|\xi_n| \leq \eta$ for all n, then

$$\lim_{n\to\infty} E(|\xi_n - \xi|) = 0,$$

which also implies

$$\lim_{n\to\infty} E(\xi_n) = E(\xi).$$

12. If Y is bounded and $\mathcal{G}$-measurable, then

$$E(XY|\mathcal{G}) = YE(X|\mathcal{G}).$$

Proof: If $Y = 1_B$ is an indicator function where $B \in \mathcal{G}$, then for any $A \in \mathcal{G}$

$$\begin{aligned}\int_A E(YX|\mathcal{G})dP = \int_A 1_B X dP &= \int_{A\cap B} XdP \\ &= \int_{A\cap B} E(X|\mathcal{G})dP \\ &= int_A 1_B E(X|\mathcal{G})dP,\end{aligned}$$

which implies $E(XY|\mathcal{G}) = YE(X|\mathcal{G})$.

By the linearity of conditional expectation, we know that the result is true if Y is a simple function (a linear combination of indicator functions).

If Y is bounded $\mathcal{G}$-measurable function, then there exists a sequence of simple functions Y_n such that $|Y_n| \leq |Y|$ and $\lim_{n\to\infty} Y_n = Y$ almost surely.

Then by dominated convergence theorem, we can show that for any $A \in \mathcal{G}$,

$$\begin{aligned}\int_A E(YX|\mathcal{G})dP = \int_A YXdP &= \lim_{n\to\infty} \int_A Y_n XdP \\ &= \lim_{n\to\infty} \int_A Y_n E(X|\mathcal{G})dP \\ &= \int_A YE(X|\mathcal{G})dP,\end{aligned}$$

which proves $E(XY|\mathcal{G}) = YE(X|\mathcal{G})$.

13. Given two σ-fields $\mathcal{B} \subset \mathcal{G}$, then

$$E(E(X|\mathcal{B})|\mathcal{G}) = E(E(X|\mathcal{G})|\mathcal{B}) = E(X|\mathcal{B}).$$

Proof: Since $\mathcal{B} \subset \mathcal{G}$, we know that $E(X|\mathcal{B})$ is $\mathcal{G}$-measurable. Then by Exercise 4, we obtain

$$E(E(X|\mathcal{B})|\mathcal{G}) = E(X|\mathcal{B}).$$

For any $A \in \mathcal{B} \subset \mathcal{G}$, we have

$$\int_A E(E(X|\mathcal{G})|\mathcal{B})dP = \int_A E(X|\mathcal{G})dP = \int_A XdP = \int_A E(X|\mathcal{B})dP,$$

which implies

$$E(E(X|\mathcal{G})|\mathcal{B}) = E(X|\mathcal{B}).$$

14. Let X and Z be such that

(i) Z is $\mathcal{G}$-measurable.

(ii) X is independent of $\mathcal{G}$.

Suppose that $E(h(X, Z)|) < \infty$. Then,

$$E(h(X, Z)|\mathcal{G}) = E(h(X, z))|_{z=Z}.$$

Proof: Denote $g(z) = E(h(X, z))$ and $\mu_X(E) = P(X \in E)$ for $E \in \mathcal{B}(\mathbb{R})$. For any $A \in \mathcal{G}$, we denote

$$Y = 1_A;$$
$$\mu_{Y,Z}(F) = P((Y, Z) \in F), \; F \in \mathcal{B}(\mathbb{R}^2).$$

Then,

$$
\begin{aligned}
\int_A E(h(X,Z)|\mathcal{G})dP &= \int_A h(X,Z)dP \\
&= E(Yh(X,Z)) \\
&= \int_{\mathbb{R}^3} yh(x,z)\mu_X(dx)\mu_{Y,Z}(dy,dz) \\
&= \int_{\mathbb{R}^2} yg(z)\mu_{Y,Z}(dy,dz) \\
&= E(Yg(Z)) \\
&= E(1_A g(Z)) \\
&= \int_A g(Z)dP \\
&= \int_A E(h(X,z))|_{z=Z}dP.
\end{aligned}
$$

15. $S \vee T$ and $S \wedge T$ are stopping times.

Proof: For all $t \geq 0$, we know that $\{S \leq t\} \in \mathcal{F}_t$ and $\{T \leq t\} \in \mathcal{F}_t$. Hence,

$$\{S \vee T \leq t\} = \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t,$$

and

$$\{S \wedge T \leq t\} = \{S \leq t\} \cup \{T \leq t\} \in \mathcal{F}_t.$$

16. Given a stopping time T,

$$\mathcal{F}_T = \{A: \; A \cap \{T \le t\} \in \mathcal{F}_t, \text{ for all } t \ge 0$$

is a σ-field.

Proof: It is obvious that Ω is in $\mathcal{F}_T$ and $\mathcal{F}_T$ is closed under intersection of countably infinite many subsets. We only need to show that $\mathcal{F}_T$ is closed under complement. If $A \in \mathcal{F}_T$ then $A \cap \{T \le t\} \in \mathcal{F}_t$, and hence

$$A^c \cap \{T \le t\} = (A \cup \{T > t\})^c = ((A \cap \{T \le t\}) \cup \{T > t\})^c \in \mathcal{F}_t.$$

17. $S \leq T \Rightarrow \mathcal{F}_S \subset \mathcal{F}_T$.

Proof: If $A \in \mathcal{F}_S$, then for any $t \geq 0$ we have $A \cap \{S \leq t\} \in \mathcal{F}_t$.

Since $S \leq T$, we also have $\{T \leq t\} \subset \{S \leq t\}$. Thus

$$A \cap \{T \leq t\} = A \cap \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t,$$

which implies $A \in \mathcal{F}_T$.

18. Let $\{X_t,\ t \geq 0\}$ be a continuous and adapted process. The *hitting time* of a set $A \subset \mathbb{R}$ is defined by

$$T_A = \inf\{t \geq 0 : \ X_t \in A\}.$$

Then, if A is open or closed, T_A is a stopping time.

Proof: If A is an open set, then for any $t > 0$, we have

$$\{T_A < t\} = \bigcup_{r \in \mathbb{Q}^+, r < t} \{X_r \in A\} \in \mathcal{F}_t.$$

By Exercise 1 in Professor Nualart's lecture notes, we can show that T_A is a stopping time.

If A is a closed set, then for any $t \geq 0$ we have

$$\{T_A \geq t\} = \bigcap_{r \in \mathbb{Q}^+, r < t} \{X_r \notin A\} \in \mathcal{F}_t.$$

Hence $\{T_A < t\} \in \mathcal{F}_t$. By Exercise 1 in Professor Nualart's lecture notes, we can show that T_A is a stopping time.

19. Let X_t be an adapted stochastic process with right-continuous paths and $T < \infty$ is a stopping time. Then the random variable

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

is $\mathcal{F}_T$-measurable.

<u>Proof:</u> For any $n \in \mathbb{N}$, define

$$T_n = \sum_{i=0}^{\infty} \frac{i+1}{2^n} 1_{\{\frac{i}{2^n} < T \le \frac{i+1}{2^n}\}}.$$

For any $t \ge 0$,

$$\{T_n \le t\} = \bigcup_{i, \frac{i+1}{2^n} \le t} \{T \le \frac{i+1}{2^n}\} \in \mathcal{F}_t.$$

Then T_n is a stopping time for each n, and moreover, $T_n \downarrow T$.

Then it suffices to show that for any open set $A \subset \mathbb{R}$

$$\{X_{T_n} \in A\} \cap \{T_n \leq t\} \in \mathcal{F}_t, \ t \geq 0.$$

In fact,

$$\begin{aligned}&\{X_{T_n} \in A\} \cap \{T_n \leq t\}\\ =\ &\cup_{\frac{k}{2^n} \leq t} (\{X_{\frac{k}{2^n}} \in A\} \cap \{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\}) \in \mathcal{F}_t.\end{aligned}$$

Since $T_n \downarrow T$ and the process X is right-continuous, we can let n go to ∞ and obtain

$$\{X_T \in A\} \cap \{T \leq t\} = \lim_{n \to \infty} \{X_{T_n} \in A\} \cap \{T_n \leq t\} \in \mathcal{F}_t.$$

Application to Brownian hitting times

Let B_t be a Brownian motion. Fix $a \in \mathbb{R}$ and consider the hitting time

$$\tau_a = \inf\{t \geq 0 : B_t = a\}.$$

Proposition

If $a < 0 < b$, then

$$P(\tau_a < \tau_b) = \frac{b}{b-a}.$$

Proof: From (4) and (5), we can get $\tau_a < \infty$ and $\tau_b < \infty$ almost surely.

Exercise 18 implies that τ_a and τ_b are two stopping times, and hence Exercise 15 implies that $\tau_a \wedge \tau_b$ is also a stopping time.

For any $t \geq 0$, $\tau_a \wedge \tau_b \wedge t$ is a bounded stopping time. Then by the optional stopping theorem, we have

$$E(B_{\tau_a \wedge \tau_b \wedge t}) = E(B_0) = 0.$$

Since

$$a \leq B_{\tau_a \wedge \tau_b \wedge t} \leq b, \ \forall t \geq 0$$

and

$$\lim_{t \to \infty} B_{\tau_a \wedge \tau_b \wedge t} = B_{\tau_a \wedge \tau_b}$$

almost surely, by the dominated convergence theorem, we have

$$E(B_{\tau_a \wedge \tau_b}) = E(\lim_{t \to \infty} B_{\tau_a \wedge \tau_b \wedge t}) = B_{\tau_a \wedge \tau_b \wedge t} E(B_{\tau_a \wedge \tau_b \wedge t}) = 0.$$

The random variable $B_{\tau_a \wedge \tau_b}$ takes only two values a and b and its distribution is given by

$$B_{\tau_a \wedge \tau_b} = \begin{cases} a, & \text{with probability } P(\tau_a < \tau_b), \\ b, & \text{with probability } P(\tau_a > \tau_b). \end{cases}$$

Hence,

$$\begin{aligned} E(B_{\tau_a \wedge \tau_b}) &= aP(\tau_a < \tau_b) + bP(\tau_a > \tau_b) \\ &= aP(\tau_a < \tau_b) + b(1 - P(\tau_a < \tau_b)) \\ &= 0. \end{aligned}$$

Solving this equation for $P(\tau_a < \tau_b)$, we obtain

$$P(\tau_a < \tau_b) = \frac{b}{b - a}.$$

Proposition

Let $T = \inf\{t \geq 0 : B_t \notin (a, b)\}$*, where* $a < 0 < b$*. Then*

$$E(T) = -ab.$$

Proof: In fact $T = \tau_a \wedge \tau_b < \infty$ almost surely.

Using that $B_t^2 - t$ is a martingale, we get, by the optional stopping theorem,

$$E(B_{T \wedge t}^2 - (T \wedge t)) = 0,$$

that is,

$$E(B_{T \wedge t}^2) = E(T \wedge t). \tag{7}$$

Since $T \wedge t \uparrow T$ as $t \uparrow \infty$, by the monotone convergence theorem we get

$$E(T) = E(\lim_{t\to\infty}(T \wedge t)) = \lim_{t\to\infty} E(T \wedge t). \tag{8}$$

Since $B^2_{T\wedge t}$ is bounded for all $t \geq 0$ and $B^2_{T\wedge t} \to B_T$ almost surely as $t \to \infty$, using the dominated convergence theorem and (7) and (8) we have

$$E(B_T^2) = E(\lim_{t\to\infty} B^2_{T\wedge t}) = \lim_{t\to\infty} E(B^2_{T\wedge t}) = E(T).$$

From the previous Proposition, we get

$$E(B_T^2) = a^2\left(\frac{b}{b-a}\right) + b^2\left(1 - \frac{b}{b-a}\right) = -ab.$$

Therefore,

$$E(T) = -ab.$$

1. Let Z be a Gaussian random variable with law $N(0, \sigma^2)$. From the expression

$$E(e^{\lambda Z}) = e^{\frac{1}{2}\lambda^2\sigma^2},$$

deduce the following formulas for the moments of Z:

$$E(Z^{2k}) = \frac{(2k)!}{2^k K!}\sigma^{2k}, \ k = 1, 2, \ldots,$$

$$E(Z^{2k-1}) = 0, \ k = 1, 2, \ldots .$$

2. Let $\{B_t,\ t \in [0,1]\}$ be a standard Brownian motion. Show that

$$\xi = \int_0^1 B_t dt$$

is a well-defined random variable. Calculate its first and second moments.