Basic properties for Brownian motion

1. Selfsimilarity:
For any \( a > 0 \), the process \( \{ a^{-\frac{1}{2}} B_{at}, t \geq 0 \} \) is also a Brownian motion.

Proof: Let \( Y_t = a^{-\frac{1}{2}} B_{at} \). It is obvious that the process \( \{ Y_t = a^{-\frac{1}{2}} B_{at}, t \geq 0 \} \) is an Gaussian process with zero mean and continuous trajectories. It is sufficient to show that for any \( 0 \leq s \leq t < +\infty \) the covariance function \( E(Y_t Y_s) = s \) which is proved as follows:

\[
E(Y_t Y_s) = E(a^{-\frac{1}{2}} B_{at} a^{-\frac{1}{2}} B_{as}) = a^{-1} E(B_{at} B_{as}) = a^{-1}(as) = s.
\]
2. For any $h > 0$, the process \{ $B_{t+h} - B_h$, $t \geq 0$ \} is a Brownian motion.

**Proof:** Let $Y_t = B_{t+h} - B_h$. We know that the process
\{ $Y_t = B_{t+h} - B_h$, $t \geq 0$ \} is an Gaussian process with zero mean and continuous trajectories. Next, we calculate its covariance function: for any $0 \leq s \leq t < +\infty$,

$$E(Y_t Y_s) = E((B_{t+h} - B_h)(B_{s+h} - B_h))$$

$$= E(B_{t+h}B_{s+h}) - E(B_hB_{s+h}) - E(B_{t+h}B_h) + E(B_hB_h)$$

$$= (s + h) - h - h + h$$

$$= s.$$

Therefore, the process \{ $B_{t+h} - B_h$, $t \geq 0$ \} is a Brownian motion.
3. The process \(-B_t, t \geq 0\) is a Brownian motion.

**Proof:** This process is a Gaussian process with zero mean and continuous trajectories, and for any \(0 \leq s \leq t < +\infty\),

\[
E((-B_t)(-B_s)) = E(B_t B_s) = s.
\]

Thus, The process \(-B_t, t \geq 0\) is a Brownian motion.
If \( \xi \) is a random variable with \( E(|\xi|^p) < \infty \), then for any \( \lambda > 0 \)

\[
P(|\xi| > \lambda) \leq \frac{E(|\xi|^p)}{\lambda^p}.
\]
Borel-Cantelli lemma

Lemma

Let $A_1, A_2, A_3, \ldots$ be a sequence of events in a probability space. If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then the probability that infinitely many of them occur is 0, that is,

$$P(\limsup_{n \to \infty} A_n) = 0.$$ 

Note that

$$\limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$
Strong law of large numbers

**Theorem**

If $\xi_1, \xi_2, \xi_3, \ldots$ is a sequence of iid random variables and $\mu = E(\xi_1) < \infty$, then

$$\frac{\sum_{i=1}^{n} \xi_i}{n} = \frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n} \to \mu$$

almost surely as $n \to \infty$. 
4. Almost surely \( \lim_{t \to \infty} \frac{B_t}{t} = 0 \) and the process

\[
X_t = \begin{cases} 
  tB_{1/t}, & t > 0; \\
  0, & t = 0
\end{cases}
\]

is a Brownian motion.

**Proof:** We first show that \( \lim_{n \to \infty} \frac{B_n}{n} = 0 \) which is true by using the strong law of large numbers since

\[
\frac{B_n}{n} = \frac{\sum_{i=1}^{n} (B_i - B_{i-1})}{n}
\]

and \( B_1 - B_0, B_2 - B_1, \ldots, B_n - B_{n-1}, \ldots \) are iid with law \( N(0,1) \).
Then for any $t > 0$ and $t \in [n, n+1)$ for some $n$, we have

$$
\left| \frac{B_t}{t} - \frac{B_n}{n} \right| = \left| \frac{B_t}{t} - \frac{B_n}{t} + \frac{B_n}{t} - \frac{B_n}{n} \right|
\leq \sup_{s \in [n,n+1]} \left| B_s - B_n \right| \frac{1}{t} + \left| B_n \right| \left( \frac{1}{n} - \frac{1}{t} \right)
\leq \frac{\sup_{s \in [n,n+1]} \left| B_s - B_n \right|}{n} + \frac{\left| B_n \right|}{n}.
\quad (1)
$$

Note that $\lim_{n \to \infty} \frac{B_n}{n} = 0$ implies

$$
\lim_{n \to \infty} \frac{\left| B_n \right|}{n} = 0.
\quad (2)
$$
Since the sequence of random variables

$$\{ \sup_{s \in [n-1,n]} |B_s - B_n|, n \geq 1 \}$$

are iid with the same law as the integrable random variable \( \sup_{s \in [0,1]} |B_s| \), using the strong law of large numbers again we can show that

$$\frac{\sum_{i=1}^{n} \sup_{s \in [i-1,i]} |B_s - B_i|}{n} \to E( \sup_{s \in [0,1]} |B_s| ) < \infty$$

almost surely, as \( n \to \infty \).
Then, the above results yields

\[
\sup_{s \in [n, n+1]} |B_s - B_n| \frac{n}{n} = \sum_{i=1}^{n} \sup_{s \in [i-1, i]} |B_s - B_i| \frac{n}{n} - \sum_{i=1}^{n-1} \sup_{s \in [i-1, i]} |B_s - B_i| \frac{n}{n-1} \left( \frac{n-1}{n} \right)
\]

\[
\rightarrow E(\sup_{s \in [0,1]} |B_s|) - E(\sup_{s \in [0,1]} |B_s|)(1) = 0,
\]

as \( n \to \infty \).

The fact of \( \frac{|B_t|}{t} \leq \left| \frac{B_t}{t} - \frac{B_n}{n} \right| + \frac{|B_n|}{n} \) together with (1), (2) and (3) imply

\[
\lim_{t \to \infty} \frac{B_t}{t} = 0.
\]
For the process \( X \), using \( \lim_{t \to \infty} \frac{B_t}{t} = 0 \) almost surely we can prove that the trajectories are continuous almost surely.

The process \( X \) is a Gaussian process with zero mean. For any \( 0 \leq s \leq t < \infty \), we have

\[
E(X_tX_s) = E((tB_{1/t})(sB_{1/s})) = (t)(s)E(B_{1/t}B_{1/s}) = (t)(s)(1/t) = s.
\]

Therefore, \( X \) is also a Brownian motion.
Fatou’s lemma

**Lemma**

- Let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$, then

\[
E(\liminf_{n \to \infty} \xi_n) \leq \liminf_{n \to \infty} E(\xi_n).
\]

- Let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$. If there exists a non-negative integrable random variable $\eta$ such that $\xi_n \leq \eta$ for all $n$, then

\[
\limsup_{n \to \infty} E(\xi_n) \leq E(\limsup_{n \to \infty} \xi_n).
\]
Hewitt-Savage 0-1 law

Theorem

Let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of independent and identically-distributed (iid) random variables taking values in a set $\mathcal{E}$. Then, any event whose occurrence or non-occurrence is determined by the values of these random variables and whose occurrence or non-occurrence is unchanged by finite permutations of the indices, has probability either 0 or 1.
5. \( \limsup_{t \to \infty} \frac{B_t}{\sqrt{t}} = \infty \) and \( \liminf_{t \to \infty} \frac{B_t}{\sqrt{t}} = -\infty \).

Proof: It suffices to show that \( \limsup_{n \to \infty} \frac{B_n}{\sqrt{n}} = \infty \) and \( \liminf_{n \to \infty} \frac{B_n}{\sqrt{n}} = -\infty \). For any positive constant \( C \), we have

\[
P \left( \limsup_{n \to \infty} \frac{B_n}{\sqrt{n}} > C \right) = P \left( \limsup_{n \to \infty} \left\{ \frac{B_n}{\sqrt{n}} > C \right\} \right)
\geq \limsup_{n \to \infty} P \left( \left\{ \frac{B_n}{\sqrt{n}} > C \right\} \right) \quad \text{(Fatou's lemma)}
= \limsup_{n \to \infty} P \left( B_1 > C \right) \quad \text{(Scaling property)}
> 0.
\]
Let $\xi_n = B_n - B_{n-1}$, $n \geq 1$. Then $\xi_1, \xi_2, \xi_3, \ldots$ is a sequence of iid random variables. Note that the event

$$\limsup_{n \to \infty} \left\{ \frac{B_n}{\sqrt{n}} > C \right\} = \limsup_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{\xi_k}{\sqrt{n}} > C \right\}$$

is unchanged by finite permutations of the indices. Hence, by Hewitt-Savage 0 – 1 law together with the last inequality on previous page, we have

$$P \left( \limsup_{n \to \infty} \left\{ \frac{B_n}{\sqrt{n}} > C \right\} \right) = 1.$$

Since $C$ is arbitrary, we can show that $\limsup_{n \to \infty} \frac{B_n}{\sqrt{n}} = \infty$.

Since $\{-B_t, t \geq 0\}$ is also a Brownian motion, we can show that

$$\liminf_{n \to \infty} \frac{B_n}{\sqrt{n}} = \liminf_{n \to \infty} \frac{-B_n}{\sqrt{n}} = -\limsup_{n \to \infty} \frac{B_n}{\sqrt{n}} = -\infty.$$
Remark: From the Exercise 5, we obtain the following result

\[ P(\limsup_{t \to \infty} B_t = +\infty) = 1, \quad (4) \]

and

\[ P(\liminf_{t \to \infty} B_t = -\infty) = 1. \quad (5) \]

Furthermore,

\[ P(\limsup_{t \to \infty} B_t = +\infty, \liminf_{t \to \infty} B_t = -\infty) = 1. \quad (6) \]
6. Show that for each fixed $t \geq 0$

$$P \left( \omega \in \Omega : \limsup_{h \to 0^+} \frac{B_{t+h} - B_t}{h} = \infty \right) = 1.$$ 

Proof: Given this Brownian motion $B$, we construct another Brownian motion $X$ defined in Exercise 4:

$$X_t = \begin{cases} tB_{1/t}, & t > 0; \\ 0, & t = 0. \end{cases}$$

Then for any $t \geq 0$, we obtain

$$\limsup_{n \to \infty} \frac{B_{t+\frac{1}{n}} - B_t}{\frac{1}{n}} = \limsup_{n \to \infty} \frac{B_{\frac{1}{n}}}{\frac{1}{n}} \quad \text{(Exercise 2)}$$

$$\geq \limsup_{n \to \infty} \sqrt{n}B_{\frac{1}{n}}$$

$$= \limsup_{n \to \infty} \frac{X_n}{\sqrt{n}} = \infty \quad \text{(Previous exercise).}$$
7. Almost surely the paths of $B$ are not differentiable at any point $t \geq 0$, more precisely, the set

$$\{ \omega \in \Omega : \text{for each } t \in [0, \infty), \text{ either } \limsup_{h \to 0^+} \frac{B_{t+h} - B_t}{h} = \infty$$

or

$$\liminf_{h \to 0^+} \frac{B_{t+h} - B_t}{h} = -\infty \}$$

contains an event $A \in \mathcal{F}$ with $P(A) = 1$.

Proof: Denote

$$D^+ B_t = \limsup_{h \to 0^+} \frac{B_{t+h} - B_t}{h}$$

and

$$D^- B_t = \liminf_{h \to 0^+} \frac{B_{t+h} - B_t}{h}.$$
It suffices to show the result on the interval $[0, 1]$. For any fixed integers $j \geq 1$ and $k \geq 1$, define

$$A_{jk} = \bigcup_{t \in [0,1]} \bigcap_{h \in [0,1/k]} \{\omega \in \Omega : |B_{t+h}(\omega) - B_t(\omega)| \leq jh\}.$$  

Note that $A_{jk}$ is not an event, that is $A_{jk} \notin \mathcal{F}$. Note also that

$$\{\omega \in \Omega : -\infty < D^+ B_t(\omega) \leq D^+ B_t(\omega) < \infty, \text{ for some } t \in [0, 1]\} = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_{jk}.$$  

Then the proof of the result will be complete if we can find an event $C \in \mathcal{F}$ with $P(C) = 0$ such that $A_{jk} \subseteq C$ for all $j$ and $k$. 
For any fixed $j$ and $k$, we consider a sample path $\omega \in A_{jk}$, that is, there exists $t \in [0, 1]$ with

$$|B_{t+h}(\omega) - B_t(\omega)| \leq jh$$

for all $h \in [0, 1/k]$.

Let $n$ be an integer with $n \geq 4k$. Then there exists some $i$, $1 \leq i \leq n$ such that $\frac{i-1}{n} \leq t < \frac{i}{n}$. Then it is also easy to verify that, for $l = 1, 2, 3$,

$$\frac{i + l}{n} - t \leq \frac{l + 1}{n} \leq \frac{1}{k}.$$

Thus,

$$|B_{\frac{i+1}{n}}(\omega) - B_{\frac{i}{n}}(\omega)| \leq |B_{\frac{i+1}{n}}(\omega) - B_t(\omega)| + |B_{\frac{i}{n}}(\omega) - B_t(\omega)| \leq \frac{2j}{n} + \frac{j}{n} = \frac{3j}{n},$$
\[
|B_{i+2}^n(\omega) - B_{i+1}^n(\omega)| \leq |B_{i+2}^n(\omega) - B_t^n(\omega)| + |B_{i+1}^n(\omega) - B_t^n(\omega)| \\
\leq \frac{3j}{n} + \frac{2j}{n} = \frac{5j}{n},
\]

and

\[
|B_{i+3}^n(\omega) - B_{i+2}^n(\omega)| \leq |B_{i+3}^n(\omega) - B_t^n(\omega)| + |B_{i+2}^n(\omega) - B_t^n(\omega)| \\
\leq \frac{4j}{n} + \frac{3j}{n} = \frac{7j}{n}.
\]

Now denote

\[
C_i^{(n)} = \bigcap_{l=1}^{3} \left\{ \omega \in \Omega : \left| B_{i+l}^n(\omega) - B_{i+l-1}^n(\omega) \right| \leq \frac{(2l + 1)j}{n} \right\}.
\]

Then, we can see that \(C_i^{(n)}\) is an event in \(\mathcal{F}\) and \(A_{jk} \subseteq \bigcup_{i=1}^{n} C_i^{(n)}\) for all \(n \geq 4k\).
Note that the random variables
\[ \sqrt{n} \left( B_{i+l} - B_{i+l-1} \right), \quad l = 1, 2, 3, \]
are iid with law $N(0, 1)$. Note also that if a random variable $\xi$ has law $N(0, 1)$ then
\[ P(|\xi| \leq x) \leq x, \quad \text{for any } x \geq 0. \]
Therefore, we have
\[ P(C_i^{(n)}) \leq \left( \frac{3j}{\sqrt{n}} \right) \left( \frac{5j}{\sqrt{n}} \right) \left( \frac{7j}{\sqrt{n}} \right) = \frac{105j^3}{n^2}, \quad i = 1, 2, \ldots, n. \]
Then
\[ P\left( \bigcup_{i=1}^{n} C_i^{(n)} \right) \leq \frac{105j^3}{\sqrt{n}}. \]
Take

\[ C = \bigcap_{n=4k}^{\infty} \bigcup_{i=1}^{n} C_i^{(n)} \in \mathcal{F}. \]

Then \( A_{jk} \subseteq C \) and it also holds that

\[ P(C) = P\left( \bigcap_{n=4k}^{\infty} \bigcup_{i=1}^{n} C_i^{(n)} \right) \leq \inf_{n \geq 4k} P\left( \bigcup_{i=1}^{n} C_i^{(n)} \right) = 0, \]

which completes the proof.
8. Linearity:

\[ E(aX + bY \mid \mathcal{G}) = aE(X \mid \mathcal{G}) + bE(Y \mid \mathcal{G}). \]

**Proof:** \(aE(X \mid \mathcal{G}) + bE(Y \mid \mathcal{G})\) is \(\mathcal{G}\)-measurable. For any \(A \in \mathcal{G}\),

\[
\begin{align*}
\int_A E(aX + bY \mid \mathcal{G}) \, dP &= \int_A (aX + bY) \, dP \\
&= a \int_A X \, dP + b \int_A Y \, dP \\
&= a \int_A E(X \mid \mathcal{G}) \, dP + b \int_A E(Y \mid \mathcal{G}) \, dP \\
&= \int_A (aE(X \mid \mathcal{G}) + bE(Y \mid \mathcal{G})) \, dP,
\end{align*}
\]

which implies

\[ E(aX + bY \mid \mathcal{G}) = aE(X \mid \mathcal{G}) + bE(Y \mid \mathcal{G}). \]
9. \( E(E(X|\mathcal{G})) = E(X) \).

**Proof:** In the definition of conditional expectation, let \( A = \Omega \), then

\[ E(E(X|\mathcal{G})) = \int_{\Omega} E(X|\mathcal{G})dP = \int_{\Omega} XdP = E(X). \]
10. If $X$ and $\mathcal{G}$ are independent, then $E(X|\mathcal{G}) = E(X)$.

Proof: It is obvious that $E(X)$ is $\mathcal{G}$-measurable. For any $A \in \mathcal{G}$,

$$\int_A E(X|\mathcal{G})dP = \int_A XdP = E(1_A X)$$

$$= E(1_A)E(X) \quad (X \text{ and } 1_A \text{ are independent})$$

$$= P(A)E(X)$$

$$= \int_A E(X)dP.$$

Thus, it implies that $E(X|\mathcal{G}) = E(X)$. 
11. If $X$ is $\mathcal{G}$-measurable, then $E(X|\mathcal{G}) = X$.

Proof: The proof follows from the fact that $X$ is $\mathcal{G}$-measurable and the definition of conditional expectation: for any $A \in \mathcal{G}$

$$\int_A E(X|\mathcal{G})dP = \int_A XdP.$$
**Theorem**

Let $\xi_1, \xi_2, \xi_3, \ldots$ be a sequence of random variables. If the sequence $\{\xi_n\}$ converges almost surely to a random variable $\xi$ and there exists a positive and integrable random variable $\eta$ such that $|\xi_n| \leq \eta$ for all $n$, then

$$
\lim_{n \to \infty} E(|\xi_n - \xi|) = 0,
$$

which also implies

$$
\lim_{n \to \infty} E(\xi_n) = E(\xi).
$$
12. If $Y$ is bounded and $\mathcal{G}$-measurable, then

$$E(XY|\mathcal{G}) = YE(X|\mathcal{G}).$$

**Proof:** If $Y = 1_B$ is an indicator function where $B \in \mathcal{G}$, then for any $A \in \mathcal{G}$

$$\int_A E(YX|\mathcal{G})dP = \int_A 1_B XdP = \int_{A \cap B}XdP$$

$$= \int_{A \cap B} E(X|\mathcal{G})dP$$

$$= \text{int}_A 1_B E(X|\mathcal{G})dP,$$

which implies $E(XY|\mathcal{G}) = YE(X|\mathcal{G})$.

By the linearity of conditional expectation, we know that the result is true if $Y$ is a simple function (a linear combination of indicator functions).
If $Y$ is bounded $\mathcal{G}$-measurable function, then there exists a sequence of simple functions $Y_n$ such that $|Y_n| \leq |Y|$ and $\lim_{n \to \infty} Y_n = Y$ almost surely.

Then by dominated convergence theorem, we can show that for any $A \in \mathcal{G}$,

$$\int_A E(YX|\mathcal{G})dP = \int_A YXdP = \lim_{n \to \infty} \int_A Y_nXdP$$

$$= \lim_{n \to \infty} \int_A Y_nE(X|\mathcal{G})dP$$

$$= \int_A YE(X|\mathcal{G})dP,$$

which proves $E(XY|\mathcal{G}) = YE(X|\mathcal{G})$. 
13. Given two $\sigma$-fields $B \subset G$, then

$$E(E(X|B)|G) = E(E(X|G)|B) = E(X|B).$$

Proof: Since $B \subset G$, we know that $E(X|B)$ is $G$-measurable. Then by Exercise 4, we obtain

$$E(E(X|B)|G) = E(X|B).$$

For any $A \in B \subset G$, we have

$$\int_A E(E(X|G)|B) dP = \int_A E(X|G) dP = \int_A X dP = \int_A E(X|B) dP,$$

which implies

$$E(E(X|G)|B) = E(X|B).$$
14. Let $X$ and $Z$ be such that

(i) $Z$ is $\mathcal{G}$-measurable.

(ii) $X$ is independent of $\mathcal{G}$.

Suppose that $E(h(X, Z) | \mathcal{G}) < \infty$. Then,

$$E(h(X, Z) | \mathcal{G}) = E(h(X, z))_{|z=Z}.$$ 

Proof: Denote $g(z) = E(h(X, z))$ and $\mu_X(E) = P(X \in E)$ for $E \in \mathcal{B}(\mathbb{R})$. For any $A \in \mathcal{G}$, we denote

$$Y = 1_A;$$

$$\mu_{Y,Z}(F) = P((Y, Z) \in F), \ F \in \mathcal{B}(\mathbb{R}^2).$$
Then,

\[
\int_A E(h(X, Z) \mid \mathcal{G}) dP = \int_A h(X, Z) dP =\]

\[
E(Yh(X, Z)) = \int_{\mathbb{R}^3} yh(x, z) \mu_X(dx) \mu_Y, Z(dy, dz) = \int_{\mathbb{R}^2} yg(z) \mu_Y, Z(dy, dz) = E(Yg(Z)) = E(1_A g(Z)) = \int_A g(Z) dP = \int_A E(h(X, z)) \ Greek\mid z=Z dP.\]
15. $S \lor T$ and $S \land T$ are stopping times.

**Proof:** For all $t \geq 0$, we know that \{ $S \leq t$ \} $\in \mathcal{F}_t$ and \{ $T \leq t$ \} $\in \mathcal{F}_t$. Hence,

$$\{ S \lor T \leq t \} = \{ S \leq t \} \cap \{ T \leq t \} \in \mathcal{F}_t,$$

and

$$\{ S \land T \leq t \} = \{ S \leq t \} \cup \{ T \leq t \} \in \mathcal{F}_t.$$
16. Given a stopping time $T$,

$$\mathcal{F}_T = \{ A : A \cap \{ T \leq t \} \in \mathcal{F}_t, \text{ for all } t \geq 0 \}$$

is a $\sigma$-field.

**Proof:** It is obvious that $\Omega$ is in $\mathcal{F}_T$ and $\mathcal{F}_T$ is closed under intersection of countably infinite many subsets. We only need to show that $\mathcal{F}_T$ is closed under complement. If $A \in \mathcal{F}_T$ then $A \cap \{ T \leq t \} \in \mathcal{F}_t$, and hence

$$A^c \cap \{ T \leq t \} = (A \cup \{ T > t \})^c = ((A \cap \{ T \leq t \}) \cup \{ T > t \})^c \in \mathcal{F}_t.$$
17. $S \leq T \Rightarrow \mathcal{F}_S \subset \mathcal{F}_T$.

Proof: If $A \in \mathcal{F}_S$, then for any $t \geq 0$ we have $A \cap \{S \leq t\} \in \mathcal{F}_t$.

Since $S \leq T$, we also have $\{T \leq t\} \subset \{S \leq t\}$. Thus

$$A \cap \{T \leq t\} = A \cap \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t,$$

which implies $A \in \mathcal{F}_T$. 

18. Let \( \{X_t, \ t \geq 0\} \) be a continuous and adapted process. The \textit{hitting time} of a set \( A \subset \mathbb{R} \) is defined by

\[
T_A = \inf\{ t \geq 0 : X_t \in A \}.
\]

Then, if \( A \) is open or closed, \( T_A \) is a stopping time.

\textbf{Proof:} If \( A \) is an open set, then for any \( t > 0 \), we have

\[
\{T_A < t\} = \bigcup_{r \in \mathbb{Q}^+, r < t} \{X_r \in A\} \in \mathcal{F}_t.
\]

By Exercise 1 in Professor Nualart’s lecture notes, we can show that \( T_A \) is a stopping time.
If $A$ is a closed set, then for any $t \geq 0$ we have

$$\{ T_A \geq t \} = \bigcap_{r \in \mathbb{Q}^+, r < t} \{ X_r \notin A \} \in \mathcal{F}_t.$$ 

Hence $\{ T_A < t \} \in \mathcal{F}_t$. By Exercise 1 in Professor Nualart’s lecture notes, we can show that $T_A$ is a stopping time.
19. Let $X_t$ be an adapted stochastic process with right-continuous paths and $T < \infty$ is a stopping time. Then the random variable

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

is $\mathcal{F}_T$-measurable.

**Proof:** For any $n \in \mathbb{N}$, define

$$T_n = \sum_{i=0}^{\infty} \frac{i + 1}{2^n} \mathbf{1}_{\left\{ \frac{i}{2^n} < T \leq \frac{i+1}{2^n} \right\}}.$$

For any $t \geq 0$,

$$\{T_n \leq t\} = \bigcup_{i, \frac{i+1}{2^n} \leq t} \{T \leq \frac{i + 1}{2^n}\} \in \mathcal{F}_t.$$

Then $T_n$ is a stopping time for each $n$, and moreover, $T_n \downarrow T$. 
Then it suffices to show that for any open set \( A \subset \mathbb{R} \)

\[
\{ X_{T_n} \in A \} \cap \{ T_n \leq t \} \in \mathcal{F}_t, \ t \geq 0.
\]

In fact,

\[
\{ X_{T_n} \in A \} \cap \{ T_n \leq t \} = \bigcup_{n \geq 1} \{ X_{k2^n} \in A \} \cap \left\{ \frac{k-1}{2^n} < T \leq \frac{k}{2^n} \right\} \in \mathcal{F}_t.
\]

Since \( T_n \downarrow T \) and the process \( X \) is right-continuous, we can let \( n \) go to \( \infty \) and obtain

\[
\{ X_T \in A \} \cap \{ T \leq t \} = \lim_{n \to \infty} \{ X_{T_n} \in A \} \cap \{ T_n \leq t \} \in \mathcal{F}_t.
\]
Let $B_t$ be a Brownian motion. Fix $a \in \mathbb{R}$ and consider the hitting time

$$\tau_a = \inf\{ t \geq 0 : B_t = a \}.$$

**Proposition**

If $a < 0 < b$, then

$$P(\tau_a < \tau_b) = \frac{b}{b - a}.$$

**Proof:** From (4) and (5), we can get $\tau_a < \infty$ and $\tau_b < \infty$ almost surely.

Exercise 18 implies that $\tau_a$ and $\tau_b$ are two stopping times, and hence Exercise 15 implies that $\tau_a \land \tau_b$ is also a stopping time.
For any $t \geq 0$, $\tau_a \land \tau_b \land t$ is a bounded stopping time. Then by the optional stopping theorem, we have

$$E(B_{\tau_a \land \tau_b \land t}) = E(B_0) = 0.$$ 

Since

$$a \leq B_{\tau_a \land \tau_b \land t} \leq b, \ \forall t \geq 0$$

and

$$\lim_{t \to \infty} B_{\tau_a \land \tau_b \land t} = B_{\tau_a \land \tau_b}$$

almost surely, by the dominated convergence theorem, we have

$$E(B_{\tau_a \land \tau_b}) = E(\lim_{t \to \infty} B_{\tau_a \land \tau_b \land t}) = B_{\tau_a \land \tau_b} \land t E(B_{\tau_a \land \tau_b \land t}) = 0.$$ 

The random variable $B_{\tau_a \land \tau_b}$ takes only two values $a$ and $b$ and its distribution is given by

$$B_{\tau_a \land \tau_b} = \begin{cases} a, & \text{with probability } P(\tau_a < \tau_b), \\ b, & \text{with probability } P(\tau_a > \tau_b). \end{cases}$$
Hence,

\[ E(B_{\tau_a \land \tau_b}) = aP(\tau_a < \tau_b) + bP(\tau_a > \tau_b) \]

\[ = aP(\tau_a < \tau_b) + b(1 - P(\tau_a < \tau_b)) \]

\[ = 0. \]

Solving this equation for \( P(\tau_a < \tau_b) \), we obtain

\[ P(\tau_a < \tau_b) = \frac{b}{b - a}. \]
Proposition

Let \( T = \inf \{ t \geq 0 : B_t \not\in (a, b) \} \), where \( a < 0 < b \). Then

\[ E(T) = -ab. \]

Proof: In fact \( T = \tau_a \wedge \tau_b < \infty \) almost surely.

Using that \( B_t^2 - t \) is a martingale, we get, by the optional stopping theorem,

\[ E(B_{T \wedge t}^2 - (T \wedge t)) = 0, \]

that is,

\[ E(B_{T \wedge t}^2) = E(T \wedge t). \quad (7) \]
Since $T \wedge t \uparrow T$ as $t \uparrow \infty$, by the monotone convergence theorem we get

$$E(T) = E\left( \lim_{t \to \infty} (T \wedge t) \right) = \lim_{t \to \infty} E(T \wedge t). \quad (8)$$

Since $B_{T \wedge t}^2$ is bounded for all $t \geq 0$ and $B_{T \wedge t}^2 \to B_T$ almost surely as $t \to \infty$, using the dominated convergence theorem and (7) and (8) we have

$$E(B_T^2) = E\left( \lim_{t \to \infty} B_{T \wedge t}^2 \right) = \lim_{t \to \infty} E(B_{T \wedge t}^2) = E(T).$$

From the previous Proposition, we get

$$E(B_T^2) = a^2 \left( \frac{b}{b-a} \right) + b^2 \left( 1 - \frac{b}{b-a} \right) = -ab.$$

Therefore,

$$E(T) = -ab.$$
1. Let $Z$ be a Gaussian random variable with law $N(0, \sigma^2)$. From the expression

$$E(e^{\lambda Z}) = e^{\frac{1}{2}\lambda^2\sigma^2},$$

deduce the following formulas for the moments of $Z$:

$$E(Z^{2k}) = \frac{(2k)!}{2^k K!} \sigma^{2k}, \; k = 1, 2, \ldots,$$

$$E(Z^{2k-1}) = 0, \; k = 1, 2, \ldots.$$
2. Let \( \{B_t, \ t \in [0, 1]\} \) be a standard Brownian motion. Show that

\[ \xi = \int_0^1 B_t \, dt \]

is a well-defined random variable. Calculate its first and second moments.