Theoretical Tutorial Session 2

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Outline

- Itô's formula
- Martingale representation theorem
- Stochastic differential equations

1. Using Itô's formula to show that $M_t = B_t^3 - 3\int_0^t B_s ds$ is a martingale.

Proof: Let $f(x) = x^3 \in C^2(\mathbb{R})$. Then

$$f'(x) = 3x^2, \text{ and } f''(x) = 6x.$$

Applying Itô's formula to $f(B_t)$ we have

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds,$$

that is,

$$B_t^3 = 3\int_0^t B_s^2 dB_s + 3\int_0^t B_s ds. \tag{1}$$

Then

$$M_t = B_t^3 - 3\int_0^t B_s ds = 3\int_0^t B_s^2 dB_s.$$

Since

$$E\left(\int_0^t (B_s^2)^2 ds\right) = E\left(\int_0^t B_s^4 ds\right) = \int_0^t 3s^2 ds < \infty$$

for all $t \geq 0$, by the basic property of indefinite Itô integral, we can show that

$$M_t = B_t^3 - 3\int_0^t B_s ds = 3\int_0^t B_s^2 dB_s$$

is a martingale.

2. Use Itô's formula to show that

$$tB_t = \int_0^t B_s ds + \int_0^t s dB_s. \tag{2}$$

Proof: Let $f(t, x) = tx$. Then $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\frac{\partial f}{\partial t}(t, x) = x,$$
$$\frac{\partial f}{\partial x}(t, x) = t,$$
$$\frac{\partial^2 f}{\partial x^2}(t, x) = 0.$$

Applying Itô's formula

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds,$$

we obtain

$$tB_t = \int_0^t B_s ds + \int_0^t s dB_s.$$

Note that the above equation give us an integration by parts formula.

3. Check if the process $X_t = B_t^3 - 3tB_t$ is a martingale.

Solution: Using (1) and (2), we can write

$$\begin{aligned} X_t &= B_t^3 - 3tB_t \\ &= 3\int_0^t B_s^2 dB_s + 3\int_0^t B_s ds - 3\left(\int_0^t B_s ds + \int_0^t s dB_s\right) \\ &= \int_0^t (3B_s^2 - 3s) dB_s. \end{aligned}$$

We can also show that

$$E\left(\int_0^t (3B_s^2 - 3s)^2 ds\right) < \infty, \ \forall t \geq 0.$$

Therefore, the process $X_t = B_t^3 - 3tB_t$ is a martingale.

4. Find the stochastic integral representation on the time interval $[0, T]$ of the square integrable random variable B_T^3.

Solution: Using (1) and (2) with $t = T$, we have

$$\begin{aligned} B_T^3 &= 3\int_0^T B_s^2 dB_s + 3\int_0^T B_s ds \\ &= 3\int_0^T B_s^2 dB_s + 3\left(TB_T - \int_0^T s dB_s\right) \\ &= 3\int_0^T B_s^2 dB_s + 3\left(T\int_0^T 1 dB_s - \int_0^T s dB_s\right) \\ &= \int_0^T (3B_s^2 + 3T - 3s) dB_s. \end{aligned}$$

The above is the integral representation for B_T^3 since the process $\{2B_s + 3T - 3s,\ s \in [0, T]\}$ in in $L^2(\mathcal{P})$ and $E(B_T^3) = 0$.

5. Verify that the following processes are martingales:

(a) $X_t = t^2 B_t - 2\int_0^t sBsds$

(b) $X_t = e^{t/2}\cos B_t$

(c) $X_t = e^{t/2}\sin B_t$

(d) $X_t = B_1(t)B_2(t)$, where B_1 and B_2 are two independent Brownian motion.

Solution 5(a): Let $f(t,x) = t^2 x$. Then $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\frac{\partial f}{\partial t}(t,x) = 2tx,$$
$$\frac{\partial f}{\partial x}(t,x) = t^2,$$
$$\frac{\partial^2 f}{\partial x^2}(t,x) = 0.$$

Applying Itô's formula

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds,$$

we get

$$t^2 B_t = \int_0^t 2sB_s ds + \int_0^t s^2 dB_s.$$

Hence, the process

$$X_t = t^2 B_t - 2\int_0^t sB_s ds = \int_0^t s^2 dB_s$$

is a martingale.

Solution 5(b): Let $f(t,x) = e^{t/2}\cos x$. Then $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\begin{aligned}\frac{\partial f}{\partial t}(t,x) &= \frac{1}{2}f(t,x),\\ \frac{\partial f}{\partial x}(t,x) &= -e^{t/2}\sin x,\\ \frac{\partial^2 f}{\partial x^2}(t,x) &= -f(t,x).\end{aligned}$$

Note that

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0, \text{ and } f(0,B_0) = 1.$$

Then we apply Itô's formula and show that the process

$$X_t = e^{t/2}\cos B_t = 1 - \int_0^t e^{s/2}\sin B_s dB_s$$

is a martingale since $E(\int_0^t e^s \sin B_s{}^2 ds) \leq \int_0^t e^s ds < \infty$ for all $t \geq 0$.

Solution 5(c): Let $f(t,x) = e^{t/2}\sin x$. Then $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\frac{\partial f}{\partial t}(t,x) = \frac{1}{2}f(t,x),$$
$$\frac{\partial f}{\partial x}(t,x) = e^{t/2}\cos x,$$
$$\frac{\partial^2 f}{\partial x^2}(t,x) = -f(t,x).$$

Note that

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0, \text{ and } f(0,B_0) = 0.$$

Then we apply Itô's formula and show that the process

$$X_t = e^{t/2}\sin B_t = \int_0^t e^{s/2}\cos B_s dB_s \tag{3}$$

is a martingale since $E(\int_0^t e^s \cos {B_s}^2 ds) \leq \int_0^t e^s ds < \infty$ for all $t \geq 0$.

Solution 5(d): For this exercise, we need to apply Itô's formula in multidimensional case. Let $f(x_1, x_2) = x_1 x_2$. Then $f \in C^2(\mathbb{R}^2)$ and

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = x_2$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = x_1$$

$$\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = 0$$

$$\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = 1$$

Applying multidimensional Itô's formula, one can obtain

$$
\begin{aligned}
f(B_1(t)B_2(t)) = f(B_1(0)B_2(0)) &+ \int_0^t \frac{\partial f}{\partial x_1}(B_1(s), B_2(s))dB_1(s) \\
&+ \int_0^t \frac{\partial f}{\partial x_2}(B_1(s), B_2(s))dB_2(s) \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_1^2}(B_1(s), B_2(s))ds \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x_2^2}(B_1(s), B_2(s))ds \\
&+ \int_0^t \frac{\partial^2 f}{\partial x_1 \partial x_2}(B_1(s), B_2(s))dB_1(s)dB_2(s),
\end{aligned}
$$

then noticing that $dB_1 dB_2 = 0$, we can show that the process

$$X_t = B_1(t)B_2(t) = \int_0^t B_2(s)dB_1(s) + \int_0^t B_1(s)dB_2(s)$$

is a martingale, since
$E\left(\int_0^t B_1(s)^2 ds\right) = E\left(\int_0^t B_2(s)^2 ds\right) = \int_0^t s ds < \infty,\ \forall t \geq 0.$

6. If $f(t,x) = e^{ax - \frac{a^2}{2}t}$ and $Y_t = f(t, B_t) = e^{aB_t - \frac{a^2}{2}t}$ where a is a constant, then prove that Y satisfies the following linear SDE:

$$Y_t = 1 + a\int_0^t Y_s dB_s. \tag{4}$$

<u>Proof:</u> Note that $f(t,x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x) &= -\frac{a^2}{2} f(t,x), \\ \frac{\partial f}{\partial x}(t,x) &= af(t,x), \\ \frac{\partial^2 f}{\partial x^2}(t,x) &= a^2 f(t,x). \end{aligned}$$

Note also that

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0.$$

Applying Itô's formula, we have

$$\begin{aligned} f(t, B_t) &= f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds \\ &= 1 + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s, \end{aligned}$$

that is

$$Y_t = 1 + a\int_0^t Y_s dB_s.$$

Remark: i). Note that

$$E\left(\int_0^t |Y_s|^2 ds\right) = E\left(\int_0^t e^{2aB_s - a^2 s} ds\right)$$
$$= \int_0^t e^{a^2 s} E\left(e^{2aB_s - \frac{(2a)^2}{2} s}\right) ds$$
$$= \int_0^t e^{a^2 s} ds < \infty,$$

for all $t \geq 0$. Hence, the Itô integral $\int_0^t Y_s dB_s$ is well-defined.

ii). The solution of the stochastic differential equation

$$dY_t = aY_t dB_t, \;\; Y_0 = 1$$

is not $Y_t = e^{aB_t}$, but $Y_t = e^{aB_t - \frac{a^2}{2} t}$.

7. Find the stochastic integral representation on the time interval $[0, T]$ of the following square integrable random variables:

(a) $F = B_T$

(b) $F = B_T^2$

(c) $F = e^{B_T}$

(d) $F = \sin B_T$

(e) $F = \int_0^T B_t dt$

(f) $F = \int_0^T tB_t^2 dt$

Solution 7(a): Since $E(B_T) = 0$, the stochastic integral representation for B_T is

$$B_T = \int_0^T 1 dB_t = E(B_T) + \int_0^T 1 dB_t.$$

Solution 7(b): Let $f(x) = x^2$. Then $f \in C^2(\mathbb{R})$ and $f'(x) = 2x$ and $f''(x) = 2$. Using Itô's formula and $E(B_T^2) = T$, we have

$$\begin{aligned} B_T^2 &= B_0^2 + \int_0^T 2B_s dB_s + \frac{1}{2}\int_0^T 2dt \\ &= \int_0^T 2B_s dB_s + T \\ &= E(B_T^2) + \int_0^T 2B_s dB_s. \end{aligned}$$

Solution 7(c): We can calculate that

$$E(e^{B_T}) = e^{\frac{T}{2}}$$

In fact, we can NOT apply Itô's formula directly to get the stochastic integral representation, since if we choose $f(x) = e^x$ and apply Itô's formula to $f(B_T)$, then we get

$$\begin{aligned} e^{B_T} &= e^{B_0} + \int_0^T e^{B_t} dB_t + \frac{1}{2} \int_0^T e^{B_t} dt \\ &= 1 + \int_0^T e^{B_t} dB_t + \frac{1}{2} \int_0^T e^{B_t} dt. \end{aligned}$$

We can not get rid of the integral with respect to dt.

Question: How can we get its stochastic integral representation?

In order to obtain the stochastic integral representation for e^{B_T}, we will need the result (4) in Exercise 6 with $a = 1$ and $t = T$:

$$e^{B_T - \frac{T}{2}} = 1 + \int_0^T e^{B_t - \frac{t}{2}} dB_t.$$

Multiplying $e^{\frac{T}{2}}$ on both sides of the above equation, we obtain the following stochastic integral representation

$$\begin{aligned} e^{B_T} &= e^{\frac{T}{2}} + e^{\frac{T}{2}} \int_0^T e^{B_t - \frac{t}{2}} dB_t \\ &= E(e^{B_T}) + \int_0^T e^{B_t + \frac{T-t}{2}} dB_t. \end{aligned}$$

Solution 7(d): Note that

$$E(\sin B_T) = 0.$$

Since $\sin x$ and e^x are closely related in

$$e^{ix} = \cos x + i \sin x,$$

we can foresee the same problem if we apply Itô's formula directly to $f(B_T) = \sin B_T$.

Instead, we make use of (3) and obtain

$$\begin{aligned} \sin B_T &= e^{-\frac{T}{2}} \int_0^T e^{\frac{t}{2}} \cos B_t dB_t \\ &= E(\sin B_T) + \int_0^T e^{-\frac{T-t}{2}} \cos B_t dB_t. \end{aligned}$$

Solution 7(e): We have

$$E\left(\int_0^T B_t dt\right) = 0.$$

Using (2) with $t = T$ we get

$$\begin{aligned}
\int_0^T B_t dt &= TB_T - \int_0^T t dB_t \\
&= T\int_0^T 1 dB_t - \int_0^T t dB_t \\
&= \int_0^T (T - t) dB_t \\
&= E\left(\int_0^T B_t dt\right) + \int_0^T (T - t) dB_t.
\end{aligned}$$

Solution 7(f): Note that

$$E\left(\int_0^T tB_t^2 dt\right) = \int_0^T tE(B_t^2)dt = \int_0^T t^2 dt = \frac{T^3}{3}.$$

From Part 7(b), we know

$$B_t^2 = 2\int_0^t B_s dB_s + t, \ \forall t \geq 0.$$

Using the above equation and then changing the order of the integrals we have

$$\begin{aligned}\int_0^T tB_t^2 dt &= \int_0^T t\left(2\int_0^t B_s dB_s + t\right) dt \\ &= \int_0^T t^2 dt + 2\int_0^T \int_0^t tB_s dB_s dt \\ &= \frac{T^3}{3} + 2\int_0^T B_s \left(\int_s^T tdt\right) dB_s \\ &= E\left(\int_0^T tB_t^2 dt\right) + \int_0^T (T^2 - s^2)B_s dB_s.\end{aligned}$$

8. Consider an n-dimensional Brownian motion $B(t) = (B_1(t), B_2(t), \cdots, B_n(t))$ and constants α_i, $i = 1, \ldots, n$. Solve the following SDE:

$$dX_t = rX_t dt + X_t \sum_{i=1}^{n} \alpha_i dB_i(t),\ X_0 = x,$$

where $x \in \mathbb{R}$.

<u>Solution:</u> The coefficients in this SDE satisfy the Lipschitz and linear growth conditions, so there exists a unique solution.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the above SDE becomes an ODE

$$dX_t = rX_t dt,\ X_0 = x.$$

and its unique solution is $X_t = xe^{rt}$.

If $\sum_{i=1}^{n} \alpha_i^2 \neq 0$, then by using the standard method mentioned in my first tutorial session we can show that the process

$$\tilde{B}_t = \frac{1}{\sqrt{\sum_{i=1}^{n} \alpha_i^2}} \sum_{i=1}^{n} \alpha_i B_i(t)$$

is a Brownian motion.

Let $Y_t = e^{-rt}$. Then Y satisfies $dY_t = -rY_t dt$. Applying multidimensional Itô's formula to $f(x, y) = xy$ we have

$$\begin{aligned} d(X_t Y_t) &= X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t dY_t + Y_t dX_t \\ &= X_t(-rY_t)dt + Y_t(rX_t dt + X_t \sum_{i=1}^{n} \alpha_i dB_i(t)) \\ &= X_t Y_t \sum_{i=1}^{n} \alpha_i dB_i(t) \\ &= \sqrt{\sum_{i=1}^{n} \alpha_i^2} (X_t Y_t) d\tilde{B}_t. \end{aligned} \tag{5}$$

Thus, $X_t Y_t$ satisfies the linear SDE (5). Note also that $X_0 Y_0 = x$. Then the solution to (5) is

$$X_t Y_t = x \exp\left\{ \sqrt{\sum_{i=1}^{n} \alpha_i^2}\, \tilde{B}_t - \frac{t \sum_{i=1}^{n} \alpha_i^2}{2} \right\}.$$

Therefore,

$$\begin{aligned} X_t &= Y_t^{-1} x \exp\left\{ \sqrt{\sum_{i=1}^{n} \alpha_i^2}\, \tilde{B}_t - \frac{t \sum_{i=1}^{n} \alpha_i^2}{2} \right\} \\ &= x \exp\left\{ rt + \sqrt{\sum_{i=1}^{n} \alpha_i^2}\, \tilde{B}_t - \frac{t \sum_{i=1}^{n} \alpha_i^2}{2} \right\} \\ &= x \exp\left\{ \sum_{i=1}^{n} \alpha_i B_i(t) + \left(r - \frac{\sum_{i=1}^{n} \alpha_i^2}{2} \right) t \right\}. \end{aligned}$$

9. Solve the following stochastic differential equations

$$dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t, \; X_0 = x > 0.$$

For which values of the parameter α the solution explodes?

Solution: Let $Y_t = e^{-\alpha B_t - \frac{\alpha^2 t}{2}}$. Then Y_t satisfies the following SDE:

$$dY_t = -\alpha Y_t dB_t, \; Y_0 = 1.$$

Using Itô's formula, we have

$$\begin{aligned} d(X_t Y_t) &= X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t(-\alpha Y_t dB_t) + Y_t \left(\frac{1}{X_t}dt + \alpha X_t dB_t \right) - \alpha^2 X_t Y_t dt \\ &= \frac{Y_t}{X_t}dt - \alpha^2 X_t Y_t dt, \end{aligned}$$

which implies

$$2(X_tY_t)d(X_tY_t) = 2Y_t^2dt - 2\alpha^2(X_tY_t)^2dt.$$

Then for each fixed ω, $(X_t(\omega)Y_t(\omega))^2$ solves the following linear ODE:

$$\dot{y} = 2Y_t^2(\omega) - 2\alpha^2 y,\ \ y(0) = x^2,$$

whose solution is given by

$$y(t) = e^{-2\alpha^2 t}\left(x^2 + 2\int_0^t e^{2\alpha^2 s} Y_s^2(\omega)\right) ds.$$

Then

$$X_t = Y_t^{-1} e^{-\alpha^2 t}\sqrt{x^2 + 2\int_0^t e^{2\alpha^2 s} Y_s^2 ds}.$$

Since the trajectories of the process Y are continuous on $[0,\infty)$ almost surely, the above integral is well defined for all $t \geq 0$, and hence X_t will not explode for any parameter α.

10. Solve the following stochastic differential equations

$$dX_t = X_t^{\gamma} dt + \alpha X_t dB_t, \; X_0 = x > 0.$$

For which values of the parameters γ, α the solution explodes?

Solution: If $\alpha = 0$, then the differential equation is an ODE

$$\frac{d}{dt} X = X^{\gamma}, \; X_0 = x > 0.$$

This is a separable equation and we know that the solution explodes when $\gamma > 1$.

If $\alpha \neq 0$ and $\gamma = 1$, then this is a linear SDE and its solution is given by

$$X_t = e^{\alpha B_t + \left(1 - \frac{\alpha^2}{2}\right) t}, \; \forall t \geq 0.$$

If $\alpha \neq 0$ and $\gamma \neq 1$, then we will use very similar steps as in Problem 9 to obtain the solution. Let $Y_t = e^{-\alpha B_t - \frac{\alpha^2 t}{2}}$. Then Y_t satisfies the following SDE:

$$dY_t = -\alpha Y_t dB_t, \ Y_0 = 1,$$

Using Itô's formula, we have

$$\begin{aligned} d(X_t Y_t) &= X_t dY_t + Y_t dX_t + dX_t dY_t \\ &= X_t(-\alpha Y_t dB_t) + Y_t \left(X_t^\gamma dt + \alpha X_t dB_t \right) - \alpha^2 X_t Y_t dt \\ &= Y_t X_t^\gamma dt - \alpha^2 X_t Y_t dt, \end{aligned}$$

which implies for each fixed $\omega \in \Omega$, $y(t) = X_t(\omega) Y_t(\omega)$ satisfies the following nonlinear ODE:

$$\dot{y} = Y_t(\omega)^{1-\gamma} y^\gamma - \alpha^2 y, \ y(0) = x,$$

or equivalently,

$$\dot{y} + \alpha^2 y = Y_t(\omega)^{1-\gamma} y^\gamma, \; y(0) = x.$$

Multiplying the above equation by $e^{\alpha^2 t}$ and denoting $z = e^{\alpha^2 t} y$, we obtain

$$\dot{z} = \left(Y_t(\omega) e^{\alpha^2 t} \right)^{1-\gamma} z^\gamma, \; z(0) = x.$$

We can separate the variables to solve this ODE as follows:

$$\frac{\dot{z}}{z^\gamma} = \left(Y_t(\omega) e^{\alpha^2 t} \right)^{1-\gamma} = e^{-\alpha(1-\gamma)B_t + \frac{\alpha^2(1-\gamma)t}{2}}, \; z(0) = x. \tag{6}$$

Note that

$$z(t) = X_t(\omega)e^{-\alpha B_t(\omega)+\frac{1}{2}\alpha^2 t}$$

and $e^{-\alpha B_t(\omega)+\frac{1}{2}\alpha^2 t}$ is continuous in t. Then, $X_t(\omega)$ explodes as $t \uparrow T(\omega)$ if and only if $z(t)$ explodes when $t \uparrow T(\omega)$.

Suppose that $X_t(\omega)$ explodes as $t \uparrow T(\omega)$ for some $T(\omega) < \infty$. Then integrating (6) on both sides, we should get

$$\int_x^\infty \frac{dz}{z^\gamma} = \int_0^T e^{-\alpha(1-\gamma)B_t + \frac{\alpha^2(1-\gamma)t}{2}} dt < \infty, \tag{7}$$

and hence, explosion might occur only if $\gamma > 1$.

For $\gamma > 1$, then

$$\int_x^\infty \frac{dz}{z^\gamma} = \frac{x^{1-\gamma}}{\gamma - 1}.$$

Thus, if $X_t(\omega)$ explodes as $t \uparrow T(\omega)$ for some $T(\omega) < \infty$, the following equation holds

$$\int_0^T e^{-\alpha(1-\gamma)B_t + \frac{\alpha^2(1-\gamma)t}{2}} dt = \frac{x^{1-\gamma}}{\gamma - 1}.$$

Note also that for each $t \geq 0$ we have

$$\begin{aligned}
& E\left(\int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds\right) \\
&= \int_0^t e^{\frac{\alpha^2(1-\gamma)^2 s}{2} + \frac{\alpha^2(1-\gamma)s}{2}} ds \\
&= \int_0^t e^{\frac{\alpha^2(1-\gamma)(2-\gamma)s}{2}} ds \\
&= \begin{cases} t, \text{ if } \gamma = 2, \\ \frac{2}{\alpha^2(1-\gamma)(2-\gamma)} \left(e^{\frac{\alpha^2(1-\gamma)(2-\gamma)t}{2}} - 1\right), \text{ if } \gamma \neq 2. \end{cases} \qquad (8)
\end{aligned}$$

So, for $\alpha \neq 0$ and $\gamma \geq 2$, we have

$$\lim_{t\to\infty} E\left(\int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds\right) = \infty. \tag{9}$$

Define

$$\tau = \inf\left\{t \geq 0, \int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds = \frac{x^{1-\gamma}}{\gamma - 1}\right\}.$$

That is, z (or equivalently, X) explodes at τ.

Then τ is a stopping time, and moreover, from (9), we get

$$P(\tau < \infty) > 0,$$

that is, X explodes on $\{\tau < \infty\}$.

For $\alpha \neq 0$ and $1 < \gamma < 2$, we get from (8)

$$\lim_{t\to\infty} E\left(\int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds\right) = \frac{2}{\alpha^2(\gamma-1)(2-\gamma)}.$$

If α and γ satisfy

$$\frac{2}{\alpha^2(\gamma-1)(2-\gamma)} > \frac{x^{1-\gamma}}{\gamma-1},$$

then we also get

$$P(\tau < \infty) > 0,$$

that is, X explodes on $\{\tau < \infty\}$ in this case.