Bridge representation and small time approximation of the joint density

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Outline

- Preliminaries
- Bridge representations for transition density
- Probabilistic derivation of the heat kernel expansion
- Extensions to processes driven by fractional Brownian motions: mixed Brownian and fractional Brownian motions with drift

Conditional expectation of Gaussian random vectors

Let $\mathbf{X} = (X_1, \cdots, X_n)'$ and $\mathbf{Y} = (Y_1, \cdots, Y_m)'$ be joint Gaussian random vectors and $\mathbf{Z} = (X_1, \cdots, X_n, Y_1, \cdots, Y_m)'$. Denote the expectations of $\mathbf{X}$, $\mathbf{Y}$ and the covariance matrix for $\mathbf{Z}$ by

$$\mathbb{E}[\mathbf{X}] = \mu_{\mathbf{X}}, \ \mathbb{E}[\mathbf{Y}] = \mu_{\mathbf{Y}}$$

and

$$\mathbf{\Sigma} = Cov\left[\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}\right] = \begin{pmatrix} \mathbf{\Sigma}_{\mathbf{XX}} & \mathbf{\Sigma}_{\mathbf{XY}} \\ \mathbf{\Sigma}_{\mathbf{YX}} & \mathbf{\Sigma}_{\mathbf{YY}} \end{pmatrix}$$

Conditional expectation of Gaussian random vectors

Lemma

Suppose that the covariance matrix $\mathbf{\Sigma}$ *is positive definite. Then, the conditional distribution of* $\mathbf{X}$ *given that* $\mathbf{Y} = \mathbf{y}$ *is n-dimensional Gaussian with expectation and covariance matrix respectively,*

$$\mathbb{E}[\mathbf{X}|\mathbf{Y} = \mathbf{y}] = \mu_{\mathbf{X}} + \mathbf{\Sigma}_{\mathbf{XY}}\mathbf{\Sigma}_{\mathbf{YY}}{}^{-1}(\mathbf{y} - \mu_{\mathbf{Y}}),$$

$$Cov[\mathbf{X}|\mathbf{Y} = \mathbf{y}] = \mathbf{\Sigma}_{\mathbf{XX}} - \mathbf{\Sigma}_{\mathbf{XY}}\mathbf{\Sigma}_{\mathbf{YY}}{}^{-1}\mathbf{\Sigma}_{\mathbf{YX}}.$$

Moreover, the Gaussian vector $\mathbf{X}$ *has the following decomposition*

$$\mathbf{X} = \mu_{\mathbf{X}} + \mathbf{\Sigma}_{\mathbf{XY}}\mathbf{\Sigma}_{\mathbf{YY}}{}^{-1}(\mathbf{Y} - \mu_{\mathbf{Y}}) + \mathbf{V},$$

where the random vector $\mathbf{V}$ *is n-dimensional Gaussian with zero expectation and the following covariance matrix*

$$Cov[\mathbf{V}] = \mathbf{\Sigma}_{\mathbf{XX}} - \mathbf{\Sigma}_{\mathbf{XY}}\mathbf{\Sigma}_{\mathbf{YY}}{}^{-1}\mathbf{\Sigma}_{\mathbf{YX}}.$$

Gaussian bridge

Let X be a $\mathbb{R}^d$-valued continuous Gaussian process with $X_0 = x$ and $\mathbb{E}[X_t] = x$ for all $t \in [0, T]$. For any given $y \in \mathbb{R}^d$, then the Gaussian process defined by

$$X_t^{x,y} = X_t + \mathbf{\Sigma}(\mathbf{t};\mathbf{T})\mathbf{\Sigma}(\mathbf{T})^{-1}(y - X_T)$$

is a Gaussian brigde of X in the sense that $X_0^{x,y} = x$ and $X_T^{x,y} = y$, where

$$\mathbf{\Sigma}(\mathbf{t};\mathbf{T}) = Cov(X_t, X_T)$$

and

$$\mathbf{\Sigma}(\mathbf{T}) = \mathbf{\Sigma}(\mathbf{T};\mathbf{T}) = Cov(X_T, X_T).$$

Moreover, we have

$$\mathbb{P}\text{-Law}(\{X_t^{x,y}\}_{t\in[0,T]}) = \mathbb{P}\text{-Law}(\{X_t\}_{t\in[0,T]}|X_T = y)$$

Nondegenerate diffusion process

Consider the diffusion process

$$\begin{cases} dS_t = a(S_t, t)dB_t + b(S_t, t)dt, \;\; t \in [0, T], \\ S_0 = s_0, \end{cases}$$

with $a(\xi, t) \geq \epsilon > 0$ for all (ξ, t).

A Brownian bridge representation for the transition density p can be derived, which will lead to the derivation of the heat kernel expansion for the transition density.

- One dimensional nondegenerate diffusion is easier to deal with because we can always unitize the diffusion coefficient by applying the Lamperti transformation.
- Such transformations generally do not exist in higher dimensions due to geometry obstructions.

Lamperti transformation

Let

$$\varphi(x,t) = \int_{s_0}^{x} \frac{d\xi}{a(\xi,t)}.$$

By Ito's formula, $X_t = \varphi(S_t, t)$ satisfies the SDE

$$dX_t = dB_t + h(X_t, t)dt$$

where $h(x,t) := \varphi_t + \frac{b}{a} - \frac{a_x}{2}$. Therefore, X_t is a Brownian motion with a drift. Define

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= e^{-\int_0^T h(X_s,s)dB_s - \frac{1}{2}\int_0^T h^2(X_s,s)ds} \\ &= e^{-\int_0^T h(X_s,s)dX_s + \frac{1}{2}\int_0^T h^2(X_s,s)ds}. \end{aligned}$$

By Girsanov's theorem, under the probability measure $\tilde{\mathbb{P}}$, X_t is a Brownian motion.

Girsanov transformation

Given any bounded measurable function f, we have

$$\mathbb{E}[f(X_T)] = \tilde{\mathbb{E}}\left[f(X_T)\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right]$$
$$= \tilde{\mathbb{E}}\left[f(X_T)e^{\int_0^T h(X_s,s)dX_s - \frac{1}{2}\int_0^T h^2(X_s,s)ds}\right].$$

In other words, let p be the transition density of X,

$$\int f(y)p(T,y|0,x)dy$$
$$= \int f(y)\tilde{\mathbb{E}}_y\left[e^{\int_0^T h(X_s,s)dX_s - \frac{1}{2}\int_0^T h^2(X_s,s)ds}\right]\frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}}dy,$$

where $\tilde{\mathbb{E}}_y[\cdot] = \tilde{\mathbb{E}}[\cdot|X_T = y]$, and hence

$$p(T, y|0, x) = \tilde{\mathbb{E}}_y \left[e^{\int_0^T h(X_s,s)dX_s - \frac{1}{2}\int_0^T h^2(X_s,s)ds} \right] \frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}}.$$

Furthermore, by applying Ito's formula, we rewrite the stochastic integral term as

$$\int_0^T h(X_s, s)dX_s = H(X_T, T) - H(X_0, 0) - \int_0^T \left[H_t(X_s, s) + \frac{h_x(X_s, s)}{2} \right] ds,$$

where $H_x(x, t) = h(x, t)$, i.e., H is an antiderivative of h (in x). Therefore,

$$e^{\int_0^T h(X_s,s)dX_s} = e^{H(X_T,T) - H(X_0,0) - \int_0^T \left[H_t(X_s,s) + \frac{h_x(X_s,s)}{2} \right] ds}.$$

Brownian bridge representation

For $T > 0$,

$$\begin{aligned} p(T, y|0, x) &= \tilde{\mathbb{E}}_y \left[e^{\int_0^T h(X_s,s)dX_s - \frac{1}{2}\int_0^T h^2(X_s,s)ds} \right] \frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}} \\ &= \frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}} e^{H(y,T)-H(x,0)} \times \\ &\quad \tilde{\mathbb{E}} \left[e^{-\frac{1}{2}\int_0^T h^2(X_s^{x,y},s)+h_x(X_s^{x,y},s)+2H_t(X_s^{x,y},s)ds} \right]. \end{aligned}$$

- $X_s^{x,y}$ is the standard Brownian bridge under $\tilde{\mathbb{P}}$ with initial and terminal points at x and y respectively in the time horizon $[0, T]$.

We consider h in the case that it is independent of time t. For small time horizon, we approximate the Brownian bridge expectation as

$$
\begin{aligned}
&\tilde{\mathbb{E}}\left[e^{-\frac{1}{2}\int_0^T h^2(X_s^{x,y})+h'(X_s^{x,y})ds}\right] \\
\sim\quad & 1-\frac{1}{2}\int_0^T \tilde{\mathbb{E}}\left[h^2(X_s^{x,y})+h'(X_s^{x,y})\right]ds \\
\sim\quad & 1-\frac{1}{2}\int_0^T \left[h^2\left(x+\frac{s}{T}(y-x)\right)+h'\left(x+\frac{s}{T}(y-x)\right)\right]ds \\
=\quad & 1-\frac{T}{2(y-x)}\int_x^y (h^2(\xi)+h'(\xi))d\xi
\end{aligned}
$$

- In the second approximation, we estimate Brownian bridge expectation by evaluating along the straight line: $x(s)=x+\frac{s}{T}(y-x)$.
- Note that $\tilde{\mathbb{E}}_y[X_s]=x(s)$ for $0\le s\le T$.

Recovery of heat kernel expansion

We end up with the following small time approximation of the transition density:

$$p(T, y|0, x) \sim \frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}} e^{H(y)-H(x)} \times \left[1 - \frac{T}{2(y-x)} \int_x^y (h^2(\xi) + h'(\xi)) d\xi\right],$$

which is exactly the heat kernel expansion up to order 1!

- The conventional way in deriving the heat kernel expansion in PDE theory is by applying the WKB ansatz.

$$p(t, x, y) \sim \frac{1}{t^{n/2}} e^{-\frac{d^2(x,y)}{2t}} \left[u_0(x, y) + u_1(x, y)t + o(t)\right],$$

where n is the dimension of the underlying space.

Fractional Brownian motion (fBM)

A centered Gaussian process $B^H = \{B_t^H;\ t \in [0, T]\}$ is called a fractional Brownian motion (fBM) with Hurst parameter $H \in (0,1)$ if it has the covariance function

$$\mathbb{E}(B_s^H B_t^H) = R_H(s, t) := \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}),$$

for all $s,\ t \in [0, T]$.

- Standard Brownian motion corresponds to $H = \frac{1}{2}$.
- B_t^H is not a semimartingale when $H \neq \frac{1}{2}$.
- B_t^H is Hölder continuous of order β in t for any $\beta < H$ almost surely.

Representation of fBM

A fBM B^H can be expressed in terms of a stochastic integral with respect to standard Brownian motion as

$$B_t^H = \int_0^t K_H(t,s) dB_s,$$

where K_H is given by

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} \right] \mathbf{1}_{[0,t]}$$

Consider the Molchan-Golosov operator

$$(\mathcal{K}_H f)(t) = \int_0^T K_H(t,s) f(s) ds, \;\; f \in L^2([0,T]). \qquad (1)$$

Mixed fBMs with drift

Consider the two dimensional stochastic system

$$\begin{cases} X_t = x_0 + \rho B_t + \sqrt{1-\rho^2} W_t + \int_0^t h_1(s, X_s, Y_s) ds, \\ \\ Y_t = y_0 + B_t^H + \int_0^t h_2(s, X_s, Y_s) ds, \end{cases} \tag{2}$$

where $(X_0, Y_0) = (x_0, y_0)$ is the initial point, $\rho \in (0, 1)$, and the two functions $h_1, h_2 : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are deterministic.

- X_t is a standard Brownian motion with drift
- Y_t is a fractional Brownian motion with drift
- Goal: Obtain a bridge representation for the joint density of (X_T, Y_T) and a small time approximation accordingly

Assumptions

(a) The functions h_1 and h_2 are Lipschitz in x, y uniformly for t. That is, there exists a constant $L > 0$ such that

$$|h_i(t, x_1, y_1) - h_i(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|), \quad i = 1, 2, \tag{3}$$

for all $t \in [0, T]$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

(b) (i) If $H > \frac{1}{2}$, there exist two constants $L > 0$ and $\gamma \in (H - \frac{1}{2}, \frac{1}{2})$ such that the function h_1 satisfies

$$|h_1(t, 0, 0)| \leq L, \ \forall t \in [0, T],$$

and the function h_2 satisfies

$$|h_2(t, x, y) - h_2(s, x, y)| \leq L|t - s|^{\gamma}, \ \forall s, t \in [0, T], \ \forall (x, y) \in \mathbb{R}^2, \tag{4}$$

i.e., h_2 is Hölder continuous in t of order γ uniformly for x and y.

(ii) If $H \leq \frac{1}{2}$, there exists a constant $L > 0$ such that

$$|h_i(t, 0, 0)| \leq L, \ \forall s, t \in [0, T], \ i = 1, 2.$$

Theorem

Let the conditions in the assumptions be satisfied. Then, there exists a positive constant δ *such that the system* (2) *has a unique solution* (X, Y) *when* $T < \delta$*. Moreover, the trajectories of* X *and* Y *satisfy* $X \in C^{\frac{1}{2}-\epsilon}([0, T])$ *and* $Y \in C^{H-\epsilon}([0, T])$ *almost surely for every* $0 < \epsilon < \min\{\frac{1}{2}, H\}$.

Sketch of Proof: Using contraction mapping theorem: Let $(x^i, y^i), i = 1, 2,$ be two stochastic processes taking values in $C([0, T])$. Define

$$\begin{cases} X_t^i = x_0 + \rho B_t + \sqrt{1-\rho^2} W_t + \int_0^t h_1(s, x_s^i, y_s^i) ds, \\ \\ Y_t^i = y_0 + B_t^H + \int_0^t h_2(s, x_s^i, y_s^i) ds, \end{cases}$$

for each $i = 1, 2$.

Novikov's condition

The processes $\tilde{h}_1$ and $\tilde{h}_2$ are determined by the following system of equations

$$\begin{aligned} \sqrt{1-\rho^2}\,\tilde{h}_1(t) + \rho\tilde{h}_2(t) &= h_1(t, X_t, Y_t), \\ \tilde{h}_2(t) &= \mathcal{K}_H^{-1}\left(\int_0^{\cdot} h_2(s, X_s, Y_s)ds\right)(t), \end{aligned}$$

where $\mathcal{K}_H^{-1}$ is the inverse of the Molchan-Golosov operator defined in (1).

Lemma

There exists a small $t_0 > 0$ such that the adapted processes $\tilde{h}_1$ and $\tilde{h}_2$ satisfy the Novikov's condition in $[0, t_0]$. That is,

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^{t_0} |\tilde{h}_1(t)|^2 dt + \frac{1}{2}\int_0^{t_0} |\tilde{h}_2(t)|^2 dt\right\}\right] < \infty. \tag{5}$$

Girsanov's Theorem

We consider all $T \leq t_0$ and define

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left\{ -\int_0^T \tilde{h}_1(t)dW_t - \frac{1}{2}\int_0^T \tilde{h}_1^2(t)dt - \int_0^T \tilde{h}_2(t)dB_t - \frac{1}{2}\int_0^T \tilde{h}_2^2(t)dt \right\},$$

Theorem

Under the probability measure $\tilde{\mathbb{P}}$, the processes $\tilde{W} = \{\tilde{W}_t = W_t + \int_0^t \tilde{h}_1(s)ds,\ t \in [0, T]\}$ and $\tilde{B} = \{\tilde{B}_t = B_t + \int_0^t \tilde{h}_2(s)ds,\ t \in [0, T]\}$ become two independent Brownian motions, and the process $\tilde{B}^H = \{\tilde{B}_t^H = B_t^H + \int_0^t K_H(t, s)\tilde{h}_2(s)ds,\ t \in [0, T]\}$ becomes a fractional Brownian motion.

We can rewrite the processes X and Y as

$$\begin{aligned} X_t &= x_0 + \rho B_t + \sqrt{1-\rho^2} W_t + \int_0^t h_1(s, X_s, Y_s) ds \\ &= x_0 + \rho \tilde{B}_t + \sqrt{1-\rho^2} \tilde{W}_t + \int_0^t \left[h_1(s, X_s, Y_s) - \rho \tilde{h}_2(s) - \sqrt{1-\rho^2}\, \tilde{h}_1(s) \right] ds \\ &= x_0 + \rho \tilde{B}_t + \sqrt{1-\rho^2} \tilde{W}_t \qquad (6) \end{aligned}$$

and

$$\begin{aligned} Y_t &= y_0 + B_t^H + \int_0^t h_2(s, X_s, Y_s) ds \\ &= y_0 + \int_0^t K_H(t,s) dB_s + \int_0^t h_2(s, X_s, Y_s) ds \\ &= y_0 + \int_0^t K_H(t,s) d\tilde{B}_s - \int_0^t K_H(t,s) \tilde{h}_2(s) ds + \int_0^t h_2(s, X_s, Y_s) ds \\ &= y_0 + \tilde{B}_t^H. \qquad (7) \end{aligned}$$

Bridge representation for the joint density

Theorem

The joint density $p_T(x,y|x_0,y_0)$ of (X_T,Y_T) at time T has the bridge representation

$$\begin{aligned} & p_T(x,y|x_0,y_0) \\ = \quad & \phi_T(x-x_0,y-y_0) \\ & \times \tilde{\mathbb{E}}_{x,y}\left[e^{\int_0^T \tilde{h}_1(t)d\tilde{W}_t - \frac{1}{2}\int_0^T \tilde{h}_1^2(t)dt + \int_0^T \tilde{h}_2(t)d\tilde{B}_t - \frac{1}{2}\int_0^T \tilde{h}_2^2(t)dt}\right], \end{aligned}$$

where ϕ_T is the bivariate Gaussian density of (X_T-x_0, Y_T-y_0) under $\tilde{P}$

$$\phi_T(\xi,\eta) = \frac{1}{2\pi T^{H+\frac{1}{2}}\sqrt{1-\rho^2\kappa_H^2}} \times$$

$$\exp\left\{-\frac{1}{2(1-\rho^2\kappa_H^2)}\left[\left(\frac{\xi}{\sqrt{T}}\right)^2 - 2\rho\kappa_H\left(\frac{\xi}{\sqrt{T}}\right)\left(\frac{\eta}{T^H}\right) + \left(\frac{\eta}{T^H}\right)^2\right]\right\}.$$

Sketch of Proof: For any bounded function f defined on $\mathbb{R}^2$, we have

$$
\begin{aligned}
&\int p_T(x, y|x_0, y_0) f(x, y) dxdy \\
= \;& \mathbb{E}[f(X_T, Y_T)] = \tilde{\mathbb{E}}\left[f(X_T, Y_T) \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \right] \\
= \;& \tilde{\mathbb{E}}\left[f(X_T, Y_T) e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2}\int_0^T \tilde{h}_1^2(t)dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2}\int_0^T \tilde{h}_2^2(t)dt} \right] \\
= \;& \int \phi_T(x - x_0, y - y_0) f(x, y) \\
& \quad \times \tilde{\mathbb{E}}_{x,y}\left[e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2}\int_0^T \tilde{h}_1^2(t)dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2}\int_0^T \tilde{h}_2^2(t)dt} \right] dxdy.
\end{aligned}
$$

Modal-path approximation

Notations:

$$\langle f\rangle = \int_0^T f(s)ds, \forall f \in L^1([0,T]).$$

and

$$\bar{\rho} = \sqrt{1-\rho^2},\ \kappa_H = c_H \frac{B\left(\frac{3}{2}-H, H+\frac{1}{2}\right)}{H+\frac{1}{2}},\ \rho_H = \rho\kappa_H, \quad \bar{\rho}_H = \sqrt{1-\rho_H^2}.$$

Approximating the random paths X_s and Y_s by their respective modes (thus the term modal-path) $\tilde{\mathbb{E}}[X_s^{x,y}]$ and $\tilde{\mathbb{E}}[Y_s^{x,y}]$, where $X^{x,y}$ and $Y^{x,y}$ is the Gaussian bridge that connect $(X_0, Y_0) = (x_0, y_0)$ and $(X_T, Y_T) = (x, y)$. Then we obtain a small time approximation of the joint probability density $p_T(x, y|x_0, y_0)$ as $T \to 0$ as

$$p_T(x, y|x_0, y_0) = \phi\left(x - x_0, y - y_0\right) e^{\omega(T)} \left\{1 + o(T^\alpha)\right\},$$

where

$$\begin{aligned}
&\omega(T) \\
= \;& \frac{1}{\bar{\rho}_H^2}\left\{\left(\frac{\bar{\rho}\langle\hat{h}_1\rangle}{\sqrt{T}} + \frac{\rho\langle\hat{h}_2\rangle}{\sqrt{T}} - \frac{\rho_H\langle\bar{h}_2\rangle}{T^H}\right)\left(\frac{x-x_0}{\sqrt{T}}\right)\right. \\
& \left. - \rho_H\left(\frac{\bar{\rho}\langle\hat{h}_1\rangle}{\sqrt{T}} + \frac{\rho\langle\hat{h}_2\rangle}{\sqrt{T}} - \frac{\rho_H\langle\bar{h}_2\rangle}{T^H}\right)\left(\frac{y-y_0}{T^H}\right) + \bar{\rho}_H^2\frac{\langle\bar{h}_2\rangle}{T^H}\left(\frac{y-y_0}{T^H}\right)\right\},
\end{aligned}$$

- When $H = \frac{1}{2}$, the modal-path approximation recovers the heat kernel expansion up to zeroth order!

Explicit expression for modal-path

The modal-path from (x_0, y_0) to (x, y) has the explicitly expression

$$\tilde{\mathbb{E}}[X_t] = x_0 + m_{11}(t)(x - x_0) + m_{12}(t)(y - y_0),$$
$$\tilde{\mathbb{E}}[Y_t] = y_0 + m_{21}(t)(x - x_0) + m_{22}(t)(y - y_0),$$

where

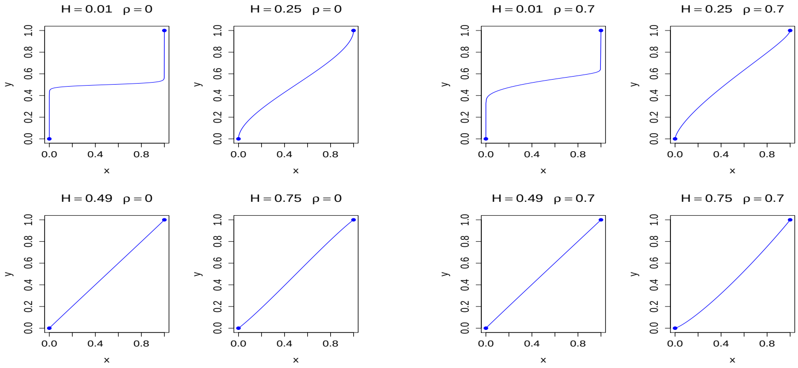
$$m_{11}(t) = \frac{1}{\bar{\rho}_H^2}\left(\frac{t}{T} - \frac{\rho\rho_H}{T^{H+\frac{1}{2}}}\int_0^t K_H(T,s)ds\right),$$

$$m_{12}(t) = \frac{1}{\bar{\rho}_H^2}\left(-\rho_H\frac{t}{T^{H+\frac{1}{2}}} + \frac{\rho}{T^{2H}}\int_0^t K_H(T,s)ds\right),$$

$$m_{21}(t) = \frac{\rho_H}{\bar{\rho}_H^2}\left(\frac{t^{H+\frac{1}{2}}}{T} - \frac{R_H(t,T)}{T^{H+\frac{1}{2}}}\right),$$

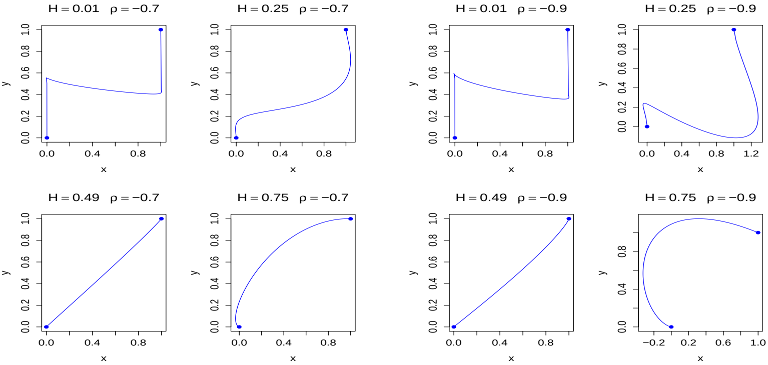
$$m_{22}(t) = \frac{1}{\bar{\rho}_H^2}\left(-\rho_H^2\left\{\frac{t}{T}\right\}^{H+\frac{1}{2}} + \frac{R_H(t,T)}{T^{2H}}\right).$$

How does the modal-path look like?



The plots of modal-paths from $(x_0, y_0) = (0, 0)$ to $(x, y) = (1, 1)$ within the time interval $[0, 1]$ with $\rho = 0$, 0.7 and Hurst exponents $H = 0.01$, 0.25, 0.49, and 0.75.

How does the modal-path look like?



The plots of modal-paths from $(x_0, y_0) = (0, 0)$ to $(x, y) = (1, 1)$ within the time interval $[0, 1]$ with $\rho = -0.7, -0.9$ and Hurst exponents $H = 0.01, 0.25, 0.49,$ and 0.75.

THANK YOU FOR YOUR ATTENTION.