A Study of Boundary-Value Problems in Interfacial Fluid Dynamics

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This work is dedicated in memory of Michael Andrew Case. A great friend whom I will always remember when I think about my graduate studies.
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Abstract
A Study of Boundary-Value Problems in Interfacial Fluid Dynamics
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We study two problems arising in interfacial fluid dynamics; a Boussinesq approximation equation derived by Bona, Chen and Saut for small amplitude long water waves, and a vortex sheet problem with fluids of the same densities. These problems are studied along with Dirichlet, Neumann, and mixed boundary conditions. We study a general elliptic partial differential equations for which we show the existence of non-periodic and periodic solutions. The proof of existence of these solutions uses techniques from the work of Duchon and Robert which relies on a fixed point type of estimate. The result is applied to the Boussinesq equations for periodic problems and the vortex sheet for non-periodic problems.
1. INTRODUCTION

Free boundary problems arise in many instances of engineering and science. Some examples include air flow around a wing of an aircraft, underground fluid flow through layers of soil and modelling of an ocean surface. We are interested in the study of two particular mathematical free boundary problems which arise in fluid mechanics. These problems are the Boussinesq approximations and a vortex sheet problem. The Boussinesq approximation equations we study are derived for water waves with long period and small amplitude. A vortex sheet is an interface between two fluids which have a discontinuity in the tangential component of the flow. For the vortex sheet problem we analyze, both fluids are incompressible, have the same density and are irrotational away from the interface. We are interested in studying these systems where the data assigned is given at two instances in time, which we will refer to as a time boundary value problem.

Boussinesq type equations are a popular subject of study since they are an approximation to the Euler’s equations for fluid flow. Using appropriate physical approximations to Euler’s equations allows us to develop well-posedness theory. Boussinesq [3], [4] first presented an approximation for a flow under water waves where the vertical component was eliminated; this approximation is known as a classical Boussinesq equation. Such an approximation resulted in construction of solitary wave solutions which provided insight into flow dominated by the horizontal components.

The Boussinesq equations that we study were derived by Bona, Chen and Saut [1]. These equations are a generalization of a classical Boussinesq equation, and we
provide this derivation in Appendix A. The equations are of the form

\[ \eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0, \quad (1.1) \]
\[ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \quad (1.2) \]

where \( u = u(x,t) \) is the velocity component in the \( x \) direction of a channel flow, \( \eta = \eta(x,t) \) is the position of the free surface of the channel and \( a, b, c \) and \( d \) are constants that depend on physical properties of the flow.

We are interested in constructing a general class of periodic solutions for these equations for data specified at \( t = 0 \) and \( t = T \) where \( T < \infty \). An example of work where periodic solutions are studied was done by Cranell [6] where she showed existence of periodic solutions satisfying certain symmetry properties. In that work although Boussinesq type equations were considered with initial data, since the solution was time-periodic, it could be consider as a time boundary value problem.

In the work of Pelloni [8] and Fokas and Pelloni [9] well-posedness is studied for an initial boundary value problem of Kdv-Kdv equations which takes \( b,d = 0 \) and \( a,c = 1 \) in equations (1.1) and (1.2). Bona, Chen, and Saut [1] and [2] showed existence of analytic solutions to initial value problem of (1.1) and (1.2) for a set of constants \( a, b, c, \) and \( d \). We extend the results of [2] to a different set of constants \( a, b, c, \) and \( d \).

Unlike Boussinesq type approximations for a water wave, the vortex sheet problem considers the full Euler’s equations for a fluid flow with an appropriate velocity discontinuity condition on the interface. The vortex sheet problem has been actively studied over the past several decades. Using asymptotic analysis Moore [15] showed evidence that the vortex sheet can develop singularities in finite time. This was numerically verified by the work of Baker, Meiron, and Orszag [14], Krasny [12], and Shelley [16]. Proof of singularity formation and ill-posedness was provided by Caflisch
and Orellana [5].

While the vortex sheet initial value problem is ill-posed in Sobolev spaces, Sulem, Sulem, Bardos and Frisch [17] have established short time existence of unique solutions in analytic function spaces. The ill-posedness of the initial value problem in Sobolev spaces can be seen to be related to the ellipticity of the system of evolution equations. Several authors have explored this ellipticity, including Wu and Lebeau and Kamotski [18], [19], [13], [11]. Wu proves, for instance, that one can specify half the initial data in order to get a solvable initial value problem. Duchon and Robert [7] proved existence for all time for solutions which have a specific type of initial data; this again can be seen as specifying half the initial data. The present work extends the work of [7]. By treating the vortex sheet as an elliptic problem with boundary data satisfying certain properties specified at $t = 0$ and $t = T$, we show existence of analytic solutions.

For both problems we rewrite the governing partial differential equations into the form of

$$v_t - Aw = F(v, w)_x, \quad (1.3)$$
$$w_t - Av = G(v, w)_x, \quad (1.4)$$

where $v, w$ are functions of $(x, y)$, and $A$ is an operator with respect to the variable $x$ which acts as a multiplier in Fourier space, meaning $\hat{(Af(x))} = \sigma_A(\xi)\hat{f}(\xi)$. We call $\sigma_A(\xi)$ to be the symbol of $A$ and for simplicity in notation will refer to it as $\sigma_A$. $F$ and $G$ are non-linearities satisfying a Lipschitz property (this property will be specified in sections 2.1 and 2.2). We will use an integrating factor type of technique to construct a solution representation of equations (1.3) and (1.4). We then define appropriate solution spaces for each problem in which the solution representation will be a contraction. The result provides us with existence of a fixed point which is a solution to our problems.
This work is broken down into the following: in chapter 2 we will show construction of a solution representation for every type of boundary conditions. This will be done without any assumptions on the solution being non-periodic or periodic. Following in sections 2.1 and 2.2 we will define the appropriate function spaces for non-periodic and periodic problems, and on these spaces the solution representation mapping will be shown to be a contraction. The developed general theory will be applied to a vortex sheet problem in chapter 3 and Boussinesq approximations in chapter 4. We will conclude with chapter 5 where we show the general theory applicable to more general types of quasi-linear elliptic equations, something that may be of use in future research.
2. GENERAL THEORY

In this chapter we develop general theory for boundary value problems of the form in equations (2.1) and (2.2). Our main goal is to develop existence and (local) uniqueness theory for these equations supplemented with some Dirichlet, Neumann, and mixed type boundary conditions. This is done by reformulating the problem as a fixed point problem on some function space. We begin by constructing a solution representation in Lemma 2.1, which we will use as a mapping for a contraction argument.

**Lemma 2.1.** Given $v, w$ functions of $(x, y)$, and $A$ is a multiplicative operator in Fourier space then

\begin{align*}
v_y - Aw &= F(v, w)_x \\
w_y - Av &= G(v, w)_x
\end{align*}

(2.1) (2.2)

can be reformulated as an integral equation

\begin{align*}
v &= S_1 f + S_2 g + \frac{1}{2} I^+(F - G) - \frac{1}{2} I^-(F + G), \quad (2.3) \\
-w &= S_1 f - S_2 g + \frac{1}{2} I^+(F - G) + \frac{1}{2} I^-(F + G) \quad (2.4)
\end{align*}

where

\begin{align*}
S_1 h(y) &= e^{-yA} h(y), \quad S_2 h(y) = e^{(y-Y)A} h(y), \quad (2.5)
\end{align*}

and

\begin{align*}
I^+ h(y) &= \int_y^0 e^{(\gamma-y)A} h_x(\gamma) d\gamma, \quad I^- h(y) = \int_y^Y e^{(y-\gamma)A} h_x(\gamma) d\gamma.
\end{align*}

and $f, g$ are determined by the appropriate boundary conditions at $y = 0$ and $y = Y$. 

Proof:

Adding (2.1) to (2.2) and subtracting (2.2) from (2.1):

\[(v + w)_y - A(v + w) = (F + G)_x, \tag{2.5}\]
\[(v - w)_y + A(v - w) = (F - G)_x. \tag{2.6}\]

Calculating the Fourier transform of (2.5) and (2.6) with respect to \(x\):

\[(\hat{v} + \hat{w})_y - \sigma_A(\hat{v} + \hat{w}) = i\xi(\hat{F} + \hat{G}), \tag{2.7}\]
\[(\hat{v} - \hat{w})_y + \sigma_A(\hat{v} - \hat{w}) = i\xi(\hat{F} - \hat{G}). \tag{2.8}\]

Along with boundary condition \((v - w)(x, 0) = 2f(x), (v + w)(x, Y) = 2g(x)\) we use integrating factor technique to solve (2.7) and (2.8):

\[(\hat{v} + \hat{w}) = 2e^{\sigma_A(y-Y)} \hat{g}(\xi) + \int_y^Y e^{(y-\gamma)\sigma_A} i\xi(\hat{F} + \hat{G}) d\gamma, \tag{2.9}\]
\[(\hat{v} - \hat{w}) = 2e^{-\sigma_A y} \hat{f}(\xi) + \int_0^y e^{(\gamma-y)\sigma_A} i\xi(\hat{F} - \hat{G}) d\gamma. \tag{2.10}\]

Adding (2.9) to (2.10) and subtracting (2.10) from (2.9):

\[\hat{v} = e^{\sigma_A(y-Y)} \hat{g}(\xi) + e^{-\sigma_A y} \hat{f}(\xi) + \frac{1}{2} \int_y^Y e^{(y-\gamma)\sigma_A} i\xi(\hat{F} + \hat{G}) d\gamma + \frac{1}{2} \int_0^y e^{(\gamma-y)\sigma_A} i\xi(\hat{F} - \hat{G}) d\gamma, \tag{2.11}\]
\[-\hat{w} = e^{\sigma_A(y-Y)} \hat{g}(\xi) - e^{-\sigma_A y} \hat{f}(\xi) - \frac{1}{2} \int_y^Y e^{(y-\gamma)\sigma_A} i\xi(\hat{F} + \hat{G}) d\gamma + \frac{1}{2} \int_0^y e^{(\gamma-y)\sigma_A} i\xi(\hat{F} - \hat{G}) d\gamma. \tag{2.12}\]
Taking the inverse Fourier transform of \( \hat{v} \) and \( \hat{w} \):

\[
v = e^{(y-Y)A}g(x) + e^{-yA}f(x)
- \frac{1}{2} \int_y^Y e^{(y-\gamma)A}(F + G)x d\gamma + \frac{1}{2} \int_0^y e^{(\gamma-y)A}(F - G)x d\gamma,
\]

(2.13)

\[
-w = e^{(y-Y)A}g(x) - e^{-yA}f(x)
+ \frac{1}{2} \int_y^Y e^{(y-\gamma)A}(F + G)x d\gamma + \frac{1}{2} \int_0^y e^{(\gamma-y)A}(F - G)x d\gamma.
\]

(2.14)

Note, the \( f \) and \( g \) in solution representation (2.3) and (2.4) are defined by combinations of \( v \) and \( w \) evaluated at \( y = 0 \) and \( y = Y \). Hence the boundary conditions that we specify will dictate what \( f \) and \( g \) are going to be. We begin with derivation of \( f \) and \( g \) for Dirichlet, Neumann and mixed boundary conditions. First, we treat four Dirichlet-type boundary conditions: we specify (A) \( v(x,0) \) and \( v(x,Y) \), (B) \( w(x,0) \) and \( w(x,Y) \), (C) \( v(x,0) \) and \( w(x,Y) \), or (D) \( w(x,0) \) and \( v(x,Y) \). We refer to these boundary value problems as Problem A, Problem B, Problem C, and Problem D, respectively. In each case when solving for \( f \) and \( g \) we will need one of the following four equations, which we get from evaluating (2.3) and (2.4) at the boundaries \( y = 0 \) and \( y = Y \):

\[
v(0) = v_0 = f + e^{-YA}g - \frac{1}{2} I_0(F + G),
\]

(2.15)

\[
v(Y) = v_Y = e^{-YA}f + g + \frac{1}{2} I_Y(F - G),
\]

(2.16)

\[
-w(0) = -w_0 = f - e^{-YA}g + \frac{1}{2} I_0(F + G),
\]

(2.17)

\[
-w(Y) = -w_Y = e^{YA}f - g + \frac{1}{2} I_Y(F - G),
\]

(2.18)

where \( I_0h = I^-h(0) \) and \( I_Yh = I^+h(Y) \).
In Problem A we solve for $f$ and $g$ in terms of $v_0$ and $v_y$ using equations (2.15) and (2.16). Using equation (2.15) we solve for $f$:

$$f = v_0 - e^{-YA}g + \frac{1}{2}I_0(F + G).$$

We plug this into equation (2.16):

$$v_Y = e^{-YA}(v_0 - e^{-YA}g + \frac{1}{2}I_0(F + G)) + g + \frac{1}{2}I_Y(F - G).$$

Solving this equation for $g$ simplifies to

$$g = (1_I - e^{-2YA})^{-1}(v_Y - \frac{1}{2}I_Y(F - G))$$

$$- (1_I - e^{-2YA})^{-1}e^{-YA}(v_0 + \frac{1}{2}I_0(F + G)), \tag{2.19}$$

where $1_I$ is the identity operator. We plug $g$ into the equation for $f$:

$$f = (1_I - e^{-2YA})^{-1}(v_0 + \frac{1}{2}I_0(F + G))$$

$$- e^{-YA}(1_I - e^{-2YA})^{-1}(v_Y - \frac{1}{2}I_Y(F - G)). \tag{2.20}$$

Similarly for Problem B where boundary conditions are $w_0$ and $w_Y$ we use equations (2.17) and (2.18) to get:

$$f = e^{-YA}(1_I - e^{-2YA})^{-1}(w_Y + \frac{1}{2}I_Y(F - G))$$

$$- (1_I - e^{-2YA})^{-1}(w_0 + \frac{1}{2}I_0(F + G)), \tag{2.21}$$

$$g = (1_I - e^{-2YA})^{-1}(w_Y + \frac{1}{2}I_Y(F - G))$$

$$- (1_I - e^{-2YA})^{-1}e^{-YA}(w_0 + \frac{1}{2}I_0(F + G)). \tag{2.22}$$
For Problem C where boundary conditions are $v_0$ and $w_Y$ we use equations (2.15) and (2.18) to get:

$$f = (1 + e^{-2YA})^{-1}(v_0 + I_0(F + G))$$
$$- e^{-YA}(1 + e^{-2YA})^{-1}(w_Y + \frac{1}{2}I_Y(F - G)),$$
$$g = (1 + e^{-2YA})^{-1}(w_Y + \frac{1}{2}I_Y(F - G))$$
$$+ (1 + e^{-2YA})^{-1}e^{-YA}(v_0 + I_0(F + G)).$$

Finally for Problem D where boundary conditions are $w_0$ and $v_Y$ we use equations (2.16) and (2.17) to get:

$$f = (1 + e^{-2YA})^{-1}(w_0 - I_0(F + G))$$
$$+ (1 + e^{-2YA})^{-1}e^{-YA}(v_Y - I_Y(F - G)),$$
$$g = (1 + e^{-2YA})^{-1}(v_Y - I_Y(F - G))$$
$$- e^{-YA}(1 + e^{-2YA})^{-1}(w_0 - I_0(F + G)).$$

Next, we treat Neumann-type boundary conditions: we specify (NA) $\nabla v(x,0) \cdot n_0$ and $\nabla v(x,Y) \cdot n_0$, (NB) $\nabla w(x,0) \cdot n_0$ and $\nabla w(x,Y) \cdot n_0$, (NC) $\nabla v(x,0) \cdot n_0$ and $\nabla w(x,Y)\cdot n_0$, (ND) $\nabla w(x,0) \cdot n_0$ and $\nabla v(x,Y)\cdot n_0$ where $n_0$ is the unit normal vector.

We refer to these boundary value problems as Problem NA, Problem NB, Problem NC and Problem ND respectively. Since these boundary conditions require taking derivatives in $y$ of equations (2.3) and (2.4) we will need derivatives of operators $I^+$ and $I^-$. Let $x(y) = z(y) = y$, and let $q(z,\gamma) = e^{(\gamma-z)A}h_x(\gamma)$; then we write $I^+h(y)$ as follows:

$$I^+h(y) = \int_0^{x(y)} q(z(y),\gamma) \, d\gamma := Q(x(y),z(y)).$$
We differentiate with respect to $y$:

\[
\frac{d}{dy} I^+ h(y) = Q_x x'(y) + Q_z z'(y) \\
= q(z(y), \gamma)|_{z(y)=y} + \int_{0}^{x(y)} q_y(z(y), \gamma) d\gamma \\
= S(0) h_x(y) + \int_{0}^{y} S_y(y - \gamma) h_x(\gamma) d\gamma \\
= h_x(y) - AI^+ h(y). \tag{2.27}
\]

In the same manner, we write $I^- h(y)$ as follows:

\[
I^- h(y) = \int_{x(y)}^{y} r(z(y), \gamma) d\gamma = R(x(y), z(y)).
\]

We differentiate with respect to $y$:

\[
\frac{d}{dy} I^- h(y) = R_x x'(y) + R_y z'(y) \\
= - r(z(y), \gamma)|_{z(y)=t} + \int_{z(y)}^{y} r_y(z(y), \gamma) d\gamma \\
= - S(0) h_x(y) + \int_{y}^{Y} S_y(\gamma - y) h_x(\gamma) d\gamma \\
= - h_x(y) + AI^- h(y). \tag{2.28}
\]

Using the results from (2.27) and (2.28) we get the derivatives in $y$ of equations (2.3) and (2.4):

\[
v_y = - AS_1 f + AS_2 g + \frac{1}{2} \left( (F - G)_x - AI^+ (F - G) \right) \\
+ \frac{1}{2} \left( (F + G)_x - AI^- (F + G) \right), \tag{2.29}
\]
\[-w_y = -AS_1 f - AS_2 g + \frac{1}{2} ((F - G)_x - AI^+(F - G)) \]
\[-\frac{1}{2} ((F + G)_x - AI^-(F + G)). \quad (2.30)\]

Now we evaluate each boundary condition from each of the four Neumann problems using (2.29) and (2.30):

\[
\tilde{v}_0 = \nabla v(x, 0) \cdot n_0 = -v_y(x, 0) = Af - Ae^{-YA}g + \frac{1}{2} AI_0(F + G) - F_x(0),
\]
\[
\tilde{v}_Y = \nabla v(x, Y) \cdot n_0 = v_y(x, Y) = -Ae^{-YA}f + Ag - \frac{1}{2} AI_Y(F - G) + F_x(Y),
\]
\[
\tilde{w}_0 = \nabla w(x, 0) \cdot n_0 = -w_y(x, 0) = -Af - Ae^{-YA}g + \frac{1}{2} AI_0(F + G) - G_x(0),
\]
\[
\tilde{w}_Y = \nabla w(x, Y) \cdot n_0 = w_y(x, Y) = Ae^{-YA}f + Ag + \frac{1}{2} I_Y(F - G) + G_x(0).
\]

We now use these equation to derive a set of equations which we will use to solve for \(f\) and \(g\) in each of the Neumann problems:

\[
v^*_0 = A^{-1}(\tilde{v}_0 + F_x(0)) = f - e^{-YA}g + \frac{1}{2} I_0(F + G), \quad (2.31)
\]
\[
v^*_Y = A^{-1}(\tilde{v}_Y - F_x(Y)) = -e^{-YA}f + g - \frac{1}{2} I_Y(F - G), \quad (2.32)
\]
\[
w^*_0 = A^{-1}(\tilde{w}_0 + G_x(0)) = f - e^{-YA}g + \frac{1}{2} I_0(F + G), \quad (2.33)
\]
\[
w^*_Y = A^{-1}(\tilde{w}_Y - G_x(Y)) = -e^{-YA}f + g + \frac{1}{2} I_Y(F - G). \quad (2.34)
\]

In Problem NA we solve for \(f\) and \(g\) in terms of \(v^*_0\) and \(v^*_Y\) using equations (2.31) and (2.32). Using equation (2.31) we solve for \(f\):

\[
f = v^*_0 + e^{-YA}g - \frac{1}{2} I_0(F + G). \quad (2.35)
\]
We plug this equation into (2.32):

\[ v^*_Y = -e^{-Ya} \left( v^*_0 + e^{-Ya} g - \frac{1}{2} I_0(F + G) \right) + g - \frac{1}{2} I_Y(F - G). \]

Using this equation we solve for \( g \):

\[
g = (1_I - e^{-2Ya})^{-1} \left( v^*_Y - \frac{1}{2} I_Y(F - G) \right)
\quad + (1_I - e^{-2Ya})^{-1} e^{-Ya} \left( v^*_0 - \frac{1}{2} I_0(F + G) \right).
\] (2.36)

We get \( f \) by plugging \( g \) into equation (2.35):

\[
f = (1_I - e^{-2Ya})^{-1} \left( v^*_0 - \frac{1}{2} I_0(F + G) \right)
\quad + e^{-Ya}(1_I - e^{-2Ya})^{-1} \left( v^*_Y - \frac{1}{2} I_Y(F - G) \right).
\] (2.37)

Similarly for Problem NB we solve for \( f \) and \( g \) in terms of \( w^*_0 \) and \( w^*_Y \) using equations (2.33) and (2.34) to get:

\[
f = - (1_I - e^{-2Ya})^{-1} \left( w^*_0 - \frac{1}{2} I_0(F + G) \right)
\quad - e^{-Ya}(1_I - e^{-2Ya})^{-1} \left( w^*_Y - \frac{1}{2} I_Y(F - G) \right),
\] (2.38)

\[
g = (1_I - e^{-2Ya})^{-1} \left( w^*_Y - \frac{1}{2} I_Y(F - G) \right)
\quad + (1_I - e^{-2Ya})^{-1} e^{-Ya} \left( w^*_0 - \frac{1}{2} I_0(F + G) \right).
\] (2.39)

For Problem NC we solve for \( f \) and \( g \) in terms of \( v^*_0 \) and \( w^*_Y \) using equations (2.31)
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and (2.34) to get

\[ f = (1 + e^{-2YA})^{-1} \left( v_0^* - \frac{1}{2} I_0(F + G) \right) \]

\[ + e^{-YA}(1 + e^{-2YA})^{-1} \left( w_Y^* - \frac{1}{2} I_Y(F - G) \right) , \]

\[ g = (1 + e^{-2YA})^{-1} \left( w_Y^* - \frac{1}{2} I_Y(F - G) \right) \]

\[ - (1 + e^{-2YA})^{-1} e^{-YA} \left( v_0^* - \frac{1}{2} I_0(F + G) \right) . \]

(2.40)

Finally, for Problem ND we solve for \( f \) and \( g \) in terms of \( w_0^* \) and \( v_Y^* \) using equations (2.33) and (2.32) to get:

\[ f = - (1 + e^{-2YA})^{-1} \left( w_0^* - \frac{1}{2} I_0(F + G) \right) \]

\[ - e^{-YA}(1 + e^{-2YA})^{-1} \left( v_Y^* + \frac{1}{2} I_Y(F - G) \right) , \]

\[ g = (1 + e^{-2YA})^{-1} \left( v_Y^* + \frac{1}{2} I_Y(F - G) \right) \]

\[ - (1 + e^{-2YA})^{-1} e^{-YA} \left( w_0^* - \frac{1}{2} I_0(F + G) \right) . \]

(2.41)

Last, we treat mixed-type boundary conditions: we specify (MA) \( v(x, 0) \) and \( \nabla v(x, Y) \cdot n_0 \), (MB) \( v(x, 0) \) and \( \nabla w(x, Y) \cdot n_0 \), (MC) \( v(x, Y) \) and \( \nabla w(x, 0) \cdot n_0 \), (MD) \( v(x, Y) \) and \( \nabla v(x, 0) \cdot n_0 \), (ME) \( w(x, 0) \) and \( \nabla w(x, Y) \cdot n_0 \), (MF) \( w(x, 0) \) and \( \nabla v(x, Y) \cdot n_0 \), (MG) \( w(x, Y) \) and \( \nabla v(x, 0) \cdot n_0 \), (MH) \( w(x, Y) \) and \( \nabla w(x, 0) \cdot n_0 \). We refer to these mixed boundary value problems as Problem MA, MB, MC, MD, ME, MF, MG, and MH respectively. In each problem we solve for \( f \) and \( g \) in terms of \( v_0 \), \( v_Y \), \( w_0 \) and \( w_Y \) in equations (2.15) to (2.18), and in terms of \( v_0^* \), \( v_Y^* \), \( w_0^* \), and \( w_Y^* \) in equations (2.31) to (2.34). For problem MA with boundary conditions \( v(x, 0) \) and \( \nabla v(x, Y) \cdot n_0 \) we use equations (2.15) and (2.32) to solve for \( f \) and \( g \). Using equation
(2.15) we solve for $f$:

$$f = v_0 + \frac{1}{2} I_0 (F + G) - e^{-YA} g. \quad (2.44)$$

Plug this equation for $f$ into equation (2.32):

$$v_Y^* = -e^{-YA} \left( v_0 + \frac{1}{2} I_0 (F + G) - e^{-YA} g \right) + g - \frac{1}{2} I_Y (F - G).$$

Solving this equation for $g$ in terms of $v_0$ and $v_Y^*$:

$$g = (1_I + e^{-2YA})^{-1} \left( v_Y^* + \frac{1}{2} I_Y (F - G) \right) \quad (2.45)$$

$$+ (1_I + e^{-2YA})^{-1} e^{-YA} \left( v_Y^* + \frac{1}{2} I_0 (F + G) \right).$$

Plugging $g$ into (2.44) we get $f$:

$$f = (1_I + e^{-2YA})^{-1} \left( v_0 + \frac{1}{2} I_0 (F + G) \right) \quad (2.46)$$

$$- e^{-YA} (1_I + e^{-2YA})^{-1} \left( v_Y^* + \frac{1}{2} I_Y (F - G) \right).$$

Similar, for Problem MB with boundary conditions $v(x, 0)$ and $\nabla w(x, Y) \cdot n_0$ we use equations (2.15) and (2.34) to get:

$$f = (1_I - e^{-2YA})^{-1} \left( v_0 + \frac{1}{2} I_0 (F + G) \right) \quad (2.47)$$

$$- e^{-YA} (1_I - e^{-2YA})^{-1} \left( w_Y^* - \frac{1}{2} I_Y (F - G) \right),$$

$$g = (1_I - e^{-2YA})^{-1} \left( w_Y^* - \frac{1}{2} I_Y (F - G) \right) \quad (2.48)$$

$$- (1_I - e^{-2YA})^{-1} e^{-YA} \left( v_0 + \frac{1}{2} I_0 (F + G) \right).$$
For Problem MC with boundary conditions $v(x, Y)$ and $\nabla w(x, 0) \cdot n_0$ we use equations (2.16) and (2.33) to get:

$$f = -(1 - e^{-2YA})^{-1} \left( w_0^* - \frac{1}{2} I_0(F + G) \right)$$

$$- (1 - e^{-2YA})^{-1} e^{-YA} \left( v_Y - \frac{1}{2} I_Y(F - G) \right)$$

$$g = (1 - e^{-2YA})^{-1} \left( v_Y - \frac{1}{2} I_Y(F - G) \right)$$

$$+ e^{-YA}(1 - e^{-2YA})^{-1} \left( w_0^* - \frac{1}{2} I_0(F + G) \right).$$

(2.49)

For Problem MD with boundary conditions $v(x, Y)$ and $\nabla v(x, 0) \cdot n_0$ we use equations (2.15) and (2.31) to get:

$$f = (1 + e^{-2YA})^{-1} \left( v_0^* - \frac{1}{2} I_0(F + G) \right)$$

$$+ (1 + e^{-2YA})^{-1} e^{-YA} \left( v_Y - \frac{1}{2} I_Y(F - G) \right)$$

$$g = (1 + e^{-2YA})^{-1} \left( v_Y - \frac{1}{2} I_Y(F - G) \right)$$

$$- e^{-YA}(1 + e^{-2YA})^{-1} \left( v_0^* - \frac{1}{2} I_0(F + G) \right).$$

(2.50)

For Problem ME with boundary conditions $w(x, 0)$ and $\nabla w(x, Y) \cdot n_0$ we use equations (2.17) and (2.34) to get:

$$f = e^{-YA}(1 + e^{-2YA})^{-1} \left( w_Y^* - \frac{1}{2} I_Y(F - G) \right)$$

$$- (1 + e^{-2YA})^{-1} \left( w_0 + \frac{1}{2} I_0(F + G) \right),$$

(2.53)

$$g = (1 + e^{-2YA})^{-1} \left( w_Y^* - \frac{1}{2} I_Y(F - G) \right)$$

$$+ (1 + e^{-2YA})^{-1} e^{-YA} \left( w_0 + \frac{1}{2} I_0(F + G) \right).$$

(2.54)
For Problem MF with boundary conditions \( w(x, 0) \) and \( \nabla v(x, Y) \cdot n_0 \) we use equations (2.17) and (2.32) to get:

\[
f = e^{-YA}(1_I - e^{-2YA})^{-1} \left( v_Y^* + \frac{1}{2} I_Y (F - G) \right) \\
- (1_I - e^{-2YA})^{-1} \left( w_0 + \frac{2}{2} I_0 (F + G) \right),
\]

(2.55)

\[
g = (1_I - e^{-2YA})^{-1} \left( v_Y^* + \frac{1}{2} I_Y (F - G) \right) \\
- (1_I - e^{-2YA})^{-1} e^{-YA} \left( w_0 + \frac{2}{2} I_0 (F + G) \right).
\]

(2.56)

For Problem MG with boundary conditions \( w(x, Y) \) and \( \nabla v(x, 0) \cdot n_0 \) we use equations (2.18) and (2.31) to get:

\[
f = (1_I - e^{-2YA})^{-1} \left( v_0^* - \frac{1}{2} I_0 (F + G) \right) \\
+ (1_I - e^{-2YA})^{-1} e^{-YA} \left( w_Y + \frac{1}{2} I_Y (F - G) \right),
\]

(2.57)

\[
g = e^{-YA}(1_I - e^{-2YA})^{-1} \left( v_0^* - \frac{1}{2} I_0 (F + G) \right) \\
+ (1_I - e^{-2YA})^{-1} \left( w_Y + \frac{1}{2} I_Y (F - G) \right).
\]

(2.58)

Finally, for Problem MH with boundary conditions \( w(x, Y) \) and \( \nabla w(x, 0) \cdot n_0 \) we use equations (2.18) and (2.33) to get:

\[
f = - (1_I + e^{-2YA})^{-1} \left( w_0^* - \frac{1}{2} I_0 (F + G) \right) \\
- (1_I + e^{-2YA})^{-1} e^{-YA} \left( w_Y + \frac{1}{2} I_Y (F - G) \right),
\]

(2.59)

\[
eg^{-YA}(1_I + e^{-2YA})^{-1} \left( w_0^* - \frac{1}{2} I_0 (F + G) \right) \\
e^{-YA}(1_I + e^{-2YA})^{-1} \left( w_Y + \frac{1}{2} I_Y (F - G) \right).
\]

(2.60)
Using the solution representation in (2.3) and (2.4) we define a mapping $T(v, w) = (v, w)$. The question of existence of a solution is then reformulated as showing that the mapping $T$ is a contraction in some function space. Next, we define the spaces on which the mapping $T$ is well-defined and contracting. Since we are interested in studying existence of non-periodic and periodic solutions, the details of domain, function spaces and contraction argument are given in sections 2.1 and 2.2 respectively.
2.1 Non-Periodic Problem

In this section we consider equations (2.1) and (2.2) where \( v \) and \( w \) are defined on the domain \((x, y) \in (-\infty, \infty) \times (0, Y)\) where \( Y < \infty \); we refer to this as the non-periodic problem. For the non-periodic problem we can treat all of the Dirichlet type of boundary conditions, Neumann problems NC and ND, and mixed boundary problems MA, MD, ME, and MH. The reason for this becomes clear when we consider the estimates in detail. We begin by introducing the spaces which we will be working with.

**Definition 2.2.** Let \( B \) denote the Banach space of functions of one variable \( x \in \mathbb{R} \), which are Fourier transforms of bounded measures, with corresponding norm
\[
\|u\|_B = \int_{\mathbb{R}} d|\hat{u}|.
\]

**Definition 2.3.** Let \( B_\rho \) denote the space of functions for \( y \geq 0 \) with values in \( B \), such that there is a bounded positive measure \( \mu \) with \(|e^{\rho|\xi|}\hat{u}(\xi, y)| \leq \mu\), for all \( y \). We call \(|u|_{B_\rho}\) the infimum of such measures \( \mu \). Then \( B_\rho \) is a Banach space with with norm
\[
\|u\|_{B_\rho} = \int_{\mathbb{R}} d|u|_{B_\rho}. \quad \text{(Note: } |u|_{B_\rho} = \sup_y |e^{\rho|\xi|}\hat{u}(\xi, y)|).\]

Note, the operators \( S_1, S_2, I^+, I^- \) are not only well defined on the space \( B_\rho \), but when \( \sigma_A > k|\xi| \) where \( k \) is some constant, also satisfy the following Lipschitz properties. Since \( y \geq 0 \)

\[
|S_1 h(y)|_{B_\rho} = \sup_y |e^{\rho|\xi|}\mathcal{F}(e^{-yA}h(y))| = \sup_y |e^{\rho|\xi|}e^{-\sigma_A y}\hat{h}(y)|
\]
\[
\leq \sup_y |e^{\rho|\xi|}\hat{h}(y)| = |h|_{B_\rho}, \quad (2.61)
\]
and since $y \leq Y$

$$|S_2 h(y)\mathcal{B}_\rho| = \sup_y |e^{\rho|\xi|} \mathcal{F}(e^{(y-Y)A} h(y))| = \sup_y |e^{\rho|\xi|} e^{\sigma A(y-Y)} \hat{h}(y)|$$

$$\leq \sup_y |e^{\rho|\xi|} \hat{h}(y)| = |h|\mathcal{B}_\rho.$$  \hfill (2.62)

Similarly if $\sigma_A \geq K_1|\xi|$ where $K_1$ is some constant

$$|I^+ h(y)\mathcal{B}_\rho| = \sup_y |e^{\rho|\xi| \hat{I}^+ h(y)}|$$

$$= \sup_y |e^{\rho|\xi|} \mathcal{F} \left( \int_0^y e^{(\gamma-y)A} \hat{h}_x(\gamma) \, d\gamma \right)|$$

$$= \sup_y |e^{\rho|\xi|} \int_0^y e^{\sigma A(\gamma-y)} (i\xi) \hat{h}(\gamma) \, d\gamma|$$

$$\leq \sup_y e^{\rho|\xi|} e^{-\sigma_A y}|\xi| \int_0^y e^{\sigma A\gamma} |\hat{h}(\gamma)| \, d\gamma$$

$$\leq \sup_y e^{-\sigma_A y}|\xi| \int_0^y e^{\sigma A\gamma} \sup_{\gamma} |e^{\rho|\xi|} \hat{h}(\gamma)| \, d\gamma$$

$$\leq |h|\mathcal{B}_\rho \sup_y e^{-\sigma_A y}|\xi| \int_0^y e^{\sigma A\gamma} \, d\gamma$$

$$= |h|\mathcal{B}_\rho \sup_y e^{-\sigma_A y}|\xi| \left( \frac{e^{\sigma A y} - 1}{\sigma_A} \right)$$

$$\leq K|h|\mathcal{B}_\rho.$$  \hfill (2.64)

where $K$ is some constant. Since $I^+$ is linear, this estimate also demonstrates that $I^+$ is Lipschitz on $\mathcal{B}_\rho$. We also define the operator $I_Y$ by

$$I_Y h = I^+ h(Y).$$

Since $I_Y$ is defined in terms of $I^+$, we see that $I_Y$ has similar estimate as in (2.64):

$$|I_Y h|\mathcal{B}_\rho = |I^+ h(Y)|\mathcal{B}_\rho \leq |I^+ h(y)|\mathcal{B}_\rho \leq K|h|\mathcal{B}_\rho.$$  \hfill (2.65)
We treat $I^-$ in a similar way

\[ |I^- h(y)|_{\mathcal{B}_\rho} = \sup_y e^{\rho|\xi|} \mathcal{F} \left( \int_y^Y e^{(y-\gamma)A} h_x(\gamma) d\gamma \right) \]

\[ \leq \sup_y e^{\rho|\xi|} \int_y^Y e^{\sigma A(y-\gamma)} (i\xi) \hat{h}(\gamma) \ d\gamma \]

\[ \leq \sup_y e^{\rho|\xi|} e^{\sigma AY} |\xi| \int_y^Y e^{-\sigma A\gamma} \hat{h}(\gamma) \ d\gamma \]

\[ \leq \sup_y e^{\sigma AY} |\xi| \int_y^Y e^{-\sigma A\gamma} \sup_{\gamma} e^{\rho|\xi|} \hat{h}(\gamma) \ d\gamma \]

\[ \leq |h|_{\mathcal{B}_\rho} \sup_y e^{\sigma AY} |\xi| \left( \frac{e^{-\sigma AY} - e^{-\sigma AY}}{-\sigma A} \right) \]

\[ \leq K|h|_{\mathcal{B}_\rho}. \quad (2.67) \]

As before, we notice that this estimate implies that $I^-$ is Lipschitz on $\mathcal{B}_\rho$. We make a further definition, letting the operator $I_0$ be

\[ I_0 h = I^- h(0). \]

Then $I_0$ has the estimate

\[ |I_0 h|_{\mathcal{B}_\rho} = |I^- h(0)|_{\mathcal{B}_\rho} \leq |I^- h(y)|_{\mathcal{B}_\rho} \leq K|h|_{\mathcal{B}_\rho}. \quad (2.68) \]

Since $S_1, S_2, I, I^-$ are well defined on $\mathcal{B}_\rho$ implies that $T(v, w) = (v, w)$ is well defined on $\mathcal{B}_\rho \times \mathcal{B}_\rho$. We want to show that the mapping $T(v, w)$ is a contraction on $\mathcal{B}_\rho \times \mathcal{B}_\rho$. For Problem A and B the $f$ and $g$ contain an operator of the form $(1_e - e^{-2YA})^{-1}$. This operator by itself is an unbounded operator. In Lemma 2.4 we provide estimates that show this operator composed with $I_0$ or $I_Y$ is bounded. We will rely on these estimates to show that $T(v, w)$ is a contraction.
Lemma 2.4. If $A$ is a mapping acting as a multiplier in Fourier space satisfying

$$k_1|\xi| \leq \sigma_A \leq k_2|\xi| \quad (2.69)$$

where $k_1$, $k_2$ are constants then

$$|(1 - e^{-2Y_A})^{-1}I_0(h)|_{\mathcal{B}_\rho} \leq k_3|h|_{\mathcal{B}_\rho}, \quad (2.70)$$

$$|(1 - e^{-2Y_A})^{-1}I_Y(h)|_{\mathcal{B}_\rho} \leq k_4|h|_{\mathcal{B}_\rho}. \quad (2.71)$$

where $k_3$ and $k_4$ are constants.

**Proof:**

Since we are working out estimates of inverse operators we rely on the following fact.

$$\mathcal{F}(A^{-1}f) = \frac{1}{\sigma_A} \hat{f}. \quad (2.72)$$

First we work out the estimate for $|(1 - e^{-2Y_A})^{-1}I_0(h)|_{\mathcal{B}_\rho}$:

$$|(1 - e^{-2Y_A})^{-1}I_0(h)|_{\mathcal{B}_\rho} = \sup_{y} \left| e^{\rho|\xi|} \left| \frac{\hat{I}_0(h)}{1 - e^{-2Y\sigma_A}} \right| \right|
= \sup_{y} \left| e^{\rho|\xi|} \left| \int_0^Y e^{-\lambda\sigma_A(i\xi)}\hat{h}(\lambda)d\lambda \right| \right|
\leq \int_0^Y e^{-\lambda\sigma_A} |\xi| |e^{\rho|\xi|\hat{h}(\lambda)}|d\lambda
\leq |h|_{\mathcal{B}_\rho} |\xi| \int_0^Y e^{-\lambda\sigma_A}d\lambda
= |h|_{\mathcal{B}_\rho} |\xi| (e^{-Y\sigma_A} - 1) \sigma_A (1 - e^{-2Y\sigma_A}).$$
Using the assumption on $A$

\[
\frac{|\xi|(e^{-Y\sigma_A} - 1)}{-\sigma_A(1 - e^{-2Y\sigma_A})} \leq \frac{1}{k_1} \frac{(e^{-Y\sigma_A} - 1)}{(e^{-2Y\sigma_A} - 1)} \leq \frac{1}{k_1} \frac{e^{-k_1Y|\xi|} - 1}{e^{-2k_2Y|\xi|} - 1}.
\]

The only point at which we need to analyze $\frac{e^{-k_1Y|\xi|} - 1}{e^{-2k_2Y|\xi|} - 1}$ is $\xi = 0$. We do so by evaluating the limit as $\xi \to 0$ by use of L’Hopital’s rule:

\[
\lim_{\xi \to 0} \frac{e^{-k_1Y|\xi|} - 1}{e^{-2k_2Y|\xi|} - 1} = \lim_{\xi \to 0} \frac{e^{-k_1Y\xi} - 1}{e^{-2k_2Y\xi} - 1} = \lim_{\xi \to 0} \frac{-k_1Ye^{-k_1Y\xi}}{-2k_2Ye^{-2k_2Y\xi}} = \frac{k_1}{2k_2},
\]

and since

\[
\lim_{\xi \to \infty} \frac{e^{-k_1Y|\xi|} - 1}{e^{-2k_2Y|\xi|} - 1} = 1,
\]

we get the estimate

\[
|(1 - e^{-2Y\sigma_A})^{-1} I_0(h)|_{B_\rho} \leq k_3 |h|_{B_\rho}.
\]

Similarly we treat $|(1 - e^{-2Y\sigma_A})^{-1} I_Y(h)|_{B_\rho}$:

\[
|(1 - e^{-2Y\sigma_A})^{-1} I_Y(h)|_{B_\rho} = \sup_y \left| e^{\rho|\xi|} \frac{I_Y(h)}{1 - e^{-2Y\sigma_A}} \right| = \sup_y \left| \frac{\int_0^Y e^{(\lambda-Y)\sigma_A}(i\xi)\hat{h}(\lambda)d\lambda}{1 - e^{-2Y\sigma_A}} \right|
\]

\[
\leq |h|_{B_\rho} \frac{|\xi| \int_0^Y e^{(\lambda-Y)\sigma_A}|d\lambda}{1 - e^{-2Y\sigma_A}} = |h|_{B_\rho} \frac{|\xi|(1 - e^{-Y\sigma_A})}{\sigma_A(1 - e^{-2Y\sigma_A})}.
\]

Similarly using the restrictions on $A$:

\[
\frac{|\xi|(1 - e^{-Y\sigma_A})}{\sigma_A(1 - e^{-2Y\sigma_A})} \leq \frac{1}{k_1} \frac{(1 - e^{-Y\sigma_A})}{(1 - e^{-2Y\sigma_A})} \leq \frac{1}{k_1} \frac{1}{1 - e^{-2k_1Y|\xi|}} \leq \frac{1}{k_1} \frac{1 - e^{-2k_1Y|\xi|}}{1 - e^{-2k_2Y|\xi|}}.
\]
We use L'Hopital’s rule to evaluate the limit of (2.73) as $\xi \to 0$ and limit as $\xi \to \infty$ to get

$$|(1 - e^{-2Y \cdot})^{-1} I_Y(h)\|_{\mathcal{B}_\rho} \leq k_4|h|_{\mathcal{B}_\rho}. \quad (2.74)$$

2.1.1 Contraction Estimate

We will now treat existence and uniqueness for each of the Dirichlet, Neumann and mixed boundary value problems in turn. Regardless of boundary conditions, using the triangle inequality the first estimate will be

$$|T(v_1, w_1) - T(v_2, w_2)|_{\mathcal{B}_\rho} \leq |S_1f_1 - S_1f_2|_{\mathcal{B}_\rho} + |S_2g_1 - S_2g_2|_{\mathcal{B}_\rho}$$

$$+ |I^+(F_1 - G_1) - I^+(F_2 - G_2)|_{\mathcal{B}_\rho} + |I^-(F_1 + G_1) - I^-(F_2 + G_2)|_{\mathcal{B}_\rho}, \quad (2.75)$$

where $F_i = F(v_i, w_i), G_i = G(v_i, w_i)$ for $i = 1, 2$. Using Lipschitz properties of $S_1$ and $S_2$ in estimates (2.61) and (2.62) implies that

$$|S_1f_1 - S_1f_2|_{\mathcal{B}_\rho} + |S_2g_1 - S_2g_2|_{\mathcal{B}_\rho} \leq |f_1 - f_2|_{\mathcal{B}_\rho} + |g_1 - g_2|_{\mathcal{B}_\rho}.$$

We begin with the estimates for Dirichlet boundary condition $(v(0), v(Y))$ using $f$ and $g$ from equations (2.19) and (2.20). First, estimate $|f_1 - f_2|_{\mathcal{B}_\rho}$. Using the triangle inequality:

$$|f_1 - f_2|_{\mathcal{B}_\rho} \leq |(1 - e^{-2Y \cdot})^{-1}(\frac{1}{2}(I_0(F_1 + G_1) - I_0(F_2 + G_2))|_{\mathcal{B}_\rho}$$

$$+ |e^{-Y \cdot}(1 - e^{-2Y \cdot})^{-1}(\frac{1}{2}(I_Y(F_1 - G_1) - I_Y(F_2 - G_2)))|_{\mathcal{B}_\rho}.$$
Using the estimate of Lemma 2.4 and the fact that $A$ satisfies (2.69) and (2.72):

\[ |f_1 - f_2|_{B_\rho} \leq K_1 |((F_1 + G_1) - (F_2 + G_2))|_{B_\rho} \]
\[ + K_2 |((F_1 - G_1) - (F_2 - G_2))|_{B_\rho}. \]  

where $K_1$ and $K_2$ are some constants. Using the triangle inequality:

\[ |f_1 - f_2|_{B_\rho} \leq K_1 (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}) \]
\[ + K_2 (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}). \]

Combining the constants:

\[ |f_1 - f_2|_{B_\rho} \leq K_3 (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}), \]

where $K_3 = K_1 + K_2$. The estimate for $|g_1 - g_2|_{B_\rho}$ is done exactly the same way with equation (2.19) to get

\[ |g_1 - g_2|_{B_\rho} \leq K_4 (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}), \]

where $K_4$ will be some constant. Using estimates from (2.64), (2.67), (2.80), (2.81), to get the estimate on (2.75):

\[ |T(v_1, w_1) - T(v_2, w_2)|_{B_\rho} \leq K (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}), \]

where $K$ is some constant depending on $K_1, K_2, K_3, K_4$.

For Dirichlet boundary conditions of the form $(w(x, 0), w(x, y))$ using equations (2.21), (2.22) the estimates for $|f_1 - f_2|_{B_\rho}, |g_1 - g_2|_{B_\rho}$ are done identically by using Lemma 2.4 to get an estimate of the form in (2.82).
For Dirichlet boundary conditions \((v(x, 0), w(x, Y))\) and \((w(x, 0), v(x, Y))\) we use equations (2.23), (2.24) and (2.25), (2.26) respectively. The difference is the estimates for \(|f_1 - f_2|_{\mathcal{B}_\rho}\), \(|g_1 - g_2|_{\mathcal{B}_\rho}\) are simpler since an operator of the form \((1_I + e^{-2Y^A})^{-1}\) appears in place of \((1_I - e^{-2Y^A})^{-1}\) hence instead of using Lemma 2.4 the operators are treated using only its symbol. Since \(\sigma_A\) is of order \(|\xi|\):

\[
\sigma_{(1_I + e^{-2Y^A})^{-1}} \leq \frac{1}{1 + e^{-2Yk|\xi|}} \leq 1,
\]

where \(k\) is some constant. Thus, in both cases we will arrive at estimates for (2.75) in the form of (2.82).

For the Neumann boundary conditions the estimates of \(|f_1 - f_2|_{\mathcal{B}_\rho}\) and \(|g_1 - g_2|_{\mathcal{B}_\rho}\) are more complex. Before we proceed, based on the definitions of \(v_0^*, v_Y^*, w_0^*,\) and \(w_Y^*\) in equations (2.31), (2.32), (2.33), and (2.34) we define:

\[
v_{i0}^* = A^{-1}(\tilde{v}_{i0} + F_{ix}(0)),
\]

\[
v_{iY}^* = A^{-1}(\tilde{v}_{iY} - F_{ix}(Y)),
\]

\[
w_{i0}^* = A^{-1}(\tilde{w}_{i0} + G_{ix}(0)),
\]

\[
w_{iY}^* = A^{-1}(\tilde{w}_{iY} - G_{ix}(Y)),
\]

where \(F_{ix} = (F_i)_x\) and \(G_{ix} = (G_i)_x\) for \(i = 1, 2\). For each of these equations we will derive estimates which use the fact that the symbol of an operator \(A^{-1}\partial_x\) is estimated:

\[
\sigma_{A^{-1}\partial_x} \leq \frac{2\pi i \xi}{k|\xi|}
\]

therefore it is a bounded operator with respect to our spaces. Using the definitions
of $v^*_{10}$, $v^*_{1Y}$, $w^*_{10}$, and $w^*_{1Y}$ we derive estimates:

$$
|v^*_{10} - v^*_{20}|_{B^\rho} = |A^{-1}\partial_x(F_1(0) - F_2(0))| \leq K|F_1 - F_2|,
$$

(2.87)

$$
|v^*_{1Y} - v^*_{2Y}|_{B^\rho} = |A^{-1}\partial_x(F_1(0) - F_2(0))| \leq K|F_1 - F_2|,
$$

(2.88)

$$
|w^*_{10} - w^*_{20}|_{B^\rho} = |A^{-1}\partial_x(G_1(0) - G_2(0))| \leq K|G_1 - G_2|,
$$

(2.89)

$$
|w^*_{1Y} - w^*_{2Y}|_{B^\rho} = |A^{-1}\partial_x(G_1(0) - G_2(0))| \leq K|G_1 - G_2|,
$$

(2.90)

where $K$ is some constant. For Problem NC using the equation for $f$ in (2.40) and triangle inequality:

$$
|f_1 - f_2|_{B^\rho} \leq |(1 + e^{-2YA})^{-1}(v^*_{10} - v^*_{20})|_{B^\rho} + |(1 + e^{-2YA})^{-1}(I_Y(F_2 + G_2) - I_Y(F_1 + G_1))|_{B^\rho} + |e^{-YA}(1 + e^{-2YA})^{-1}(w^*_{1Y} - w^*_{2Y})|_{B^\rho} + |e^{-YA}(1 + e^{-2YA})^{-1}(I_Y(F_2 - G_2) - I_Y(F_1 - G_1))|_{B^\rho}.
$$

Notice, since

$$
\sigma_{(1 + e^{-2YA})^{-1}} \leq \frac{1}{1 + e^{-k|\xi|}} < 1
$$

(where $k$ is some constant), and since

$$
\sigma_{e^{-YA}} \leq \frac{1}{e^{k|\xi|}} < 1,
$$

the only difference from what we did in the Dirichlet type problems is estimating $|v^*_{10} - v^*_{20}|_{B^\rho}$ and $|w^*_{1Y} - w^*_{2Y}|_{B^\rho}$, using estimates (2.87) and (2.90) along with (2.65) and (2.68):

$$
|f_1 - f_2|_{B^\rho} \leq K_1|F_1 - F_2|_{B^\rho} + K_2|G_1 - G_2|_{B^\rho}.
$$
The estimate for \( |g_1 - g_2|_{B_\rho} \) is analyzed using exactly the same technique, only using equation (2.41) for \( g \). This implies that the estimate for (2.75) is in the form of (2.82). Problem ND is done exactly the same way using \( f \) and \( g \) from (2.42) and (2.43) respectively, and for terms that arise using estimates (2.88) and (2.89). The reason we avoid Problems NA and NB is because operators of the form \((1_I - e^{-2Y_A})^{-1}\) are unbounded and when applied to either \( v_0^*, v_Y^*, w_0^*, \) or \( w_Y^* \) will remain unbounded.

For the mixed Problems MA, MD, ME, and MH the estimates on \( |f_1 - f_2|_{B_\rho} \) and \( |g_1 - g_2|_{B_\rho} \) are done using the corresponding equations for \( f \) and \( g \) with exactly the same technique as for the described Dirichlet and Neumann problems, along with one of the estimates (2.87) to (2.90). The mixed problems that we avoid in this section are left out for the same reason as for Neumann problems.

For every considered type of Dirichlet, Neuman and mixed boundary conditions we arrive at an estimate of the form (2.82), we need \( F \) and \( G \) to be contractions so that \( T \) is a contraction. We state the contraction property \( F \) and \( G \) need to satisfy:

**Property 2.5.** An operator \( F \) has a contracting property on \( B_\rho \times B_\rho \) if

\[
|F(v_1, w_1) - F(v_2, w_2)|_{B_\rho} \leq A_F(\mu) \ast |(v_1, w_1) - (v_2, w_2)|_{B_\rho \times B_\rho}
\]

where \((v_1, w_1), (v_2, w_2) \in B_\rho \times B_\rho\), \( \mu \) is a positive bounded measure satisfying \( \int d\mu < 1 \) and such that the pointwise inequalities \( |(v_1, w_1)|_{B_\rho} \leq \mu \) and \( |(v_2, w_2)|_{B_\rho} \leq \mu \) hold, and \( A_F \) is a continuous function with \( A_F(0) = 0 \).

If \( F \) and \( G \) satisfy property 2.5 estimate (2.82) becomes

\[
|T(v_1, w_1) - T(v_2, w_2)|_{B_\rho} \leq A(\mu) \ast |(v_1, w_1) - (v_2, w_2)|_{B_\rho}
\]

where \( A(\mu) \) is a continuous functions with \( A(0) = 0 \). If we consider a ball in \( B_\rho \) of radius \( r \) such that if \( |\mu| < r \) then \( |A(\mu)| < 1 \), then \( T \) will have to be a contraction
in that ball and the fixed point of $T$ will have to be a solution to (2.1), (2.2) with discussed types of boundary data. We summarize the result in Theorem 2.6

**Theorem 2.6.** Let $F$ and $G$ satisfy Property 2.5 and $Y < \infty$. Then there exists $\epsilon > 0$ such that if $\|v_0\|_B, \|v_Y\|_B \leq \epsilon$, and $(1_I - e^{-2AT})^{-1}v_0 \in B$ and $(1_I - e^{-2AT})^{-1}v_Y \in B$, then the system

\[
\begin{align*}
v_y - Aw &= F(v, w)_x \\
w_y - Av &= G(v, w)_x \\
v(x, 0) &= v_0 \quad v(x, Y) = v_Y
\end{align*}
\]

has a solution in $B_\rho \times B_\rho$. For Dirichlet boundary value problems with data $(w(0), w(Y))$ we specify the corresponding data $w_0$ and $w_Y$ at the boundaries. For $(v(0), w(Y))$ $(w(0), v(Y))$ we specify the corresponding data $v_0$, $v_Y$, $w_0$ or $w_Y$ and we remove the condition that $(1_I - e^{-2AT})^{-1}$ needs to map boundary data into $B$.

For Neumann and mixed boundary conditions we summarize the result in theorem 2.7.

**Theorem 2.7.** Let $F$ and $G$ satisfy Property 2.5 and $Y < \infty$. Then there exists $\epsilon > 0$ such that if $\|\tilde{v}_0\|_B, \|\tilde{w}_Y\|_B \leq \epsilon$, the system

\[
\begin{align*}
v_y - Aw &= F(v, w)_x \\
w_y - Av &= G(v, w)_x \\
\nabla v(x, 0) \cdot n_0 &= \tilde{v}_0 \quad \nabla w(x, Y) \cdot n_0 = \tilde{w}_Y
\end{align*}
\]

has a solution in $B_\rho \times B_\rho$. For Neumann boundary value problems with boundaries at $(\nabla w(x, 0) \cdot n_0, \nabla v(x, Y) \cdot n_0)$ we specify the corresponding data $\tilde{v}_Y$, $\tilde{w}_0$ at the boundaries. For mixed boundary conditions $(v(x, 0), \nabla v(x, Y) \cdot n_0, (\nabla v(x, 0) \cdot n_0, v(x, Y)),$
(w(x, 0), \nabla w(x, Y) \cdot n_0), and (\nabla w(x, 0) \cdot n_0, w(x, Y)) we specify the corresponding data \( v_0, v_Y, w_0, w_Y, \tilde{v}_0, \tilde{v}_Y, \tilde{w}_0 \) or \( \tilde{w}_Y \) at the boundaries.
2.2 Periodic Problem

In this section we consider equations (2.1) and (2.2) where \( v \) and \( w \) are defined on the domain \((x, y) \in (0, 2\pi) \times (0, Y)\) where \( Y < \infty \). Here as in the previous section the technique for showing existence of the solution to the boundary value problems is discussed. The difference is we are interested in the solution \((v, w)\) to be periodic in \( x \). Also, every type of boundary condition presented earlier in this chapter is considered.

We start with the definitions of appropriate spaces.

**Definition 2.8.** Let \( B \) be the Banach space of periodic functions with the norm \( \|u\| = \sum_{n \neq 0} |\hat{u}(n)| \), where \( \hat{u}(n) \) is the \( n \)th Fourier coefficient of \( u \in B \), such that \( \hat{u}(0) = 0 \).

Notation: When considering the periodic problem \( \hat{u}(n) = \hat{u} \) and \( F(u)(n) = F(u) \) will refer to the \( n \)th Fourier coefficient of \( u(x) \).

**Definition 2.9.** Let \( B_\rho \) be a space of continuous functions of \( y \geq 0 \) with values in \( B \) such that there exists \( \mu \in \ell^1 \) with \( |e^{\rho|n|}\hat{u}(n, y)| \leq \mu(n) \) for all \( y \). We call \( |u|_{B_\rho} \) the infimum of such \( \mu \). \( B_\rho \) is a Banach space with norm \( \sum |u|_{B_\rho} \). (Note: \( |u|_{B_\rho} = \sup_y |e^{\rho|n|}\hat{u}(n, y)| \).)

Remark: We have an algebra property for \( B_\rho \), which follows from the inequality

\[ |uv|_\rho \leq |u|_\rho \star |v|_\rho. \]

Note the operators \( S_1, S_2, I^+, I^- \) are not only well defined on the space \( B_\rho \), but when \( \sigma_A > k|n| \) where \( k \) is some constant, also satisfy the following Lipschitz properties.

Since \( y \geq 0 \)

\[ |S_1 h(y)|_{B_\rho} = \sup_y |e^{\rho|n|} F(e^{-yA} h(y))| = \sup_y |e^{\rho|n|} e^{-\sigma_A y} \hat{h}(y)| \]

\[ \leq \sup_y |e^{\rho|n|} \hat{h}(y)| = |h|_{B_\rho}, \quad (2.91) \]

\[ |S_2 h(y)|_{B_\rho} = \sup_y |e^{\rho|n|} F(e^{yA} h(y))| = \sup_y |e^{\rho|n|} e^{\sigma_A y} \hat{h}(y)| \]

\[ \leq \sup_y |e^{\rho|n|} \hat{h}(y)| = |h|_{B_\rho}, \quad (2.92) \]
and since \( y \leq Y \)

\[
|S_2 h(y)|_{B_{\rho}} = \sup_y |e^{\rho|n|} F(e^{(y-Y)A} h(y))| = \sup_y |e^{\rho|n|} e^{\sigma_A(y-Y)} \hat{h}(y)| \quad (2.93)
\]

\[
\leq \sup_y |e^{\rho|n|} \hat{h}(y)| = |h|_{B_{\rho}}. \quad (2.94)
\]

Similarly if \( \sigma_A \geq K_1|n| \) where \( K_1 \) is some constant

\[
|I^+ h(y)|_{B_{\rho}} = \sup_y \left| e^{\rho|n|} \int_0^y e^{\sigma_A(y-y)} h_x(\gamma) \, d\gamma \right|
\]

\[
= \sup_y \left| e^{\rho|n|} \int_0^y e^{\sigma_A(y-y)} (in) \hat{h}(\gamma) \, d\gamma \right|
\]

\[
\leq \sup_y e^{\rho|n|} e^{-\sigma_A|y|} |n| \int_0^y e^{\sigma_A\gamma} |\hat{h}(\gamma)| \, d\gamma
\]

\[
\leq \sup_y e^{-\sigma_A|y|} |n| \int_0^y e^{\sigma_A\gamma} \sup_{\gamma} |e^{\rho|n|} \hat{h}(\gamma)| \, d\gamma
\]

\[
\leq |h|_{B_{\rho}} \sup_y e^{-\sigma_A|y|} |n| \int_0^y e^{\sigma_A\gamma} \, d\gamma
\]

\[
= |h|_{B_{\rho}} \sup_y e^{-\sigma_A|y|} |n| \left( \frac{e^{\sigma_A|y|} - 1}{\sigma_A} \right)
\]

\[
\leq K|h|_{B_{\rho}}, \quad (2.95)
\]

where \( K \) is some constant. Notice, the estimate is derived in exactly the same way as the similar estimate in (2.64) for the non-periodic problem. Since \( I^+ \) is linear, this estimate also demonstrates that \( I^+ \) is Lipschitz on \( B_{\rho} \). We also define the operator \( I_Y \) by

\[
I_Y h = I^+ h(Y).
\]

Since \( I_Y \) is defined in terms of \( I^+_Y \), we see that \( I_Y \) has a similar estimate to (2.95)

\[
|I_Y h|_{B_{\rho}} = |I^+ h(Y)|_{B_{\rho}} \leq |I^+ h(y)|_{B_{\rho}} \leq K|h|_{B_{\rho}}. \quad (2.96)
\]
The estimate for $I^-$ is derived identically to (2.67), only we use $n$ instead of $\xi$:

$$|I^-h(y)|_{\mathcal{B}_{\rho}} \leq K|h|_{\mathcal{B}_{\rho}}. \tag{2.97}$$

As before, we notice that this estimate implies that $I^-$ is Lipschitz on $\mathcal{B}_{\rho}$. We make a further definition, letting the operator $I_0$ be

$$I_0h = I^-h(0).$$

Then $I_0$ has the estimate

$$|I_0h|_{\mathcal{B}_{\rho}} = |I^-h(0)|_{\mathcal{B}_{\rho}} \leq |I^-h(y)|_{\mathcal{B}_{\rho}} \leq K|h|_{\mathcal{B}_{\rho}}. \tag{2.98}$$

Since $S_1, S_2, I^+, I^-$ are well defined on $\mathcal{B}_{\rho}$ implies that $T(v, w) = (v, w)$ is well defined on $\mathcal{B}_{\rho} \times \mathcal{B}_{\rho}$.

As in the non-periodic problem the rest of the work is dedicated to establishing that the mapping $T(v, w)$ is a contraction on $\mathcal{B}_{\rho} \times \mathcal{B}_{\rho}$. In the non-periodic problem in Lemma 2.4 we established an important estimate involving operator $(1 - e^{2Y^A})^{-1}$ acting on operators $I^+$ and $I^-$. Since we have removed the $n = 0$ mode in our function spaces, the symbol:

$$\sigma(1 - e^{2Y^A})^{-1} \leq \frac{1}{1 - e^{kY|n|}}$$

is never unbounded which implies that the operator $(1 - e^{2Y^A})^{-1}$ will no longer be unbounded. For the periodic problem we could derive similar kind of estimate as before, which would be useful if we were not removing the $n = 0$ mode from our function space. However, it turns out in the application to the Boussinesq equations that we need to remove the $n = 0$ mode based on the operators $F$ and $G$. Hence it becomes pointless for us to develop theory that considers $n = 0$. In the following
section we discuss the details of the contraction argument for Dirichlet, Neumann, and mixed boundary conditions.

### 2.2.1 Contraction Estimate

We will now treat existence and uniqueness for each of the boundary value problems in turn. Regardless of boundary conditions, using the triangle inequality the first estimate will be

\[
|T(v_1, w_1) - T(v_2, w_2)|_{\mathcal{B}_\rho} \leq |S_1 f_1 - S_1 f_2|_{\mathcal{B}_\rho} + |S_2 g_1 - S_2 g_2|_{\mathcal{B}_\rho},
\]

\[
+ |I^+(F_1 - G_1) - I^+(F_2 - G_2)|_{\mathcal{B}_\rho} + |I^-(F_1 + G_1) - I^-(F_2 + G_2)|_{\mathcal{B}_\rho}, \tag{2.99}
\]

where \( F_i = F(v_i, w_i), \ G_i = G(v_i, w_i) \) for \( i = 1, 2 \). Using Lipschitz properties of \( S_1 \) and \( S_2 \) in estimates (2.92), (2.94) implies that

\[
|S_1 f_1 - S_1 f_2|_{\mathcal{B}_\rho} + |S_2 g_1 - S_2 g_2|_{\mathcal{B}_\rho} \leq |f_1 - f_2|_{\mathcal{B}_\rho} + |g_1 - g_2|_{\mathcal{B}_\rho}.
\]

We begin with the estimates for Dirichlet boundary condition \((v(0), v(Y))\) using \( f \) and \( g \) from equations (2.20) and (2.19). First, estimate \(|f_1 - f_2|_{\mathcal{B}_\rho}\). Using the triangle inequality:

\[
|f_1 - f_2|_{\mathcal{B}_\rho} \leq |(1 - e^{-2AY})^{-1}(\frac{1}{2}(I_0(F_1 + G_1) - I_0(F_2 + G_2)))|_{\mathcal{B}_\rho},
\]

\[
+ |e^{-YA}(1 - e^{-2YA})^{-1}(\frac{1}{2}(I_Y(F_1 - G_1) - I_Y(F_2 - G_2)))|_{\mathcal{B}_\rho}.
\]
The symbol $\sigma_{(1-e^{-2YA})^{-1}} \leq \frac{1}{(1-e^{-2k|n|})} < \infty$ for all $n \neq 0$ and $\sigma_{e^{-YA}} \leq e^{-k|n|} < \infty$ implies:

$$|f_1 - f_2|_{B_\rho} \leq K_1 \frac{1}{2} (I_0(F_1 + G_1) - I_0(F_2 + G_2))|_{B_\rho} + K_2 \frac{1}{2} (I_Y(F_1 - G_1) - I_Y(F_2 - G_2))|_{B_\rho}. \quad (2.100)$$

where $K_1$ and $K_2$ are some constants. Using the triangle inequality

$$|f_1 - f_2|_{B_\rho} \leq K_1 \frac{1}{2} (|I_0(F_1 - F_2)|_{B_\rho} + |I_0(G_1 - G_2)|_{B_\rho}) + K_2 \frac{1}{2} (|I_Y(F_1 - F_2)|_{B_\rho} + |I_Y(G_1 - G_2)|_{B_\rho}). \quad (2.101)$$

Using estimates from (2.95), (2.96) and (2.97), (2.98):

$$|f_1 - f_2|_{B_\rho} \leq K_3 (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}), \quad (2.102)$$

where $K_3 = K_1 + K_2$. The estimate for $|g_1 - g_2|_{B_\rho}$ is done exactly the same way with equation (2.19) to get

$$|g_1 - g_2|_{B_\rho} \leq K_4 (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}), \quad (2.103)$$

where $K_4$ will be some constant. Using estimates from (2.95) and (2.97) along with (2.102) and (2.103), the estimate on (2.99):

$$|T(v_1, w_1) - T(v_2, w_2)|_{B_\rho} \leq K (|F_1 - F_2|_{B_\rho} + |G_1 - G_2|_{B_\rho}), \quad (2.104)$$

where $K$ is some constant depending on $K_1$, $K_2$, $K_3$, $K_4$.

For Dirichlet boundary conditions of the form $(w(0), w(y))$ using equations (2.21), (2.22) the estimates for $|f_1 - f_2|_{B_\rho}$, $|g_1 - g_2|_{B_\rho}$ are done nearly identically to get an
estimate of (2.99) in the form of (2.104).

For Dirichlet boundary conditions of the form \((v(0), w(Y))\) and \((w(0), v(Y))\) we use equations (2.23), (2.24) and (2.25), (2.26) respectively. The difference is the estimate for \(|f_1 - f_2|_{B_{\rho}}\), \(|g_1 - g_2|_{B_{\rho}}\) have an operator of the form \((1 + e^{-2Y\Lambda})^{-1}\) which is treated using only its symbol:

\[
\sigma_{(1 + e^{-2Y\Lambda})^{-1}} \leq \frac{1}{1 + e^{-2YC|n|}} \leq 1,
\]

where \(C\) is some constant. Thus, in both cases we will arrive at estimates for (2.99) in the form of (2.104).

In the Neummann boundary value problem just as for non-periodic problems we need to do more work when dealing with estimates for \(|f_1 - f_2|_{B_{\rho}}\) and \(|g_1 - g_2|_{B_{\rho}}\). Before we proceed based on definitions of \(v_0^*, v_Y^*, w_0^*, \) and \(w_Y^*\) in equations (2.31), (2.32), (2.33), and (2.34) we define:

\[
\begin{align*}
v_{i0}^* &= A^{-1}(\tilde{v}_0 + F_{ix}(0)), \\
v_{iY}^* &= A^{-1}(\tilde{v}_Y - F_{ix}(Y)), \\
w_{i0}^* &= A^{-1}(\tilde{w}_0 + G_{ix}(0)), \\
w_{iY}^* &= A^{-1}(\tilde{w}_Y - G_{ix}(Y)),
\end{align*}
\]

where \(F_{ix} = (F_i)_x\) and \(G_{ix} = (G_i)_x\) for \(i = 1, 2\). For each of these we will derive estimates which use the fact that the symbol of an operator \(A^{-1}\partial_x\) is:

\[
\sigma_{A^{-1}\partial_x} \leq \frac{2\pi in}{k|n|}
\]

hence it is a bounded operator with respect to our spaces. Using the definitions of
We derive estimates:

\[ |v_{10}^* - v_{20}^*|_{\mathcal{B}_p} = |A^1 \partial_x (F_1(0) - F_2(0))| \leq K|F_1 - F_2|, \tag{2.109} \]
\[ |v_{1Y}^* - v_{2Y}^*|_{\mathcal{B}_p} = |A^1 \partial_x (F_1(0) - F_2(0))| \leq K|F_1 - F_2|, \tag{2.110} \]
\[ |w_{10}^* - w_{20}^*|_{\mathcal{B}_p} = |A^1 \partial_x (G_1(0) - G_2(0))| \leq K|G_1 - G_2|, \tag{2.111} \]
\[ |w_{1Y}^* - w_{2Y}^*|_{\mathcal{B}_p} = |A^1 \partial_x (G_1(0) - G_2(0))| \leq K|G_1 - G_2|, \tag{2.112} \]

where \( K \) is some constant. For Neumann Problem NA, using \( f \) from equation (2.37):

\[ |f_1 - f_2|_{\mathcal{B}_p} \leq |(1 - e^{-2Y A})^{-1}(v_{10}^* - v_{20}^*)|_{\mathcal{B}_p} \]
\[ + |(1 - e^{-2Y A})^{-1}(I_0(F_1 + G_1) - I_0(F_1 + G_2))|_{\mathcal{B}_p} \]
\[ + |e^{-2Y A}(1 - e^{-2Y A})^{-1}(v_{1Y}^* - v_{2Y}^*)|_{\mathcal{B}_p} \]
\[ + |e^{-2Y A}(1 - e^{-2Y A})^{-1}(I_Y(F_1 - G_1) - I_Y(F_2 - G_2))|_{\mathcal{B}_p}. \tag{2.113} \]

As discussed earlier the symbols \( \sigma(1 - e^{-2Y A})^{-1} \) and \( \sigma e^{-2Y A} \) are bounded hence the corresponding operators are bounded:

\[ |f_1 - f_2|_{\mathcal{B}_p} \leq K_1|v_{10}^* - v_{20}^*|_{\mathcal{B}_p} + K_2|v_{1Y}^* - v_{2Y}^*|_{\mathcal{B}_p} \]
\[ + K_4|I_0(F_1 + G_1) - I_0(F_1 + G_2)|_{\mathcal{B}_p} \]
\[ + K_5|I_Y(F_1 - G_1) - I_Y(F_2 - G_2)|_{\mathcal{B}_p}. \tag{2.114} \]
The estimate for $|g_1 - g_2|_{\mathcal{B}_\rho}$ is done in exactly the same way using (2.36) for $g$ to get:

$$|g_1 - g_2|_{\mathcal{B}_\rho} = K_7 \left( |F_1 - F_2|_{\mathcal{B}_\rho} + |G_1 - G_2|_{\mathcal{B}_\rho} \right).$$  \hspace{1cm} (2.115)

These estimates show that (2.99) can be estimated in the form of (2.104) for Neumann Problem NA. For Neumann Problem NB the estimate in (2.99) can be shown to be in the form of (2.104) using exactly the same procedure using equations for $f$ and $g$ in (2.38) and (2.39), and for $|w^*_{10} - w^*_{20}|_{\mathcal{B}_\rho}$ and $|w^*_{1Y} - w^*_{2Y}|_{\mathcal{B}_\rho}$ which will arise in those estimates, we use (2.111) and (2.112). For Neumann Problem NC and ND these type of estimates are achieved using equations for $f$ and $g$ in (2.40), (2.41) and (2.42), (2.43) respectively, along with one of the appropriate estimates (2.109) to (2.112). The difference in the Problem NC and ND is the operator $(1 + e^{-2AY})^{-1}$ which is bounded based on its symbol $\sigma((1 + e^{-2AY})^{-1} \leq \frac{1}{1 + e^{-2AY|m|}} < \infty$.

The mixed problems are treated in exactly the same way as the Dirichlet and Neumann problems where $|f_1 - f_2|_{\mathcal{B}_\rho}$ and $|g_1 - g_2|_{\mathcal{B}_\rho}$ are estimated using one of the appropriate equations for $f$ and $g$ derived in the beginning of this chapter.

Now that we have shown that for every type of Dirichlet, Neumann and mixed boundary conditions we arrive at an estimate of the form (2.104), we need $F$ and $G$ to be contractions for $T$ to be a contraction. We state the contraction property $F$ and $G$ need to satisfy:

**Property 2.10.** An operator $F$ has a contracting property on $\mathcal{B}_\rho \times \mathcal{B}_\rho$ if

$$|F(v_1, w_1) - F(v_2, w_2)|_{\mathcal{B}_\rho} \leq A_F(\mu) \ast |(v_1, w_1) - (v_2, w_2)|_{\mathcal{B}_\rho \times \mathcal{B}_\rho},$$

where $(v_1, w_1), (v_2, w_2) \in \mathcal{B}_\rho \times \mathcal{B}_\rho$, $\mu \in \ell^1$ is positive and such that the pointwise inequalities $|(v_1, w_1)|_{\mathcal{B}_\rho} \leq \mu$ and $|(v_2, w_2)|_{\mathcal{B}_\rho} \leq \mu$ hold, and $A_F$ is a continuous function with $A_F(0) = 0$. 
If $F$ and $G$ satisfy property 2.10 estimate (2.104) becomes

$$|T(v_1, w_1) - T(v_2, w_2)|_{B^\rho} \leq A(\mu) \ast |(v_1, w_1) - (v_2, w_2)|_{B^\rho},$$

where $A(\mu)$ is a continuous function with $A(0) = 0$. If we consider a ball in $B^\rho$ of radius $r$ such that if $|\mu| < r$ then $|A(\mu)| < 1$, then $T$ will have to be a contraction in that ball and the fixed point of $T$ will have to be a solution to (1.1) and (1.2) with Dirichlet boundary data. We summarize the result in theorem 2.11.

**Theorem 2.11.** Let $F$ and $G$ satisfy Property 2.10 and $Y < \infty$. Then there exists $\epsilon > 0$ such that if $||v_0||_B, ||v_Y||_B \leq \epsilon$, the system

$$v_y - Aw = F(v, w)_x,$$

$$w_y - Av = G(v, w)_x,$$

$$v(x, 0) = v_0 \quad v(x, Y) = v_Y,$$

has a (locally) unique periodic solution in a ball of $B^\rho \times B^\rho$.

For Dirichlet boundary data of the form $(v(x, 0), w(x, Y)), (w(x, 0), v(x, Y))$ and $(w(0), w(y))$ the data $v_0, v_Y, w_0$ and $w_y$ is specified with the corresponding boundaries.

For Neumann and mixed boundary conditions we summarize the result in theorem 2.12.

**Theorem 2.12.** Let $F$ and $G$ satisfy Property 2.10 and $Y < \infty$. Then there exists $\epsilon > 0$ such that if $||\tilde{v}_0||_B, ||\tilde{v}_Y||_B \leq \epsilon$, the system

$$v_y - Aw = F(v, w)_x$$

$$w_y - Av = G(v, w)_x$$

$$\nabla v(x, 0) \cdot n_0 = \tilde{v}_0 \quad \nabla v(x, Y) \cdot n_0 = \tilde{v}_Y$$
has a (locally) unique periodic solution in a ball of $B_\rho \times B_\rho$.

For Neumann boundary value problems with boundaries at $(\nabla w(x,0) \cdot n_0, \nabla w(x,Y) \cdot n_0)$, $(\nabla v(x,0) \cdot n_0, \nabla w(x,Y) \cdot n_0)$ and $(\nabla w(x,0) \cdot n_0, \nabla v(x,Y) \cdot n_0)$ we specify corresponding data $\tilde{v}_0, \tilde{v}_Y$, $\tilde{w}_0$ or $\tilde{w}_Y$ at the boundaries. For mixed boundary conditions $(v(x,0), \nabla v(x,Y) \cdot n_0)$, $(v(x,0), \nabla w(x,Y) \cdot n_0)$, $(\nabla w(x,0) \cdot n_0, v(x,Y))$, $(\nabla v(x,0) \cdot n_0, v(x,Y))$, $(w(x,0), \nabla w(x,Y) \cdot n_0)$, $(w(x,0), \nabla v(x,Y) \cdot n_0)$, $(\nabla v(x,0) \cdot n_0, w(x,Y))$ and $(\nabla w(x,0) \cdot n_0, w(x,Y))$ we specify the corresponding data $v_0$, $v_Y$, $w_0$, $w_Y$, $\tilde{v}_0$, $\tilde{v}_Y$, $\tilde{w}_0$ or $\tilde{w}_Y$ at the boundaries.
3. VORTEX SHEET

A vortex sheet is the interface between two fluid flows that have a discontinuity in the tangential velocity. For the particular vortex sheet that we consider, both fluids are incompressible and irrotational away from the interface and the densities of the two fluids are the same. The fluid flow is modelled by the Euler equations

\begin{align*}
u_t + u \cdot \nabla u &= -\nabla p, \quad (3.1) \\
\nabla \cdot u &= 0, \quad (3.2)
\end{align*}

where \( u \) and \( p \) are the velocity and density of the fluid. The discontinuity in the velocity along the interface is defined to be

\[ [u] = u_1 - u_2 \]

where \( u_1 \) and \( u_2 \) are the velocities of the two fluids. Since the discontinuity is only in the tangential component the discontinuity condition simplifies to

\[ [u] = \Omega \hat{t} \]

where \( \Omega \) is the vortex sheet strength defined by

\[
< \text{curl}(u), \varphi > = \int_{\mathbb{R}} \Omega(t,x)\varphi(x,y(t,x))dx
\]

where \((x,y)\) are the coordinates of the boundary in \(\mathbb{R}^2\) and \( \varphi \) is any smooth compactly supported test function. In the work of [17] the vortex sheet was formulated in terms
of principle value integrals

\[ y_t = -y_x V_1 + V_2, \]  
\[ \Omega_t + (\Omega V_1) = 0, \]  

where

\[ V_1 = -\frac{1}{2\pi} PV \int \frac{y(t, x) - y(t, x')}{(x - x')^2 + (y(t, x) - y(t, x'))^2} \Omega(t, x') dx', \]  
\[ V_2 = \frac{1}{2\pi} PV \int \frac{x - x'}{(x - x')^2 + (y(t, x) - y(t, x'))^2} \Omega(t, x') dx', \]  

are the components of average velocity of the fluids on both sides of the interface. In the work done by [7] the equations (3.3) and (3.4) were rewritten as

\[ y_{xt} - \Lambda w = F(y_x, w)_x, \]  
\[ w_t - \Lambda y_x = G(y_x, w)_x, \]  

where \( \Omega = 2(1+w), \Lambda u(x) = \frac{1}{\pi} PV \int \frac{u(x)-u(x')}{(x-x')^2} dx' \) which is the Hilbert transform with a derivative in \( x \) and

\[ F = \frac{1}{\pi} PV \int \left( \frac{1}{1+p^2 - 1} \right) \frac{1+w(x')}{x-x'} dx' + \frac{1}{\pi} y_x PV \int \frac{p}{1+p^2} \frac{1+w(x')}{x-x'} dx', \]  
\[ G = \frac{1}{\pi} PV \int \left( \frac{p}{1+p^2 - p} \right) \frac{dx'}{x-x'} + \frac{1}{\pi} w PV \int \frac{p}{1+p^2} \frac{1+w(x')}{x-x'} dx' \]  
\[ + \frac{1}{\pi} PV \int \frac{w(x')}{1+p^2} \frac{dx'}{x-x'} , \]  

where \( p = \frac{y(x)-y(x')}{x-x'} \). In the work of [7] they were able to show that equations (3.5) and (3.6) have a unique solution for all time in an analytic function space with initial data \( y_x(0) \) having a sufficiently small Fourier transform. If we change the notation in
equation (3.5) and (3.6) to $v = y_x$ then

\begin{align*}
v_t - \Lambda w &= F_x(v, w), \quad (3.7) \\
w_t - \Lambda v &= G_x(v, w). \quad (3.8)
\end{align*}

We use the results from [7] that $F$, and $G$ satisfy the Lipschitz property since

\begin{align*}
|F(y_{1x}, w_1) - F(y_{2x}, w_2)|_B &\leq A_F(\mu) * |(y_{1x}, w_1) - (y_{2x}, w_2)|_B, \\
|G(y_{1x}, w_1) - G(y_{2x}, w_2)|_B &\leq A_G(\mu) * |(y_{1x}, w_1) - (y_{2x}, w_2)|_B,
\end{align*}

where $\mu$ is a positive bounded measure satisfying $|y_{ix}|_B \leq \mu$, $|w_i|_B \leq \mu$ and

\begin{align*}
A_F &= 24\mu + 8 \sum_{j=2}^{\infty} (j + 1)^2 \mu^{*j}, \\
A_G &= 8\mu + 9\mu^{*2} + 70\mu^{*3} + 19\mu^{*4} + 4 \sum_{j=2}^{\infty} (2j^2 + j + 1) \mu^{*j}.
\end{align*}

$A_F$ is derived in [7], and we derive $A_G$ in Appendix C. Since $A_F$ and $A_G$ are continuous and $A_F(0) = 0$ and $A_G(0) = 0$, $F$ and $G$ satisfy property 2.5. Using Theorem 2.6 implies that the vortex sheet time boundary value problem has a unique solution in $B_\rho \times B_\rho$. We summarize the results in Theorems 3.1 and 3.2

**Theorem 3.1.** Let $T > 0$ be given. Let $y_{x0}$ and $y_{xT}$ be given such that $(1_I - e^{-2\Lambda T})^{-1}y_{x0} \in B$ and $(1_I - e^{-2\Lambda T})^{-1}y_{xT} \in B$. There exists $\epsilon > 0$ such that if $\|y_{x0}\|_B, \|y_{xT}\|_B \leq \epsilon$, then the system

\begin{align*}
y_{xt} - \Lambda w &= F_x(v, w), \quad (3.9) \\
w_t - \Lambda y_{xt} &= G_x(v, w), \quad (3.10) \\
y_{x}(x, 0) &= y_{x0}, \quad y_{x}(x, T) = y_{xT},
\end{align*}
has a solution in $\mathcal{B}_\rho \times \mathcal{B}_\rho$.

Alternatively, if $w_0$ and $w_T$ satisfy $(1 - e^{-2AT})^{-1}w_0 \in B$ and $(1 - e^{-2AT})^{-1}w_T \in B$. Then there exists $\epsilon > 0$ such that if $||w_0||_B, ||w_T||_B \leq \epsilon$, then the system given by (3.9) and (3.10), with the boundary conditions

$$w(x,0) = w_0(x), \quad w(x,T) = w_T(x),$$

has a solution in $\mathcal{B}_\rho \times \mathcal{B}_\rho$.

For boundary conditions

$$w(x,0) = w_0(x), \quad y_x(x,T) = y_{xT}$$

and

$$y_x(x,0) = y_{x0}, \quad w(x,T) = w_T(x)$$

we specify the data $y_{x0}$, $y_{xT}$, $w_0$ or $w_T$ and remove the condition that the operator $(1 - e^{-2AT})^{-1}$ have to map boundary data into $B$.

**Theorem 3.2.** Let $T > 0$ be given. There exists $\epsilon > 0$ such that if $||y_{x0}||_B, ||w_T||_B \leq \epsilon$, the system

$$y_{xt} - \Lambda w = F_x(v,w),$$
$$w_t - \Lambda y_x = G_x(v,w),$$
$$y_x(x,0) = y_{x0}(x) \quad w(x,T) = w_T(x),$$

has a solution in $\mathcal{B}_\rho \times \mathcal{B}_\rho$.

**Remark:** For Neumann problem NC with data $(\nabla y_x(x,0) \cdot n_0, \nabla w(x,T) \cdot n_0)$ we specify $\tilde{y}_{x0}$ and $\tilde{w}_Y$. For Neumann problem ND with boundaries at $(\nabla w(x,0) \cdot$
\( n_0, \nabla y_x(x, T) \cdot n_0 \) we specify the corresponding data \( \tilde{y}_{xT}, \tilde{w}_0 \) at the boundaries. For mixed boundary conditions \((y_x(x, 0), \nabla y_x(x, T) \cdot n_0), (\nabla y_x(x, 0) \cdot n_0, y_x(x, T)), (w(x, 0), \nabla w(x, T) \cdot n_0), \) and \((\nabla w(x, 0) \cdot n_0, w(x, T))\) we specify the corresponding data \( y_{x0}, y_{xT}, w_0, w_T, \tilde{y}_{x0}, \tilde{y}_{xT}, \tilde{w}_0 \) or \( \tilde{w}_T \) at the boundaries.
4. BOUSSINESQ SYSTEMS

The Boussinesq equations we use are derived in Bona, Chen, Saut [1] for small amplitude long surface waves. They are an approximation to the Euler equations of fluid under appropriate physical conditions with a free boundary condition. The equations are of the form:

\[ \eta_t + w_x + (w\eta)_x + aw_{xxx} - b\eta_{xx}t = 0, \]  
\[ w_t + \eta_x + w\eta_x + c\eta_{xxx} - dw_{xx}t = 0, \]  

where \( w \) is related to the horizontal velocity of the fluid, \( \eta \) is related to the position of the free surface and \( a, b, c, \) and \( d \) are parameters governed by the physical approximations, the parameters satisfy equations (4.3) to (4.5) in order for the system to be physical:

\[ a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \]  
\[ c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \]  
\[ a + b + c + d = \frac{1}{3}, \]  

where \( \theta \in [0, 1] \). In [1], [2] the equations (4.1) and (4.2) are treated for a specific range of parameters \( a, b, c, \) and \( d \) as an initial value problem, these parameters are provided below:

(1) \( a < 0, c < 0, b = 0, d > 0 \) \hspace{1cm} (2) \( b = 0, a = c > 0, d > 0 \) 
(3) \( d = 0, a = c > 0, b > 0 \) \hspace{1cm} (4) \( a < 0, c < 0, b > 0, d > 0 \)
(5) \(a > c > 0, b > 0, d > 0\)  
(6) \(b = c = 0, a < 0, d > 0\)  
(7) \(a = b = 0, c < 0, d > 0\)  
(8) \(c = d = 0, a < 0, b > 0\)  
(9) \(a = c > 0, b = d < 0\)  
(10) \(a = c = d = 0, b > 0\)  
(11) \(a = b = c = 0, d > 0\)  
(12) \(a = 0, b > 0, c < 0, d > 0\)  
(13) \(c = 0, a < 0, b > 0, d > 0\).

Since we treat the problem as a time boundary value problem we are able to get existence of a periodic solution for a different set of parameters:

\[
\begin{align*}
    a &< 0, \; c > 0, \; b \geq 0, \; d \geq 0, & (4.6) \\
    a &< 0, \; c < 0, \; b > 0, \; d < \frac{-1}{4\pi^2}, & (4.7) \\
    a &< 0, \; c < 0, \; b < \frac{-1}{4\pi^2}, \; d \geq 0. & (4.8)
\end{align*}
\]

Remark: Some of the values we work with are physical according with equations (4.3) to (4.5). For example

\[
a = -1, \; b = 1, \; c = \frac{2}{9}, \; d = \frac{1}{9},
\]

with \(\theta = \sqrt{\frac{2}{3}}\) satisfy (4.3) to (4.5) and belong to the set described in (4.6). Similarly

\[
a = -\frac{1}{12}, \; b = \frac{1}{12}, \; c = -\frac{1}{2}, \; d = 1,
\]

with \(\theta = 0\) satisfy (4.3) to (4.5) and belong to the set described in (4.8). However, the set described in (4.7) will never satisfy (4.4).

Our goal is to write equations (4.1) and (4.2) in the form of

\[
v_t - Av = F(v, w)_x, \quad (4.9)
\]
\[
w_t - Av = G(v, w)_x. \quad (4.10)
\]
Since we want to keep with the physical meaning of the Boussinesq equations we will change the notation from $y$ to $t$, and treat the boundary to be $t = 0$ and $t = T$. We begin by rewriting (4.1) and (4.2):

\begin{align*}
(1 - b \partial_{xx}) \eta_t + (\partial_x + a \partial_{xxx}) w &= -(w \eta)_x, \quad (4.11) \\
(1 - d \partial_{xx}) w_t + (\partial_x + c \partial_{xxx}) \eta &= -(\frac{w^2}{2})_x. \quad (4.12)
\end{align*}

Let $u = H \eta$ which implies that $\eta = -Hu$ where $H$ is the Hilbert transform in $x$ defined by a principal value integral $H f = \text{PV} \int_0^{2\pi} f(x') \cot(\frac{x-x'}{2})dx'$. We rewrite (4.11) as:

\begin{align*}
-H(1 - b \partial_{xx}) u_t + (\partial_x + a \partial_{xxx}) w &= (w(Hu))_x, \\
(1 - b \partial_{xx}) u_t + (\Lambda + a \Lambda \partial_{xx}) w &= H (w(Hu))_x, \\
(1 - b \partial_{xx}) u_t + (\Lambda - a \Lambda^3) w &= H (w(Hu))_x,
\end{align*}

where $\Lambda h = H \partial_x h$. Similarly we can rewrite (4.12) as:

\begin{align*}
(1 - d \partial_{xx}) w_t - H(\partial_x + c \partial_{xxx}) u &= -\left(\frac{w^2}{2}\right)_x, \\
(1 - d \partial_{xx}) w_t - (\Lambda + c \Lambda \partial_{xx}) u &= -\left(\frac{w^2}{2}\right)_x, \\
(1 - d \partial_{xx}) w_t - (\Lambda - c \Lambda^3) u &= -\left(\frac{w^2}{2}\right)_x.
\end{align*}

Now we are considering the Boussinesq equations in the form:

\begin{align*}
(1 - b \partial_{xx}) u_t + (\Lambda - a \Lambda^3) w &= H (w(Hu))_x, \quad (4.13) \\
(1 - d \partial_{xx}) w_t - (\Lambda - c \Lambda^3) u &= -\left(\frac{w^2}{2}\right)_x. \quad (4.14)
\end{align*}

Let $\Phi_1 = (1 - b \partial_{xx})$, $\Phi_2 = (1 - d \partial_{xx})$, $A_1 = (-\Lambda + a \Lambda^3)$, $A_2 = (\Lambda - c \Lambda^3)$. Applying
Φ_1^{-1} to (4.13) and Φ_2^{-1} to (4.14) we get (4.15), (4.16).

\[ u_t - \Phi_1^{-1} A_1 w = \Phi_1^{-1} H (w(Hu))_x, \quad (4.15) \]
\[ w_t - \Phi_2^{-1} A_2 u = -\Phi_2^{-1} (\frac{w^2}{2})_x. \quad (4.16) \]

Define the invertible operator Θ such that \( v = \Theta u \) and \( u = \Theta^{-1} v \), where the symbol of Θ is a multiplier in Fourier space, and

\[ \sigma_{\Theta \Phi_1^{-1} A_1} = \sigma_{\Phi_2^{-1} A_2 \Theta^{-1}}; \]

hence

\[ \sigma_\Theta = \sqrt{\frac{\sigma_{\Phi_2^{-1} A_2}}{\sigma_{\Phi_1^{-1} A_1}}}. \]

We know that

\[ \sigma_{A_1} = -2\pi |n| + 8\pi^3 a |n|^3, \quad \sigma_{A_2} = 2\pi |n| - 8\pi^3 c |n|^3, \]
\[ \sigma_{\Phi_1^{-1}} = \frac{1}{1 + 4\pi^2 b n^2}, \quad \sigma_{\Phi_2^{-1}} = \frac{1}{1 + 4\pi^2 d n^2}, \]

hence

\[ \sigma_\Theta = \sqrt{\frac{(1 + 4\pi^2 b n^2)(4\pi^2 c n^2 - 1)}{(1 + 4\pi^2 d n^2)(1 - 4\pi^2 a n^2)}}. \]

This implies that

\[ \sigma_{\Theta \Phi_1^{-1} A_1} = \sigma_{\Phi_2^{-1} A_2 \Theta^{-1}} = 2\pi |n| \sqrt{\frac{(1 - 4\pi^2 a n^2)(4\pi^2 c n^2 - 1)}{(1 + 4\pi^2 d n^2)(1 + 4\pi^2 b n^2)}}. \]
Finally we can use $\Theta$ to reformulate (4.15) and (4.16) into (4.17) and (4.18).

\[
v_t - \Theta \Phi_1^{-1} A_1 w = \Theta \Phi_1^{-1} H \left( w(HB^{-1}v) \right)_x, \tag{4.17}
\]
\[
w_t - \Phi_2^{-1} A_2 \Theta^{-1} v = -\Phi_2^{-1} \left( \frac{w^2}{2} \right)_x. \tag{4.18}
\]

Note (4.17) and (4.18) are in the form of (4.9) and (4.10) with $A = \Theta \Phi_1^{-1} A_1 = \Phi_2^{-1} A_2 \Theta^{-1}$, $F(v, w) = \Theta \Phi_1^{-1} H \left( w(H\Theta^{-1}v) \right)$ and $G(v, w) = -\Phi_2^{-1} \left( \frac{w^2}{2} \right)$. In order to apply the theory developed in section 2.2 we need $F, G$ to satisfy Property 2.10 and operator $A$ to have symbol $k_1|n| \leq \sigma_A \leq k_2|n|$, where $k_1$ and $k_2$ are some constants.

Let us consider parameters $a, b, c, d$ in (4.6), (4.7), and (4.8). Since

\[
\sigma_A = 2\pi|n| \sqrt{\frac{(1 - 4\pi^2 an^2)(4\pi^2 cn^2 - 1)}{(1 + 4\pi^2 dn^2)(1 + 4\pi^2 bn^2)}},
\]

analyzing only the square root portion of $\sigma_A$ we are guaranteed to avoid complex numbers only if $n \neq 0$. This is the reason why in the general theory section for periodic problems we remove the $n = 0$ mode, and also is the reason why we can not apply the non-periodic theory to the Boussinesq equations. Evaluating the limit as $n \to \infty$ of the square root portion:

\[
\lim_{n \to \infty} \sqrt{\frac{(1 - 4\pi^2 an^2)(4\pi^2 cn^2 - 1)}{(1 + 4\pi^2 dn^2)(1 + 4\pi^2 bn^2)}} = \sqrt{\frac{-ac}{db}},
\]

and since there are no other singularities results in $k_1|n| \leq \sigma_A \leq k_2|n|$. Since $F, G$ are constructed from $\Theta, \Phi_1$ and $\Phi_2$ we first analyse these operators separately.

**Property of $\Theta$**

\[
|\Theta h_1 - \Theta h_2|_{B^p} \leq |\Theta(h_1 - h_2)|_{B^p} = \sup_t |e^{\rho|n| \sigma_\Theta(h_1 - h_2)}|,
\]
where
\[
\sigma_\Theta = \sqrt{\frac{(1 + 4\pi^2bn^2)(4\pi^2cn^2 - 1)}{(1 + 4\pi^2dn^2)(1 - 4\pi^2an^2)}} \geq 0,
\]
for \(a, b, c, d\) in (4.6), (4.7), and (4.8). Since \(n\) is an integer value and \(n \neq 0\) we get that \(\sigma_\Theta\) is well defined and since \(\lim_{n \to \infty} \sigma_\Theta = \sqrt{\frac{bc}{-ad}}\) implies that \(\sigma_\Theta\) is bounded by some constant. Hence
\[
|\Theta h_1 - \Theta h_2|_{B_\rho} \leq C|h_1 - h_2|_{B_\rho},
\]
where \(C\) is some constant.

Similarly for \(\sigma_{\Theta^{-1}} \geq 0\) and \(\lim_{n \to \infty} \sigma_\Theta = \sqrt{\frac{-ad}{bc}}\). Hence
\[
|\Theta^{-1} h_1 - \Theta^{-1} h_2|_{B_\rho} \leq D|h_1 - h_2|_{B_\rho},
\]
where \(D = \frac{1}{C}\) is some constant.

**Property of \(\Phi_1\) and \(\Phi_2\)**

\[
|\Phi_1^{-1} h_1 - \Phi_1^{-1} h_2|_{B_\rho} = |\sigma_{\Phi_1^{-1}}(h_1 - h_2)|_{B_\rho},
\]
\[
= \sup_t |e^{\rho |t|} \sigma_{\Phi_1^{-1}}(h_1 - h_2)|,
\]
where \(\sigma_{\Phi_1^{-1}} = \frac{1}{1 + 4\pi^2bn^2}\) and \(b \geq 0\) or \(b < -\frac{1}{4\pi^2}\), implies that
\[
|\Phi_1^{-1} h_1 - \Phi_1^{-1} h_2|_{B_\rho} \leq |h_1 - h_2|_{B_\rho}.
\]

Similarly \(\sigma_{\Phi_2} = \frac{1}{1 + 4\pi^2dn^2}\) where \(d \geq 0\) or \(d < -\frac{1}{4\pi^2}\), implies that
\[
|\Phi_2^{-1} h_1 - \Phi_2^{-1} h_2|_{B_\rho} \leq |h_1 - h_2|_{B_\rho}.
\]
CHAPTER 4. BOUSSINESQ SYSTEMS

Property of $F$ and $G$

Let $F(v, w) = \Theta \Phi^{-1} H (w(H \Theta^{-1} v))$.

$$|F_1 - F_2|_{\mathcal{B}_p} = |\Theta \Phi^{-1} H (w_1(H \Theta^{-1} v_1)) - \Theta \Phi^{-1} H (w_2(H \Theta^{-1} v_2))|_{\mathcal{B}_p},$$

$$= |\Theta \Phi^{-1} H (w_1(H \Theta^{-1} v_1) - w_2(H \Theta^{-1} v_2))|_{\mathcal{B}_p}.$$

Using the properties of $\Theta$ and $\Phi_1$:

$$|F_1 - F_2|_{\mathcal{B}_p} = C|w_1(H \Theta^{-1} v_1) - w_2(H \Theta^{-1} v_2)|_{\mathcal{B}_p},$$

$$\leq C|w_1(H \Theta^{-1} v_1) - w_2(H \Theta^{-1} v_1)|_{\mathcal{B}_p}$$

$$+ C|w_2(H \Theta^{-1} v_1) - w_2(H \Theta^{-1} v_2)|_{\mathcal{B}_p},$$

$$= C|H \Theta^{-1} v_1(w_1 - w_2)|_{\mathcal{B}_p} + C|w_2 H \Theta^{-1}(v_1 - v_2)|_{\mathcal{B}_p},$$

$$\leq |v_1(w_1 - w_2)|_{\mathcal{B}_p} + C|w_2|_{\mathcal{B}_p} * |H \Theta^{-1}(v_1 - v_2)|_{\mathcal{B}_p},$$

$$\leq |v_1|_{\mathcal{B}_p} * |w_1 - w_2|_{\mathcal{B}_p} + |w_2|_{\mathcal{B}_p} * |v_1 - v_2|_{\mathcal{B}_p},$$

$$\leq \mu \max(|v_1 - v_2|_{\mathcal{B}_p}, |w_1 - w_2|_{\mathcal{B}_p}),$$

$$= \mu \max(|(v_1, w_1) - (v_2, w_2)|_{\mathcal{B}_p}).$$

Let $G = -\Phi_2^{-1} w^2$.

$$|G_1 - G_2|_{\mathcal{B}_p} = |\Phi_2^{-1} w_2^2 - \Phi_2^{-1} w_1^2|_{\mathcal{B}_p},$$

$$\leq |\frac{w_2^2 - w_1^2}{2}|_{\mathcal{B}_p} = \frac{1}{2}|w_2^2 - w_1^2|_{\mathcal{B}_p} + \frac{1}{2}|w_1 w_2 - w_2^2|_{\mathcal{B}_p},$$

$$\leq \frac{1}{2}|w_2|_{\mathcal{B}_p} * |w_1 - w_2|_{\mathcal{B}_p} + \frac{1}{2}|w_1|_{\mathcal{B}_p} * |w_1 - w_2|_{\mathcal{B}_p},$$

$$\leq \mu \max(|w_1 - w_2|_{\mathcal{B}_p}).$$

This implies that $F$ and $G$ satisfy property 2.10 hence we can apply theorem 2.11 to
equations (4.17) and (4.18) with Dirichlet boundary conditions to get existence of a periodic solution. Since \( \eta = -H\Theta^{-1}v \) and \( \Theta \) and \( H \) are multipliers in Fourier space with bounded symbols, if there is an \( \epsilon > 0 \) such that \( ||v(0)||_B < \epsilon \) and \( ||v(T)||_B < \epsilon \), then there is some \( \epsilon^* > 0 \) such that \( ||\eta(0)||_B < \epsilon^* \) and \( ||\eta(T)||_B < \epsilon^* \). We state the result for equations (4.1) and (4.2) in theorem 4.1.

**Theorem 4.1.** There exists \( \epsilon > 0 \) such that if \( ||\eta_0||_B, ||\eta_T||_B \leq \epsilon \), the system

\[
\begin{align*}
\eta_t + w_x + (w\eta)_x + aw_{xxx} - b\eta_{xxt} &= 0, \quad (4.19) \\
\eta_t + w_x + ww_x + c\eta_{xxx} - dw_{xxt} &= 0, \quad (4.20) \\
\eta(0) &= \eta_0, \quad \eta(T) = \eta_T, \quad (4.21)
\end{align*}
\]

with \( a, b, c, d \) satisfying (4.6), (4.7), or (4.8), has a periodic solution in \( B_\rho \times B_\rho \).

For boundary value problems with data specified at \((\eta(0), w(T)), (w(0), \eta(T))\) and \((w(0), w(T))\) the data \( \eta_0, \eta_T, w_0, w_T \) needs to be specified with the corresponding boundary.

For Neumann boundary value problems with boundary data \((\nabla \eta(x, 0) \cdot n_0, \nabla w(x, Y) \cdot n_0), (\nabla w(x, 0) \cdot n_0, \nabla w(x, Y) \cdot n_0), (\nabla \eta(x, 0) \cdot n_0, \nabla w(x, Y) \cdot n_0)\) and \((\nabla w(x, 0) \cdot n_0, \nabla \eta(x, Y) \cdot n_0)\) we specify corresponding data \( \tilde{\eta}_0, \tilde{\eta}_Y, \tilde{w}_0 \) or \( \tilde{w}_Y \) at the boundaries.

For mixed boundary value problems with boundary data \((\eta(x, 0), \nabla \eta(x, Y) \cdot n_0), (\eta(x, 0), \nabla w(x, Y) \cdot n_0), (\nabla w(x, 0) \cdot n_0, \eta(x, Y)), (\nabla \eta(x, 0) \cdot n_0, \eta(x, Y)), (\nabla \eta(x, 0) \cdot n_0, \eta(x, Y)), (w(x, 0), \nabla \eta(x, Y) \cdot n_0), (w(x, 0), \nabla \eta(x, Y) \cdot n_0), (\nabla \eta(x, 0) \cdot n_0, w(x, Y))\) and \((\nabla w(x, 0) \cdot n_0, w(x, Y))\) we specify the corresponding data \( \eta_0, \eta_Y, w_0, w_Y, \tilde{\eta}_0, \tilde{\eta}_Y, \tilde{w}_0 \) or \( \tilde{w}_Y \) at the boundaries.
5. QUASILINEAR POISSON’S EQUATION

The general theory presented in chapter 2 was applied to two problems arising in interfacial fluid dynamics. We want to conclude this work by presenting a series of quasi-linear elliptic partial differential equations that can be shown to be well-posed using theory from chapter 2. The reason why we think this is useful, is in general existence theory for elliptic partial differential equations is treated using a maximal principal type argument, which is extensively studied in Gilbarg and Trudinger [10] and contraction type arguments are used for hyperbolic equations.

The partial differential equation that we consider has the form

\[ v_{yy} + v_{xx} = K(v), \]  

(5.1)
on domain \((x,y) \in [0,2\pi] \times [0,Y] \), for some \(Y > 0\), and we take periodic boundary conditions in \(x\). The nonlinear operator \(K\) includes second derivatives of \(v\); specifically, we decompose \(K\) as

\[ K(v) = (F(v))_{xy} + \Lambda(G(v))_x, \]  

(5.2)

where the operator \(\Lambda = H\partial_x\), with \(H\) being the Hilbert transform. In order to connect theory developed in chapter 2 we will need to rewrite equations (5.3) and (5.4) into (5.1).

\[ v_y - \Lambda w = (F(v,w))_x, \]  

(5.3)

\[ w_y - \Lambda v = (G(v,w))_x. \]  

(5.4)

To get (5.1) from (5.3), (5.4), we restrict to the case that \(F\) and \(G\) depend only on \(v\)
and not on \( w \), we differentiate (5.3) with respect to \( y \), and then

\[
v_{yy} - \Lambda w_y = F_{xy}
\]

replace \( w_y \) using (5.4)

\[
v_{yy} - \Lambda^2 v = (F(v))_{xy} + \Lambda(G(v))_x.
\] (5.5)

Then, we need only observe that \( \Lambda^2 h = -h_{xx} \), which can be seen clearly from the symbol. Next, we construct examples.

Let \( F(v) = \sum_{k=2}^{m} \alpha_k v^k \) where the \( \alpha_k \) are any constants. We show now that \( F \) satisfies Property 2.10:

\[
|F(v_1) - F(v_2)|_{B_\rho} = \left| \sum_{k=2}^{m} \alpha_k (v_1^k - v_2^k) \right|_{B_\rho},
\] (5.6)

\[
= \left| \sum_{k=2}^{m} \alpha_k \sum_{j=1}^{k} v_1^{j-1} v_2^{k-j} (v_1 - v_2) \right|_{B_\rho} \] (5.7)

\[
\leq |v_1 - v_2|_{B_\rho} \sum_{k=2}^{m} \alpha_k \sum_{j=1}^{k} |v_1|_{B_\rho}^{(j-1)} |v_2|_{B_\rho}^{(k-j)}
\] (5.8)

\[
\leq |v_1 - v_2|_{B_\rho} \sum_{k=2}^{m} \alpha_k \sum_{j=1}^{k} \mu^{(k-1)}
\] (5.9)

\[
\leq |v_1 - v_2|_{B_\rho} \sum_{k=2}^{m} k\alpha_k \mu^{(k-1)} = A(\mu) \ast |v_1 - v_2|_{B_\rho}. \] (5.10)

Notice \( A(\mu) \) is continuous and \( A(0) = 0 \) hence \( F \) satisfies Property 2.10. We now define \( G \) to be \( G = -H \sum_{k=2}^{m} \beta_k v^k \) where \( H \) is again the Hilbert transform, and the
\( \beta_k \) are any constants. Since the symbol of Hilbert transform is \( i \text{sgn}(n) \), we have

\[
|G(v_1) - G(v_2)|_{B_\rho} = \left| H \sum_{k=2}^{m} \beta_k(v_1^k - v_2^k) \right|_{B_\rho} \\
= \sup_y \left| i \text{sgn}(n) e^{\rho|n|} \mathcal{F} \left( \sum_{k=2}^{m} \beta_k(v_1^k - v_2^k) \right) \right| \\
\leq \sup_y \left| e^{\rho|n|} \mathcal{F} \left( \sum_{k=2}^{m} \beta_k(v_1^k - v_2^k) \right) \right| = \left| \sum_{k=2}^{m} \beta_k(v_1^k - v_2^k) \right|_{B_\rho} \\
\leq |v_1 - v_2|_{B_\rho} \ast \sum_{k=2}^{m} k \beta_k \mu^{\ast(k-1)} = A(\mu) \ast |v_1 - v_2|_{B_\rho}.
\]

If we do choose \( F(v) = \sum_{k=2}^{m} \alpha_k v^k \), then

\[
(F(v))_x = \sum_{k=2}^{m} k \alpha_k v^{(k-1)} v_x, \\
(F(v))_{xy} = \sum_{k=2}^{m} k(k-1) \alpha_k v^{(k-2)} v_x v_y + k \alpha v^{(k-1)} v_{xy}.
\]

Likewise, if we do choose \( G(v) = -H \sum_{k=2}^{m} \beta_k v^k \), then

\[
\Lambda G_x = -H^2 \partial_x^2 \sum_{k=2}^{m} \beta_k v^k = \partial_x^2 \sum_{k=1}^{m} \beta_k v^k \\
= \sum_{k=2}^{m} k(k-1) \beta_k v^{(k-2)} v_x^2 + k \beta_k v^{(k-1)} v_{xx}.
\]

Hence the following equation, along with proper boundary conditions, will have a solution in some ball of \( B_\rho \):

\[
v_{yy} + v_{xx} = \left( \sum_{k=2}^{m} k(k-1) \alpha_k v^{(k-2)} v_x v_y + k\alpha v^{(k-1)} v_{xy} \right) \\
+ \left( \sum_{k=2}^{m} k(k-1) \beta_k v^{(k-2)} v_x^2 + k \beta_k v^{(k-1)} v_{xx} \right). \quad (5.11)
\]
As our second example, we take $F = 0$ and $G = \partial_x^{-2}(q(v))$, where $q(v)$ is again any second-degree (or higher) polynomial, $q(v) = \sum_{k=2}^{m} \beta_k v^k$. The operator $\partial_x^{-1}$ is the zero-mean integration operator, which satisfies

$$\sigma_{\partial_x^{-1}} = \frac{1}{\sigma_{\partial_x}} = \frac{1}{in}.$$

We have

$$\Lambda(G(v))_x = H\partial_x^{-2}\partial_x^2(q(v)) = H(q(v)).$$

For the Lipschitz estimate, we use the fact that we only need consider $|n| \geq 1$:

$$|G(v_1) - G(v_2)| \leq \left| \partial_x^{-2} \sum_{k=2}^{m} \beta_k (v_1^k - v_2^k) \right|_{B_\rho} \leq \left| \sum_{k=2}^{m} \beta_k (v_1^k - v_2^k) \right|_{B_\rho} \leq A(\mu) * |v_1 - v_2|_{B_\rho}.$$

This estimate implies that there exists a locally unique solution for the boundary value problems associated to the equation

$$v_{xx} + v_{yy} = H \sum_{k=2}^{m} \beta_k v^k.$$

For our third example, we take $F(v) = \cosh(v)$ and $G = 0$. We show that $F$ satisfies Property 2.10:

$$|F(v_1) - F(v_2)|_{B_\rho} = \left| \sum_{k=0}^{\infty} \frac{1}{2k!} (v_1^k - v_2^k) \right|_{B_\rho} = \left| \sum_{k=1}^{2k} v_1^{j-1} v_2^{2k-j} (v_1 - v_2) \right|_{B_\rho} \leq |v_1 - v_2|_{B_\rho} * \sum_{k=1}^{\infty} \frac{\mu^{(2k-1)}}{(2k-1)!} = |v_1 - v_2|_{B_\rho} * \sum_{k=0}^{\infty} \frac{\mu^{(2k+1)}}{(2k + 1)!}.$$
We let $A_F(\mu) = \sum_{k=0}^{\infty} \frac{\mu^{*(2k+1)}}{(2k+1)!}$ and show that $A_F$ is continuous. Let $\mu = \hat{u}$ and $\nu = \hat{v}$ then

$$|A_F(\mu) - A_F(\nu)| = \left| \sum_{k=0}^{\infty} \frac{\mu^{*(2k+1)} - \nu^{*(2k+1)}}{(2k+1)!} \right| = \left| \sum_{k=0}^{\infty} \frac{F\left(u^{(2k+1)} - v^{(2k+1)}\right)}{(2k+1)!} \right|$$

$$= \left| \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} F\left((u - v) \sum_{j=1}^{2k+1} u^{j-1} v^{2k+1-j} \right) \right|. $$

If $\mu, \nu \in \ell^1$, then $u$ and $v$ are bounded ($u < C_1$, $v < C_2$ where $C_1$ and $C_2$ are some finite constants). Hence $\max\{u, v\} = C$ where $C = \max\{C_1, C_2\}$

$$|A_F(\mu) - A_F(\nu)| \leq \left| \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} F\left((u - v) \sum_{j=1}^{2k+1} C^{2k} \right) \right|$$

$$= \left| \sum_{k=0}^{\infty} \frac{C^{2k}}{2k!} (\mu - \nu) \right| = \cosh(C)|\mu - \nu|.$$ 

Hence $A_F$ is continuous and notice that $A_F(0) = 0$. This estimate implies that there exists a locally unique solution for the boundary value problems associated to the equation

$$v_{xx} + v_{yy} = \sinh(v)v_{xy} + \cosh(v)v_xv_y.$$ 

As a fourth and final example, we let $F = 0$ and $G = H(\sin(v) - v)$. We must verify that $G$ satisfies Property 2.10, but we leave out the details since this is so similar to the previous example. This implies that we have a locally unique solution of the boundary value problems for the equation

$$(\sin(v))_{xx} + v_{yy} = 0.$$

Of course, there are many possible choices of $F$ and $G$; the important feature that
these nonlinearities must have is that they must satisfy Property 2.10. As we have seen, this allows for a variety of pseudodifferential operators (with respect to \( x \)) to be included.
List of References


In this appendix we reproduce the derivation of the Bona-Chen-Saut Boussinesq approximation from [1]. Let $\Omega_t \subset \mathbb{R}^3$ be occupied by inviscid and incompressible fluid; then the fluid motion in $\Omega_t$ is governed by the Euler equations

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \nabla p = -g \vec{k},$$

$$\nabla \cdot \vec{v} = 0,$$  \hspace{1cm} (12)  \hspace{1cm} (13)

where $g$ is the gravity and $\vec{k}$ is the $z$ direction. We also assume initial velocity field is irrotational which means we can represent velocity $\vec{v}$ in terms of a potential $\phi$ by $\vec{v} = \nabla \phi$, which along with the incompressibility condition (13) implies that

$$\nabla^2 \phi = 0.$$  \hspace{1cm} (14)

The boundary of $\Omega_t$ has a fixed surface $z = -h(x,y)$ and a free surface $z = \eta(x,y,t)$. The fixed surface must satisfy the impermeability condition $\vec{v} \cdot \vec{n} = 0$ where $\vec{n}$ is the unit normal to the surface. This condition results in

$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \text{ on } z = -h.$$  \hspace{1cm} (15)

The free surface is a material surface which must satisfy $\frac{D(\eta - z)}{Dt} = 0$, which results in

$$\frac{D\eta}{Dt} = \eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0 \text{ on } z = \eta.$$  \hspace{1cm} (16)
Assuming pressure on free surface is equal to ambient air pressure we get the Bernoulli condition:

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + gz = 0 \quad \text{on} \quad z = \eta.
\]  (17)

We are interested in an open channel where fluid motion is irrotational, inviscid, and uniform in the cross channel direction which implies \( h_x = 0, \ h_y = 0, \) and \( \phi_y = 0. \) These assumptions reduce equations (14), (15), (16), and (17) to:

\[
\phi_{xx} + \phi_{zz} = 0 \quad \text{in} \quad -h < z < \eta(x,t), \quad (18)
\]

\[
\phi_z = 0 \quad \text{on} \quad z = -h, \quad (19)
\]

\[
\eta_t + \phi_x \eta_x - \phi_z = 0 \quad \text{on} \quad z = \eta(x,t), \quad (20)
\]

\[
\phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) + gz = 0 \quad \text{on} \quad z = \eta(x,t). \quad (21)
\]

We non-dimensionalize equations (18)-(21) using

\[
x = l \tilde{x}, \quad z = h(\tilde{z} - 1), \quad \eta = A \tilde{\eta}, \quad t = \frac{l}{c_0} \tilde{t}, \quad \phi = \frac{gA l}{c_0} \tilde{\phi}
\]

where \( c_0 = \sqrt{gh}. \) Since we are interested in small amplitude and large wave length \( l >> h, \) and we take the depth of fixed surface to be much larger then the amplitude of the free surface, so \( A << h. \) The non-dimensional version of equations (18)-(21)
are:

\[ \beta \phi_{xx} + \phi_{zz} = 0 \quad \text{in} \quad 0 < z < 1 + \alpha \eta(x,t), \quad (22) \]

\[ \phi_z = 0 \quad \text{on} \quad z = 0, \quad (23) \]

\[ \eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta} \phi_z = 0 \quad \text{on} \quad z = 1 + \alpha \eta(x,t), \quad (24) \]

\[ \eta + \phi_t + \frac{1}{2} \alpha \phi_x^2 + \frac{\alpha}{2\beta} \phi_z^2 = 0 \quad \text{on} \quad z = 1 + \alpha \eta(x,t), \quad (25) \]

where \( \alpha = \frac{A}{h} \) and \( \beta = \frac{2}{T_f} \). \( S = \frac{\alpha}{\beta} \) is known as the Stokes number and we consider it to be of order 1. Next, we consider an expansion of \( \phi \):

\[ \phi(x, z, t) = \sum_{m=0}^{\infty} f_m(x, t) z^m, \]

plugging this expansion into (22)

\[ -\beta \sum_{m=0}^{\infty} (f_m(x, t))_{xx} z^m = \sum_{m=2}^{\infty} m(m - 1) f_m(x, t) z^{m-2} \]

\[ = \sum_{m=0}^{\infty} (m + 2)(m + 1) f_{m+2} z^m. \]

This results in:

\[ (m + 2)(m + 1) f_{m+2} = -\beta (f_m(x, t))_{xx}. \quad (26) \]
Let $F = f_0$ denote the velocity potential at $z = 0$, using (26)

\[ m = 0 : \quad 2 \ast 1 f_2 = -\beta (f_0)_{xx} \]
\[ f_2 = \frac{-\beta}{2 \ast 1} (f_0)_{xx} \]

\[ m = 2 : \quad 4 \ast 3 f_4 = -\beta (f_2)_{xx} = \frac{\beta^2}{2 \ast 1} (f_0)_{xxxx} \]
\[ f_4 = \frac{\beta^2}{4!} (f_0)_{xxxx} \]

\[ \vdots \]
\[ f_{2k}(x, t) = \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k} F(x, t)}{\partial x^{2k}}. \]

Using (23) on $z = 0$:

\[ \sum_{m=1}^{\infty} m f_m z^{m-1} = \sum_{m=0}^{\infty} (m + 1) f_{m+1} z^m = 0, \]

implies $f_1 = 0$, and applying (26) repeatedly produces $f_{2k+1} = 0$. We rewrite the expansion of $\phi$:

\[ \phi(x, z, t) = \sum_{k=0}^{\infty} \frac{(-1)^k \beta^k}{(2k)!} \frac{\partial^{2k} F(x, t)}{\partial x^{2k}} z^{2k}. \]  

(27)

Since (24) and (25) are defined on $z = 1 + \alpha \eta(x, t)$, we plug (27) into (24) and (25) and replace $z$ with $1 + \alpha \eta(x, t)$ to produce (28) and (29):
\[
\eta_t + \alpha \eta x \sum_{k=0}^{\infty} \left( \frac{(-1)^k \partial^{2k+1} F}{(2k)! \partial x^{2k+1}} (1 + \alpha \eta)^{2k} \right) \beta^k \\
+ \sum_{k=0}^{\infty} \left( \frac{(-1)^k \partial^{2k+2} F}{(2k+1)! \partial x^{2k+2}} (1 + \alpha \eta)^{2k+1} \right) \beta^k = 0 \\
(28)
\]

\[
\eta + \sum_{k=0}^{\infty} \left( \frac{(-1)^k \partial^{2k+1} F}{(2k)! \partial x^{2k+1}} (1 + \alpha \eta)^{2k} \right) \beta^k \\
+ \frac{1}{2} \alpha \left\{ \sum_{k=0}^{\infty} \left( \frac{(-1)^k \partial^{2k+1} F}{(2k)! \partial x^{2k+1}} (1 + \alpha \eta)^{2k} \right) \beta^k \right\}^2 \\
+ \frac{1}{2} \alpha \beta \left\{ \sum_{k=0}^{\infty} \left( \frac{(-1)^k \partial^{2k+2} F}{(2k+1)! \partial x^{2k+2}} (1 + \alpha \eta)^{2k+1} \right) \beta^k \right\}^2 = 0 \\
(29)
\]

Equations (28) and (29) are general expansions for the Euler equations with the specified physical approximations. Keeping the terms in the expansions which are up to degree \(n\) in \(\alpha\) and \(\beta\) and then the taking derivative of equation (29) with respect to \(x\) we will arrive at \(n\)th order Boussinesq approximation. As an example the zero order expansion would be:

\[
\eta_t + \partial^2 F = 0, \\
\eta_x + \partial^2 F = 0.
\]

Since \(F(x,t)\) is the velocity potential, \(\partial F/\partial x\) is the velocity component in the \(x\) direction. Hence we get

\[
\eta_t + u_x = 0, \\
\eta_x + u_t = 0,
\]
which is the linear wave equation. Similarly the first order approximation is

\[ \eta_t + u_x + \alpha \eta u_x + \alpha \eta_x u - \frac{1}{6} \beta u_{xx} = 0, \]

\[ \eta_x + u_t + \alpha u u_x - \frac{1}{2} u_{xxt} = 0. \]
APPENDIX B: Birkhoff-Rott Integral

We are interested in deriving average velocity components $V_1$ and $V_2$ of the fluid flow near the vortex sheet interface. We begin with the assumption that the flow away from the interface is irrotational and incompressible. This implies the existence of a stream function $\psi$ where $\nabla^\perp(\psi) = v$. Taking the curl of this equation:

$$\Delta \psi = \text{curl}(v) = \Omega.$$

The solution to this Poisson equation is:

$$\psi(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x - x'| \Omega(t, x') dx'.$$

Applying $\nabla^\perp$ to this equation:

$$v(t, x) = \int_{\mathbb{R}^2} K(x - x') \Omega(t, x') dx',$$

with

$$K(x) = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right)^t.$$
Since the velocity discontinuity is in the tangential component only, using Plemelj formulae in taking the limit approaching the vortex sheet interface results in:

\[
V_1^\pm = -\frac{1}{2\pi} PV \int \frac{x_2(t, x_1) - x_2(t, x_1')}{(x_1 - x_1')^2 + (x_2(t, x_1) - x_2(t, x_1'))^2} \Omega(t, x_1') dx_1' \pm \Omega(t, x_1) \vec{t}_1,
\]

\[
V_2^\pm = \frac{1}{2\pi} PV \int \frac{x_1 - x_1'}{(x_1 - x_1')^2 + (x_2(t, x_1) - x_2(t, x_1'))^2} \Omega(t, x_1') dx_1' \pm \Omega(t, x_1) \vec{t}_2.
\]

where \( V_i^\pm \) corresponds to fluid velocity components above and below the interface, and \( \vec{t}_1, \vec{t}_2 \) are the components of the unit tangent vector to the interface. Taking the average:

\[
V_1 = -\frac{1}{2\pi} PV \int \frac{x_2(t, x_1) - x_2(t, x_1')}{(x_1 - x_1')^2 + (x_2(t, x_1) - x_2(t, x_1'))^2} \Omega(t, x_1') dx_1',
\]

\[
V_2 = \frac{1}{2\pi} PV \int \frac{x_1 - x_1'}{(x_1 - x_1')^2 + (x_2(t, x_1) - x_2(t, x_1'))^2} \Omega(t, x_1') dx_1',
\]

which are the velocity components in equations described in chapter 3.
APPENDIX C: Vortex Sheet $G$ estimate

In this Appendix we want to derive $A_G$ which is presented in chapter 3, which implies that $G$ satisfies the contracting property. We begin with defining $G$:

$$
G = \frac{1}{\pi} PV \int \left( \frac{p}{1 + p^2} - p \right) \frac{dx'}{x - x'} + \frac{1}{\pi} w PV \int \frac{p}{1 + p^2} \frac{1 + w(x')}{x - x'} dx' + \frac{1}{\pi} PV \int \frac{p}{1 + p^2} \frac{w(x')}{x - x'} dx',
$$

(30)

where $p = \frac{y(x) - y(x')}{x - x'}$. Next we want to rewrite $G$ as a series. Let

$$
R_k(y_1x, \ldots, y_kx) \Omega(x) = \frac{1}{\pi} \int \frac{1}{k} (p_1 \cdots p_k)x \Omega(x') dx',
$$

(31)

where $p_i = \frac{y_i(x) - y_i(x')}{x - x'}$. In [7] $R_K$ is shown to satisfy

$$
|R_k(y_1x, \ldots, y_kx)|_{B_\rho} \leq 2e^{-2\pi \xi} ||y_1x||_{B_\rho} \ast \cdots \ast ||y_kx||_{B_\rho} \ast ||\Omega||_{B_\rho}
$$

(32)

Define

$$
T_j(y_x) \Omega(x) = \frac{1}{\pi} PV \int \left( \frac{y(x) - y(x')}{x - x'} \right) \frac{\Omega(x')}{x - x'} dx'.
$$

(33)

Notice that $T_0$ is a Hilbert transform:

$$
T_0(y_x) \Omega(x) = H \Omega(x) = \frac{1}{\pi} \int \frac{\Omega(x')}{x - x'} dx'.
$$

(34)
Each $T_j$ can be expanded in terms of $R_k$'s:

$$T_j(y_x)\Omega(x) = y^j_x H\Omega - \sum_{k=1}^{j} y^{j-k}_x R_k(y_x)\Omega \quad (35)$$

Now that we can rewrite $G$ in terms of $R_k$ we work out the derivation by breaking it down into ten estimates. Estimate 10 is the estimate which shows that $G$ satisfies the contraction property.

Note: For simplicity of notation $|\cdot| = |\cdot|_{S_{\rho}}$

**Estimate 1**

$$|w_1 H y_{1x} - w_2 H y_{2x}| \leq |w_1 H y_{1x} - w_2 H y_{1x}| + |w_2 H y_{1x} - w_2 H y_{2x}|$$

$$\leq |H y_{1x}| * |w_1 - w_2| + |w_2| * |H(y_{1x} - y_{2x})|$$

$$\leq \mu * \nu + \mu * \nu = 2\mu * \nu.$$

**Estimate 2**

$$|R_1(y_{2x})w_2 - R_1(y_{1x})w_2| = \left| \frac{1}{\pi} \int (p_2)_{x} w_2(x')dx' - \frac{1}{\pi} \int (p_1)_{x} w_2(x')dx' \right|$$

$$= \left| \frac{1}{\pi} \int (p_2 - p_1)_{x} w_2(x')dx' \right|.$$

Since

$$p_2 - p_1 = \left( \frac{y_2(x) - y_2(x')}{x - x'} - \frac{y_1(x) - y_1(x')}{x - x'} \right) = \frac{(y_2(x) - y_1(x)) - (y_2(x') - y_1(x'))}{x - x'},$$

$$|R_1(y_{2x})w_2 - R_1(y_{1x})w_2| = |R_1(y_{2x} - y_{1x})w_2|$$

$$\leq 2 * |y_{2x} - y_{1x}| * |w_2| \leq 2 * \mu * \nu.$$. 
**Estimate 3**

Using Estimate 2,

\[
|T_1(y_{1x})w_2 - T_1(y_{2x})w_2| \leq |y_{1x}Hw_2 - y_{2x}Hw_2| + |R_1(y_{2x})w_2 - R_1(y_{1x})w_2| \\
\leq \mu \ast \nu + 2 \ast \mu \ast \nu = 3 \mu \ast \nu.
\]

**Estimate 4**

Using Estimates 2 and 3,

\[
|T_1(y_{1x})w_1 - T_1(y_{2x})w_2| \leq |T_1(y_{1x})w_1 - T_1(y_{1x})w_2| + |T_1(y_{1x})w_2 - T_1(y_{2x})w_2| \\
\leq |T_1(y_{1x})(w_1 - w_2)| + |T_1(y_{1x})w_2 - T_1(y_{2x})w_2| \\
\leq 3 \ast |y_{1x}| \ast |w_1 - w_2| + 3 \ast \mu \ast \nu = 6 \ast \mu \ast \nu.
\]

**Estimate 5**

Using Estimate 4 and \(|\Omega| \leq \mu + \delta|,

\[
|\Omega_1 T_1(y_{1x})w_1 - \Omega_2 T_1(y_{2x})w_2| \leq |\Omega_1 T_1(y_{1x})w_1 - \Omega_1 T_1(y_{2x})w_2| \\
\quad + |\Omega_1 T_1(y_{2x})w_2 - \Omega_2 T_1(y_{2x})w_2| \\
\leq |\Omega_1| \ast |T_1(y_{1x})w_1 - T_1(y_{2x})w_2| \\
\quad + |T_1(y_{2x})w_2| \ast |\Omega_1 - \Omega_2| \\
\leq (\mu + \delta) \ast (6 \mu \ast \nu) + 3 |y_{2x}| \ast |w_2| \ast \nu \\
\leq 6 \mu^{\prime 2} \ast \nu + 6 \mu \ast \nu + 3 \mu^{\prime 2} \ast \nu = 9 \mu^{\prime 2} \ast \nu + 6 \mu \ast \nu.
\]
Estimate 6

\[ |R_k(y_{2x})\Omega_1 - R_k(y_{1x})\Omega_1| = \frac{1}{\pi} \int \frac{1}{k} (p^k_{2})_x \Omega_1(x') dx' - \frac{1}{\pi} \int \frac{1}{k} (p^k_{1})_x \Omega_1(x') dx' \]

\[ = \frac{1}{\pi} \int \frac{1}{k} (p^k_2 - p^k_1)_x \Omega_1(x') dx' \]

\[ = \frac{1}{\pi} \int \frac{1}{k} \left( (p_2 - p_1) \sum_{i=1}^{k} p^{k-i}_2 p^{i-1}_1 \right) \Omega_1(x') dx' \]

\[ \leq \sum_{i=1}^{k} R_k(y_{2x} - y_{1x}, y_{2x}, ..., y_{2x}, y_{1x}, ..., y_{1x})\Omega_1 \]

\[ \leq \sum_{i=1}^{k} 2 |y_{2x} - y_{1x}| * |y_{2x}|^{k-i} * |y_{1x}|^{i-1} * |\Omega_1| \]

\[ \leq \sum_{i=1}^{k} 2 \mu^{*k-1} * \nu * (\mu + \delta) = 2k \mu^{*k-1} * \nu * (\mu + \delta) \]

\[ \leq 2k \mu^{*k} * \nu + 2k \mu^{*k-1} * \nu. \]

Estimate 7

Using Estimate 6,

\[ |R_k(y_{2x})\Omega_2 - R_k(y_{1x})\Omega_1| \leq |R_k(y_{2x})(\Omega_2 - \Omega_1)| + |R_k(y_{2x})\Omega_1 - R_k(y_{1x})\Omega_1| \]

\[ \leq 2|y_{2x}|^{*k} * |\Omega_2 - \Omega_1| + 2k \mu^{*k} * \nu + 2k \mu^{*k-1} * \nu \]

\[ \leq 2 \mu^{*k} * \nu + 2k \mu^{*k} * \nu + 2k \mu^{*k-1} * \nu \]

\[ = 2(k + 1) \mu^{*k} * \nu + 2k \mu^{*k-1} * \nu. \]
Estimate 8

\[
|T_j(y_{1x})\Omega_1 - T_j(y_{2x})\Omega_2| \leq |y_{1x}^j H\Omega_1 - y_{2x}^j H\Omega_2|
+ \sum_{k=1}^j y_{2x}^{j-k} R_k(y_{2x})\Omega_2 - \sum_{k=1}^j y_{1x}^{j-k} R_k(y_{1x})\Omega_1|
\leq |y_{1x}^j H\Omega_1 - y_{2x}^j H\Omega_1| + |y_{2x}^j H\Omega_1 - y_{2x}^j H\Omega_2|
+ \sum_{k=1}^j y_{1x}^{j-k} R_k(y_{1x})\Omega_2 - \sum_{k=1}^j y_{1x}^{j-k} R_k(y_{2x})\Omega_1|
\]

\[
\leq |\Omega_1| * |y_{1x}^j - y_{2x}^j| + |y_{2x}^j| * |\Omega_1 - \Omega_2|
+ \sum_{k=1}^j |y_{2x}^{j-k} - y_{1x}^{j-k}| * |R_k(y_{2x})\Omega_2|
+ \sum_{k=1}^j |y_{1x}^{j-k}| * |R_k(y_{2x})\Omega_2 - R_k(y_{1x})\Omega_1|.
\]

Using Estimate 7,

\[
|T_j(y_{1x})\Omega_1 - T_j(y_{2x})\Omega_2| \leq (\mu + \delta) * |(y_{1x} - y_{2x})\sum_{i=1}^j y_{1x}^{j-i} y_{2x}^{i-1}| + \mu * \nu
+ \sum_{k=1}^j |(y_{2x} - y_{1x})\sum_{i=1}^{j-k} y_{2x}^{j-k-i} y_{1x}^{i-1}| * |R_k(y_{2x})\Omega_2|
+ \sum_{k=1}^j \mu^{*(j-k)} * (2(k-1)\mu^* \nu + 2k\mu^{*(k-1)} \nu)
\leq (\mu + \delta) * (j\mu^{*(j-1)} \nu) + \mu * \nu
+ \sum_{k=1}^j 2(j-k)\mu^{*(j-k-1)} \nu * (2\mu^{*j-1} * |\Omega_2|)
+ \sum_{k=1}^j 2(k-1)\mu^{*j} \nu + 2k\mu^{*j-1} \nu.
\]
Continue:

\[ |T_j(y_{1x})\Omega_1 - T_j(y_{2x})\Omega_2| \leq j\mu^{*j} \nu + j\mu^{*j-1} + \mu^{*j} \nu
\]
\[ + \sum_{k=1}^{j-1} 2k\mu^{*j-1} \nu \mu \delta \]
\[ + \sum_{k=1}^{j-1} 2k\mu^{*j} \nu + \sum_{k=1}^j 2k\mu^{*j-1} \nu \]
\[ \leq (j + 1)\mu^{*j} \nu + j\mu^{*j-1} \nu
\]
\[ + j(j - 1)\mu^{*j} \nu + j(j - 1)\mu^{*j-1} \nu
\]
\[ + j(j - 1)\mu^{*j} \nu + j(j + 1)\mu^{*j-1} \nu
\]
\[ = (2j^2 - j + 1)\mu^{*j} \nu + (2j^2 + j)\mu^{*j-1} \nu. \]

Estimate 9

\[ |\Omega_1 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x})\Omega_1 - \Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{2x})\Omega_2| \]
\[ \leq |\Omega_1 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x})\Omega_1 - \Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x})\Omega_1|
\]
\[ + |\Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x})\Omega_1 - \Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{2x})\Omega_2|
\]
\[ \leq \left( |\Omega_1 - \Omega_2| \sum_{j=3}^{\infty} |T_j(y_{1x})\Omega_1| \right) + \left( |\Omega_2| \sum_{j=3}^{\infty} |T_j(y_{1x})\Omega_1 - T_j(y_{2x})\Omega_2| \right). \]
Using Estimate 8,

\[
\left| \Omega_1 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x}) \Omega_1 - \Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{2x}) \Omega_2 \right| 
\leq \nu \sum_{j=3}^{\infty} (1 + 2j) |y_{1x}|^x_j * |\Omega_1| 
+ (\mu + \delta) \sum_{j=3}^{\infty} (2j^2 - j + 1) \mu^{x_j} \nu + (2j^2 + j) \mu^{x_j-1} \nu 
\leq \nu \sum_{j=3}^{\infty} (1 + 2j) \mu^{x_j} (\mu + \delta) + \nu \sum_{j=3}^{\infty} (2j^2 - j + 1) \mu^{x_j+1} + \nu \sum_{j=3}^{\infty} (2j^2 + j) \mu^{x_j} 
+ \nu \sum_{j=3}^{\infty} (2j^2 - j + 1) \mu^{x_j} + \nu \sum_{j=3}^{\infty} (2j^2 + j) \mu^{x_j-1} 
= \nu \sum_{j=3}^{\infty} (2j^2 + j + 2) \mu^{x_j+1} + \nu \sum_{j=3}^{\infty} (4j^2 + 2j + 2) \mu^{x_j} + \nu \sum_{j=3}^{\infty} (2j^2 + j) \mu^{x_j-1}.
\]

Estimate 10

\[
G(y_{1x}, w - 1) - G(y_{2x}, w_2) = \left( w_1 H y_{1x} + \Omega_1 T_1(y_{1x}) w_1 + \Omega_1 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x}) \Omega_1 \right) 
- \left( w_2 H y_{2x} + \Omega_2 T_2(y_{2x}) w_2 + \Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{2x}) \Omega_2 \right).
\]

Using the triangle inequality,

\[
|G(y_{1x}, w_1) - G(y_{2x}, w_2)| \leq |w_1 H y_{1x} - w_2 H y_{2x}| + |\Omega_1 T_1(y_{1x}) w_1 - \Omega_2 T_2(y_{2x}) w_2| 
+ |\Omega_1 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{1x}) \Omega_1 - \Omega_2 \sum_{j=3}^{\infty} \epsilon_j T_j(y_{2x}) \Omega_2|.
\]
Using Estimates 1, 5 and 9,

\[ |G(y_{1x}, w_1) - G(y_{2x}, w_2)| \leq 8\mu \nu + 9\mu^2\nu + \nu \sum_{j=3}^{\infty} (2j^2 + j + 2)\mu^{*j+1} \]

\[ + \nu \sum_{j=3}^{\infty} (4j^2 + 2j + 2)\mu^{*j} + \nu \sum_{j=3}^{\infty} (2j^2 + j)\mu^{*j-1} \]

\[ = A(\mu) \nu, \]

where

\[ A(\mu) = 8\mu + 9\mu^2 + \sum_{j=3}^{\infty} (2j^2 + j + 2)\mu^{*j+1} + \sum_{j=3}^{\infty} (4j^2 + 2j + 2)\mu^{*j} + \sum_{j=3}^{\infty} (2j^2 + j)\mu^{*j-1}. \]
Vita


He then attended Rensselaer Polytechnic Institute in Troy, New York. During his time in Rensselaer he began his studies in electrical engineering, but a passion for mathematics led him to instead pursue a degree in mathematics in the last year of his undergraduate studies. He graduated in May, 2005 with B.S. in Mathematics and minors in Electrical Engineering and Economics.

In August 2005, he moved to Clemson, South Carolina to continue his education by pursuing graduate studies in mathematics. During his first year at Clemson University, he met his advisor David Ambrose at a seminar for first year graduate students. Under Professor Ambrose’s supervision, he completed an M.S. in May 2007 and decided to continue to work on a Ph.D. dissertation.

In August 2008, he chose to move with his advisor to Drexel University to continue to work on his thesis. Following his graduation in June 2011, he plans on using his knowledge of mathematics for . . .