

## THE PARTIAL SUMS OF THE HARMONIC SERIES

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

is called harmonic, it diverges to infinity. Its partial sums

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}, \quad n = 1, 2, 3, \dots,$$

(*harmonic numbers*) form a monotone sequence increasing without bound.

The integral estimates

$$1 + \frac{1}{2} + \dots + \frac{1}{n} > \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

and

$$\frac{1}{2} + \dots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n$$

are justified geometrically. Combined together, they give

$$\ln(n+1) < H_n < 1 + \ln n, \quad n > 1.$$

Therefore  $H_n$  tend to infinity at the same rate as  $\ln n$ , which is fairly slow. For instance, the sum of the first million terms is

$$H_{1000000} < 6 \ln 10 + 1 \approx 14.8.$$

Consider now the differences  $\delta_n = H_n - \ln n$ . Since

$$\ln\left(1 + \frac{1}{n}\right) < H_n - \ln n < 1, \quad n > 1,$$

we conclude that every  $\delta_n$  is a positive number not exceeding 1.

Observe that

$$\begin{aligned} \delta_n - \delta_{n+1} &= (H_n - \ln n) - (H_{n+1} - \ln(n+1)) \\ &= \ln(n+1) - \ln n - \frac{1}{n+1} \\ &= \int_n^{n+1} \frac{dx}{x} - \frac{1}{n+1} > 0 \end{aligned}$$

(draw a picture to verify the last inequality). So  $\delta_n > 0$  are monotone decreasing.

By the Monotone Sequence Theorem,  $\delta_n$  must converge as  $n \rightarrow \infty$ . The limit

$$\gamma = \lim_{n \rightarrow \infty} \delta_n = \lim_{n \rightarrow \infty} (H_n - \ln n)$$

is called the *Euler constant* (Euler, 1735), its value is about  $\gamma \approx .5772$ .

Thus, for large  $n$ , we have a convenient approximate equality

$$H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} \approx \ln n + \gamma.$$

It is not known to this day whether  $\gamma$  is rational or irrational.