Numerical Analysis
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THE ORDER OF CONVERGENCE FOR THE SECANT METHOD.

Suppose that we are solving the equation $f(x) = 0$ using the secant method. Let the iterations

(1) $$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \ldots,$$

be successful and approach a solution α, $f(\alpha) = 0$, as $n \to \infty$. How fast do they converge? Can we find the exponent p such that

$$|x_{n+1} - \alpha| \approx C|x_n - \alpha|^p,$$

as we did for bisections and Newton's method?
Yes we can, but the error analysis is a bit more involved.

Equation (1) expresses the iterate x_{n+1} as a function of x_n and x_{n-1}.

Let $x_n = \alpha + \epsilon_n$. Since $x_n \to \alpha$, the sequence of errors ϵ_n approaches 0 as $n \to \infty$. In terms of α and ϵ_n the formula becomes

(2) $$\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha + \epsilon_n)(\epsilon_n - \epsilon_{n-1})}{f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1})}.$$

Assume that $f(x)$ is a two times differentiable function and that $f'(\alpha), f''(\alpha) \neq 0$.
Write Taylor's formula:

$$f(\alpha + \epsilon) = f(\alpha) + f'(\alpha)\epsilon + \frac{f''(\alpha)}{2}\,\epsilon^2 + R_2(\epsilon).$$

Here $f(\alpha) = 0$, ϵ is small, and $R_2(\epsilon)$ is the remainder term. Recall that $R_2(\epsilon)$ vanishes at $\epsilon = 0$ at a faster rate than ϵ^2. Neglecting the terms of order higher than 2 in ϵ, we have the approximation

$$f(\alpha + \epsilon) \approx f'(\alpha)\epsilon + \frac{f''(\alpha)}{2}\,\epsilon^2.$$

Set for brevity

$$M = \frac{f''(\alpha)}{2f'(\alpha)},$$

and use the approximate equalities

$$f(\alpha + \epsilon_n) \approx \epsilon_n f'(\alpha)\big(1 + M\epsilon_n\big)$$
$$f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1}) \approx f'(\alpha)(\epsilon_n - \epsilon_{n-1})\big(1 + M(\epsilon_n + \epsilon_{n-1})\big)$$

to simplify equation (2):

$$\begin{aligned}
\epsilon_{n+1} &\approx \epsilon_n - \frac{\epsilon_n f'(\alpha)(1 + M\epsilon_n)(\epsilon_n - \epsilon_{n-1})}{f'(\alpha)(\epsilon_n - \epsilon_{n-1})\big(1 + M(\epsilon_n + \epsilon_{n-1})\big)} \\
&= \epsilon_n - \frac{\epsilon_n(1 + M\epsilon_n)}{1 + M(\epsilon_n + \epsilon_{n-1})} \\
&= \frac{\epsilon_{n-1}\epsilon_n M}{1 + M(\epsilon_n + \epsilon_{n-1})} \\
&\approx \epsilon_{n-1}\epsilon_n M.
\end{aligned}$$

We have obtained a relation for the errors,

$$\epsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)}\ \epsilon_{n-1}\epsilon_n, \tag{3}$$

where the terms of order higher than 2 in ϵ are neglected.

Compare (3) to the corresponding formula for Newton's method:

$$\epsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)}\ \epsilon_n^2.$$

Formula (3) tells us that, as $n \to \infty$, the error tends to zero faster than a linear function and yet not quadratically. What exactly is the rule

$$|\epsilon_{n+1}| \approx C|\epsilon_n|^p$$

determined by (3)? Argue as follows. If $|\epsilon_{n+1}| \approx C|\epsilon_n|^p$ then

$$\begin{aligned} C|\epsilon_n|^p &\approx |M||\epsilon_{n-1}||\epsilon_n| \\ |\epsilon_n|^{p-1} &\approx \frac{|M|}{C}\ |\epsilon_{n-1}| \\ |\epsilon_n| &\approx \left(\frac{|M|}{C}\right)^{\frac{1}{p-1}} |\epsilon_{n-1}|^{\frac{1}{p-1}}. \end{aligned}$$

Therefore, $C = \left(\frac{|M|}{C}\right)^{\frac{1}{p-1}}$ and $p = \frac{1}{p-1}$. Because $p > 0$, the condition on p gives

$$p = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

The condition on C then implies that

$$C^p = |M| \quad \text{or} \quad C = |M|^{1/p} = \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{p-1}.$$

We conclude that for the secant method

$$|x_{n+1} - \alpha| \approx \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{\frac{\sqrt{5}-1}{2}} |x_n - \alpha|^{\frac{\sqrt{5}+1}{2}}.$$

Evidently, the order of convergence is generally lower than for Newton's method. However the derivatives $f'(x_n)$ need not be evaluated, and this is a definite computational advantage.

Remark. We can give a linear estimate of the error $\alpha - x_n$ in terms of the iterates. It is valid for both Newton and secant methods. By the mean value theorem,

$$f(\alpha) - f(x_n) = f'(c_n)(\alpha - x_n),$$

where c_n lies between x_n and α. So, if $x_n \to \alpha$, then $c_n \approx x_n$ for large n, and we have

$$\begin{aligned} \alpha - x_n &= -\frac{f(x_n)}{f'(c_n)} \\ &\approx -\frac{f(x_n)}{f'(x_n)} \\ &\approx -f(x_n)\, \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \\ &= x_{n+1} - x_n. \end{aligned}$$

Thus

$$\alpha - x_n \approx x_{n+1} - x_n.$$

Although this formula is not deep enough to exhibit the order of convergence, it is simple and allows to judge the absolute error of x_n after $(n+1)$ iterations.