Consider coin tossing with probability of heads $p$, $0 < p < 1$, and probability of tails $q = 1 - p$. Let $\hat{m} = \hat{m}(n)$ be the number of heads in $n$ independent tosses. Then $\hat{m}$ is a sum of $n$ independent copies of a Bernoulli random variable ($1/0$ indicator of whether the coin lands heads up) and has a binomial distribution. We have

$$P(\hat{m} = k) = \binom{n}{k} p^k q^{n-k}, \quad \mathbb{E}[\hat{m}] = pn, \quad \text{and} \quad \text{Var}(\hat{m}) = pqn.$$

When $n$ is large, computing the probability that $\hat{m}$ falls within certain bounds, $k_1$ and $k_2$, may involve very large numbers. For instance,

$$\binom{100}{50} = \frac{100!}{50!^2} \approx 9.33 \times 10^{157}.$$

Thus the evaluation of an exact answer,

$$P(k_1 \leq \hat{m} \leq k_2) = \sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k},$$

is a computational challenge. This puts some limitations on the use of Bernoulli’s formula.

An alternative is provided by normal approximation: for large $n$, the rescaled random variable $\frac{\hat{m} - pn}{\sqrt{pqn}}$ is very close to the standard normal $Z \sim \mathcal{N}(0,1)$ in distribution. This was proved by de Moivre (1730) for $p = 1/2$, and by Laplace (1812) for a general $0 < p < 1$. One distinguishes between the local

$$P(\hat{m} = k) \approx \frac{1}{\sqrt{2\pi}} e^{-(k-np)^2/2npq} \frac{1}{\sqrt{npq}}$$

and the integral

$$P(k_1 \leq \hat{m} \leq k_2) \approx \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-x^2/2} dx, \quad x_1 = \frac{k_1 - np}{\sqrt{npq}}, \quad x_2 = \frac{k_2 - np}{\sqrt{npq}}$$

approximation forms of the de Moivre–Laplace limit theorem. Each is a consequence of the other.

Thus, for large $n$, the distribution of $\hat{m}$ is nearly $\mathcal{N}(pn, pqn)$. This allows a quick approximate answer in terms of well-understood $\Phi(x)$, the cumulative distribution function of $Z$.

**Example** [Rice, page 187] Suppose that a coin is tossed 100 times and lands heads up 60 times. Should we suspect a bias?

If the coin is fair, the probability that $\hat{m} = 50$ is $\binom{100}{50}/2^{100} \approx 0.0796$, and the probability that $\hat{m} = 60$ is $\binom{100}{60}/2^{100} \approx 0.0108$. What does this say? Is $\hat{m} = 60$ a likely or an unlikely outcome?

Let us take an approximation approach. For a fair coin, we have $\mathbb{E}[\hat{m}] = 0.5 \cdot 100 = 50$ and $\text{Var}(\hat{m}) = 0.25 \cdot 100 = 25$. Since the number of trials is large, the normalized binomial variable $(\hat{m} - 50)/5$ should be well approximated by $Z$:

$$P(\hat{m} \geq 60) = P \left( \frac{\hat{m} - 50}{5} \geq 2 \right) \approx P(Z \geq 2) = 1 - \Phi(2) < 0.05/2 = 0.025.$$

The estimate suggests that the likelihood of the event $\hat{m} = 60$ is small (after all, $Z = 2$ is two standard deviations away from the mean). So suspecting a bias would not be unreasonable.