The Arc Length of a Parabola

Let us calculate the length of the parabolic arc $y = x^2,\ 0 \le x \le a$.

According to the arc length formula,

$$L(a) = \int_0^a \sqrt{1 + y'(x)^2}\ dx = \int_0^a \sqrt{1 + (2x)^2}\ dx.$$

Replacing $2x$ by x, we may write $L(a) = \dfrac{1}{2} \displaystyle\int_0^{2a} \sqrt{1 + x^2}\ dx.$

Thus the task is to find the antiderivative of $\sqrt{1 + x^2}$.

This is often done by setting $x = \sinh t$ or $x = \tan t$.

We will obtain the answer by manipulating square roots.

Two observations are needed:

$$\begin{aligned}\left(x\sqrt{x^2+1}\right)' &= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} \\ &= \sqrt{x^2+1} + \sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}} \\ &= 2\sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}}\ .\end{aligned}$$

$$\begin{aligned}\left(\ln\left(x + \sqrt{1+x^2}\right)\right)' &= \frac{1 + x/\sqrt{1+x^2}}{x + \sqrt{1+x^2}} \\ &= \frac{1}{\sqrt{1+x^2}}\ .\end{aligned}$$

Hence $2\sqrt{1+x^2} = \left(x\sqrt{x^2+1}\right)' + \left(\ln\left(x + \sqrt{1+x^2}\right)\right)'$.

Hence $\displaystyle\int \sqrt{1+x^2}\ dx = \frac{1}{2}\ x\sqrt{1+x^2} + \frac{1}{2}\ \ln\left(x + \sqrt{1+x^2}\right) + C.$

It follows that $L(a) = \dfrac{1}{2}\ a\sqrt{1+4a^2} + \dfrac{1}{4}\ \ln\left(2a + \sqrt{1+4a^2}\right).$

In particular, $L(1) = \dfrac{1}{2}\ \sqrt{5} + \dfrac{1}{4}\ \ln(2 + \sqrt{5}) \approx 1.48.$