The cross product transformation

Choose a nonzero (force) vector in $\mathbb{R}^3$, say $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. For $x$ in $\mathbb{R}^3$ (position vector), consider the linear transformation

$$T(x) = x \times v,$$

where $\times$ is the cross product operation. Geometrically, $x \times v$ is a vector orthogonal to both $x$ and $v$ and such that $x, v, x \times v$ form a right-hand triple:

![Cross product diagram]

The length of $x \times v$ is $\|x\| \|v\| \sin \alpha$, where $\alpha$ is the angle between $x$ and $v$. In particular, $x \times v$ is the zero vector whenever $x$ is a scalar multiple of $v$. Also note that $x \times v = -v \times x$.

For the standard basis vectors, we have: $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$.

In terms of coordinates,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} x_2v_3 - x_3v_2 \\ -x_1v_3 + x_3v_1 \\ x_1v_2 - x_2v_1 \end{bmatrix}.$$

Using geometry, we see that $\ker T$ is the line through the origin spanned by $v$ and $\text{im } T$ is the plane through the origin orthogonal to $v$. The standard matrix of $T$ is

$$A = \begin{bmatrix} e_1 \times v & e_2 \times v & e_3 \times v \end{bmatrix} = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

We have $\ker T = \ker A = \text{span} \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$ and $\text{im } T = \text{im } A = \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$.

What are the eigenvalues of $A$? Since $A$ is not invertible, one of its eigenvalues is $\lambda = 0$. The corresponding eigenspace is one-dimensional:

$$\mathcal{E}_0 = \ker (A - 0 \cdot I) = \ker A = \text{span} \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right).$$

For every $x$ in $\mathcal{E}_0$, we have $Ax = 0x = 0$. 

Does $A$ have other eigenvalues?

$$
\det(A - \lambda I) = \begin{vmatrix}
-\lambda & 3 & -2 \\
-3 & -\lambda & 1 \\
2 & -1 & -\lambda
\end{vmatrix} = (-\lambda) \begin{vmatrix}
-\lambda & 1 \\
2 & -\lambda
\end{vmatrix} - 3 \begin{vmatrix}
-3 & 1 \\
2 & -\lambda
\end{vmatrix} + (-2) \begin{vmatrix}
-3 & -\lambda \\
2 & -1
\end{vmatrix}
$$

$$
= -\lambda^3 - \lambda - 9\lambda + 6 - 6 - 4\lambda
$$

$$
= -\lambda(\lambda^2 + 14).
$$

Since $\lambda^2 + 14 = 0$ has no real roots, $A$ has no real eigenvalues other than $\lambda = 0$.

Furthermore, because $E_0$ is one-dimensional, it is not possible to find a basis for $\mathbb{R}^3$ consisting of the eigenvectors of $A$; not even two linearly independent real eigenvectors.

However, we may still be able to find a basis for $\mathbb{R}^3$ more suitable than standard for the purposes of representing our transformation $T$. Let us exploit the fact that $\ker T$ and $\im T$ are orthogonal subspaces. Consider, for instance, the following choice:

$$
\mathcal{B} : \quad w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.
$$

It is easy to check that $\mathcal{B}$ is indeed a basis for $\mathbb{R}^3$. Note that $w_1 = v$ is in $\ker A$, $w_1$ is orthogonal to both $w_2$ and $w_3$, and $w_2$, $w_3$ is a basis for $\im A$.

We have

$$
T(w_1) = w_1 \times w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0w_1 + 0w_2 + 0w_3
$$

$$
T(w_2) = w_2 \times w_1 = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} = 0w_1 + 6w_2 - 5w_3
$$

$$
T(w_3) = w_3 \times w_1 = \begin{bmatrix} -2 \\ 10 \\ -6 \end{bmatrix} = 0w_1 + 10w_2 - 6w_3.
$$

So that $T$ in the $\mathcal{B}$-basis is represented by $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 10 \\ 0 & -5 & -6 \end{bmatrix}$.

In many respects, $B$ is simpler than $A$, but it lacks skew-symmetry. Perhaps, a better choice would be to make $w_2$, $w_3$ an orthogonal basis for $\im A$.

Consider

$$
\mathcal{\tilde{B}} : \quad w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{w}_3 = w_1 \times w_2 = \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix},
$$

an orthogonal basis for $\mathbb{R}^3$. A quick check shows that, in the $\mathcal{\tilde{B}}$-basis, $T$ is represented by

$$
\mathcal{\tilde{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 14 \\ 0 & -1 & 0 \end{bmatrix}.
$$
Let us finally adjust the lengths to obtain an orthonormal basis for $\mathbb{R}^3$:

$$B_0 : \begin{align*}
u_1 &= \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \\
u_2 &= \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \\
u_3 &= \frac{1}{\sqrt{70}} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}.
\end{align*}$$

Do you see what is then the matrix of $T$?

Answer: $B_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{14} \\ 0 & -\sqrt{14} & 0 \end{bmatrix}$. The skew-symmetry is restored.

The action of $T$ can be read off this matrix. Given any $x = c_1u_1 + c_2u_2 + c_3u_3$, $T$ eliminates its $u_1$-component (orthogonal projection onto the $u_2u_3$-plane), stretches the $u_2u_3$-component by a factor of $\parallel v \parallel = \sqrt{14}$ (force magnitude) and rotates the result by an angle of $90^\circ$ about the $u_1$-axis (clockwise). In the language of physics/mechanics, $T$ gives the torque.

Note that since the basis $B_0$ is orthonormal, the coordinates with respect to $B_0$ are easy to find, the similarity matrix $S = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ is orthogonal, $S^{-1} = S^T$, and we have $A = SB_0S^T$. 