The matrix of differentiation

Differentiation is a linear operation:
\[(f(x) + g(x))' = f'(x) + g'(x) \quad \text{and} \quad (cf(x))' = cf'(x).\]

Does it have a matrix?

In brief, the answer is yes. We need, however, to agree on the domain of the operation and decide on how to interpret functions as vectors. Consider an illustration.

Let \(P_2\) be the collection of all polynomials of degree at most 2, with real coefficients. Every polynomial \(p(x) = a + bx + cx^2\) is completely determined by the vector \(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\) of its coefficients. The constant polynomial 1 corresponds to \(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\), \(x\) to \(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\), and \(x^2\) to \(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\). Every polynomial in \(P_2\) is a linear combination of 1, \(x\), and \(x^2\), just as every vector in \(\mathbb{R}^3\) is a linear combination of the orts \(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\), \(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\), and \(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\). In this sense, \(P_2\) is very much like \(\mathbb{R}^3\): addition and scaling work in the same way.

View \(P_2\) as the domain of the derivative operation. Differentiation maps 1 to 0, \(x\) to 1, and \(x^2\) to 2\(x\). Using the above vector interpretation, we may write this correspondence as
\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.
\]

Thus the derivative operation on \(P_2\) is represented by the matrix \(D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\).

The action \(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}\) does agree with the formula \((a + bx + cx^2)' = b + 2cx\).

\(D\) is clearly not invertible: it lacks a pivot. It carries out a linear transformation of \(\mathbb{R}^3\) which is neither one-to-one nor onto. Indeed, every vector of the form \(\begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}\) is mapped to \(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\), and no vector with nonzero third component is in the range of \(D\).

The range of \(D\) consists of vectors with zero third component. That’s right, the derivative of a polynomial of degree at most 2 is of degree at most 1. Note that \(D^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\) represents the second derivative on \(P_2\), and, of course, \(D^3\) is the zero transformation.

Using the above framework, we may represent differentiation on \(P_3\) (polynomials of degree at most 3) by \(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\), differentiation on \(P_4\) (polynomials of degree at most 4) by \(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}\), and so on. The pattern is clear.