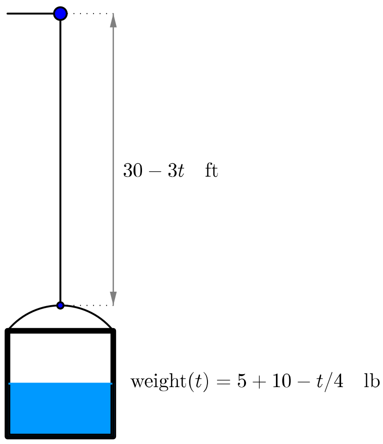
Integral Calculus

Grinshpan

The leaking bucket problem

A 5 lb bucket containing 10 lb of water is hanging at the end of a 30 ft rope which weighs $1/2$ lb/ft. The other end of the rope is attached to a pulley. The rope is wound onto the pulley at a rate of 3 ft/s causing the bucket to be lifted. Find the work done in winding the rope onto the pulley if the water leaks out of the bucket at a rate of $1/4$ lb/s.



Let's deal with the rope first. A small section of the rope of length dx ft positioned x ft below the top weighs $\frac{1}{2}\,dx$ lb. So the work needed to lift it is $dW = \frac{1}{2}\,x\,dx$ ft·lb. The total work needed to lift the rope is therefore

$$W_{\text{rope}} = \int_0^{30} \frac{1}{2}\,x\,dx = 225 \text{ ft} \cdot \text{lb}.$$

Now for the bucket (without water). Since its weight is constant, the work needed is

$$W_{\text{bucket}} = 30 \cdot 5 = 150 \text{ ft} \cdot \text{lb}.$$

Now for the water (without the bucket). Its weight varies with position. Over a small displacement of length dx ft, the weight is nearly constant. So the work needed to lift the water from level x ft to level $x - dx$ ft is the weight of water at level x multiplied by dx. The total work is

$$W_{\text{water}} = \int_0^{30} [\text{weight at level } x]\;dx$$

The position depends on time. The distance to the top t seconds from the start is $x = 30 - 3t$ ft. The weight of the water t seconds from the start is $10 - t/4$ lb. It takes 10 seconds to complete the lift, which is not enough time for the bucket to become empty. Changing the variable, we have:

$$W_{\text{water}} = \int_{10}^{0} [10 - t/4]\;(-3dt) = 262.5 \text{ ft} \cdot \text{lb}.$$

The answer is $W = W_{\text{rope}} + W_{\text{bucket}} + W_{\text{water}}$.