The simplex method in matrix notation

Arrange the variable tuples

\[(x_1, \ldots, x_n, w_1, \ldots, w_m) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}),\]
\[(z_1, \ldots, z_n, y_1, \ldots, y_m) = (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}).\]

Let \(B\) and \(N\) be the current lists of basic and nonbasic indices, respectively.

Arrange the data entries according to the index lists:

\[x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}_{(m+n)\times 1}, \quad A = \begin{bmatrix} A_B & A_N \end{bmatrix} = [B \quad N]_{m \times (m+n)}, \quad c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}_{(m+n)\times 1}, \quad z = \begin{bmatrix} z_B \\ z_N \end{bmatrix}_{(m+n)\times 1}.\]

Let \(x^*\) and \(z^*\) be the points associated with the current primal and dual dictionaries,

\[x^* = \begin{bmatrix} x_B^* \\ x_N^* \end{bmatrix}, \quad z^* = \begin{bmatrix} z_B^* \\ z_N^* \end{bmatrix} = \begin{bmatrix} 0 \\ z_N^* \end{bmatrix}.\]

These dictionaries are then of the form:

Primal

\[\begin{align*}
\zeta &= \zeta^* - z_N^* \top x_N \\
x_B &= x_B^* - K \cdot x_N \\
\end{align*} \]

Dual

\[\begin{align*}
-\xi &= -\zeta^* - x_B^* \top z_B \\
z_N &= z_N^* + K^\top z_B.
\end{align*}\]

We can write down the expressions for \(\zeta, x_B^*, z_N^*, \zeta^*, K\) in terms of \(b, c_B, c_N, B, N\).

Indeed, since

\[Ax = [B \quad N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = Bx_B + Nx_N = b\]

and since \(B\) is invertible, we have

\[x_B = B^{-1}b - B^{-1}Nx_N.\]

Furthermore,

\[\zeta = c^\top x = \begin{bmatrix} c_B^\top & c_N^\top \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = c_B^\top x_B + c_N^\top x_N = c_B^\top (B^{-1}b - B^{-1}Nx_N) + c_N^\top x_N = c_B^\top B^{-1}b - ((B^{-1}N)^\top c_B - c_N)^\top x_N.\]

Thus

\[x_B^* = B^{-1}b \quad \text{is the vector of current values of basic variables},\]
\[K = B^{-1}N \quad \text{is the matrix of current constraint coefficients},\]
\[\zeta^* = c_B^\top B^{-1}b \quad \text{is the current objective function value},\]
\[z_N^* = (B^{-1}N)^\top c_B - c_N \quad \text{is the vector of current values of dual basic variables}.\]

Note that the complementarity condition \(x_j^*z_j^* = 0, \ j = 1, \ldots, n+m,\) is a direct consequence of how the variables are arrangement. If the primal dictionary is feasible, \(x_B^* \geq 0,\) and if the dual dictionary is feasible, \(z_N^* \geq 0,\) then the primal dictionary is optimal.
The $m \times n$ matrix $K = B^{-1}N$ of constraint coefficients carries important information about the dictionary. If the current lists of basic and nonbasic indices are $B = (\beta_1, \ldots, \beta_m)$ and $N = (\nu_1, \ldots, \nu_n)$, then the equality $x_B = x_B^* - Kx_N$ takes the form

$$x_{\beta_1} = x_{\beta_1}^* - K_{11}x_{\nu_1} - \ldots - K_{1n}x_{\nu_n},$$
$$x_{\beta_2} = x_{\beta_2}^* - K_{21}x_{\nu_1} - \ldots - K_{2n}x_{\nu_n},$$
$$\vdots \quad \vdots \quad \vdots$$
$$x_{\beta_m} = x_{\beta_m}^* - K_{m1}x_{\nu_1} - \ldots - K_{mn}x_{\nu_n}.$$

Let $x_B^* \geq 0$. Suppose that $x_{\nu_j} = t$ increases from zero, while $x_{\nu} = 0$ for all other $\nu \in N$. We then have $x_B = x_B^* - tK_j$, where $K_j$ is the $j$th column of $K$. To preserve feasibility, we must ensure that

$$x_{\beta_i}^* - tK_{ij} \geq 0, \quad i = 1, \ldots, m.$$

If $K_{ij} \leq 0$, then $x_{\beta_i}^* - tK_{ij} \geq 0$ gives no bound on $t$, and if $K_{ij} > 0$, then $t \leq \frac{x_{\beta_i}^*}{K_{ij}}$.

Thus the new value of $x_{\nu_j}$ is given by

$$t = \min \frac{x_{\beta_i}^*}{K_{ij}},$$

where the minimum is taken over all $i = 1, \ldots, m$ such that $K_{ij} > 0$. If $K_j \leq 0$, the problem is unbounded. Otherwise, for any $i$ for which the minimum is attained, the variable $x_{\beta_i}$ can be chosen as the leaving variable.

**Step summary**

- **Entering variable:** select any $x_{\nu_j}$ such that $(z_N^*)_{\nu_j} < 0$
- **Step direction:** $(B^{-1}N)_j = K_j$
- **Step size:** $t = \min \frac{x_{\beta_i}^*}{K_{ij}}$ over all $i = 1, \ldots, m$ such that $K_{ij} > 0$
- **Leaving variable:** select any $x_{\beta_i}$ such that $t = \frac{x_{\beta_i}^*}{K_{ij}}$
- **New basic vector:** replace $x_B^*$ by $x_B^* - tK_j$, set $(x_B^*)_i = t$
- **Dual step direction:** $-(N^T B^{-1})_i = -(K^*)_i$
- **Dual step size:** $s = \frac{z_{\nu_j}^*}{K_{ij}}$
- **New dual basic vector:** replace $z_N^*$ by $z_N^* - s(K^*)_i$, set $(z_N^*)_j = s$
- **Index update:** interchange $\beta_i$ in $B$ and $\nu_j$ in $N$
- **Matrix update:** interchange the $i$th column of $B$ and $j$th column of $N$