Linear Algebra

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Permutation matrices

A permutation matrix is a square matrix obtained from the same size identity matrix by a permutation of rows. Such a matrix is always row equivalent to an identity.

Every row and every column of a permutation matrix contain exactly one nonzero entry, which is 1. There are two 2×2 permutation matrices: $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$.
There are six 3×3 permutation matrices. There are $n!$ permutation matrices of size n.

Every permutation matrix is a product of elementary row-interchange matrices. The elementary matrix factors may be chosen to only involve adjacent rows. For instance,

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&\underline{1}\\0&\underline{1}&0\end{bmatrix}\begin{bmatrix}0&\underline{1}&0\\\underline{1}&0&0\\0&0&1\end{bmatrix}.$$

Since interchanging ith and jth rows of an identity is equivalent to interchanging its ith and jth columns, every *elementary* permutation matrix is symmetric, $P^\top = P$.
A general permutation matrix is not symmetric.

Since interchanging two rows is a self-reverse operation, every *elementary* permutation matrix is invertible and agrees with its inverse, $P = P^{-1}$ or $P^2 = I$.
A general permutation matrix does not agree with its inverse.

A product of permutation matrices is again a permutation matrix.
The inverse of a permutation matrix is again a permutation matrix. In fact, $P^{-1} = P^\top$.

Left multiplication by a permutation matrix rearranges the corresponding rows:

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_2\\x_3\\x_1\end{bmatrix}, \qquad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}a&a&a\\b&b&b\\c&c&c\end{bmatrix} = \begin{bmatrix}b&b&b\\c&c&c\\a&a&a\end{bmatrix}.$$

Right multiplication by a permutation matrix rearranges the corresponding columns:

$$\begin{bmatrix}a&b&c\\a&b&c\\a&b&c\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} = \begin{bmatrix}c&a&b\\c&a&b\\c&a&b\end{bmatrix}, \qquad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} = \begin{bmatrix}f&d&e\\i&g&h\\c&a&b\end{bmatrix}.$$

Some power of a permutation matrix is the identity. For instance,

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Here $P^3 = I$ or $P^2 = P^{-1} = P^\top$.