The set of rational numbers is not $G_δ$

By Baire’s theorem, the interval $[0, 1]$ is not a countable union of nowhere-dense sets. Consider $E = \mathbb{Q} \cap [0, 1]$, the set of rational numbers in $[0, 1]$.

$E$ is a countable union of singletons, so it is a countable union of nowhere-dense sets. So the complement $[0, 1] \setminus E$ is not a countable union of nowhere-dense sets.

Claim: $E$ is not a $G_δ$ set.

Proof: Suppose for the sake of contradiction that $E$ is $G_δ$.

Then $E$ is a countable intersection of open sets $G_k$ in $[0, 1]$.

Since $E$ is dense in $[0, 1]$, the same is true of each $G_k$.

So the complements $[0, 1] \setminus G_k$ are closed and nowhere-dense.

So $[0, 1] \setminus E$ is a countable union of nowhere-dense sets. A contradiction. □

Thus the rational numbers form a first category (meager) set, which is $F_σ$ but not $G_δ$, and the irrational numbers form a second category (residual) set, which is $G_δ$ but not $F_σ$. 