Numerical Analysis

Grinshpan

TAYLOR POLYNOMIALS

The remainder term

Let f be an n-times differentiable function defined on an interval, and let x_0 and x be points in its domain. Consider the n-th Taylor polynomial of f centered at x_0 and evaluated at x,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Our goal is to get a hold of the remainder term $f(x) - T_n(x)$.

The formulas stated below are valid under the assumption that the n-th derivative $f^{(n)}$ is continuously differentiable. Actually, they hold under a slightly weaker but more technical assumption that $f^{(n)}$ is continuous on the closed interval with endpoints x_0 and x and is differentiable in the open interval with those endpoints.

AN INTEGRAL FORM OF THE REMAINDER

$$f(x) - T_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(s)(x - s)^n ds.$$

THE LAGRANGE FORM OF THE REMAINDER

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1},$$

for some c in the open interval with endpoints x_0 and x.

Both formulas are tied to the Mean-Value Theorem and may be used for error estimates.

EXAMPLE The n-th Taylor polynomial of $f(x) = e^x$ at $x_0 = 0$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots + \frac{1}{n!} x^n.$$

Set $x = 1$. Then, by the Lagrange remainder formula,

$$e^1 - T_n(1) = \frac{e^c}{(n+1)!} \ ,$$

where $c = c(n)$ satisfies $0 < c < 1$. The sums $T_n(1)$ are monotonically increasing with n. Using that $1 < e^c < 3$, we obtain simple bounds

$$\frac{1}{(n+1)!} \ < \ e - T_n(1) \ < \ \frac{3}{(n+1)!} \ .$$

In particular, the limit of $T_n(1)$, as $n \to \infty$, is e. It is easy to check that $3/13! < 5 \times 10^{-10}$, and so $T_{12}(1) = \sum_{k=0}^{12} \frac{1}{k!} = 2.718281828...$ has 10 significant digits of e.