Classification of stationary points: an example

Consider the function \( f(x, y) = xy - x^3 - y^2 \).

This is a polynomial in two variables of degree 3.

To find its stationary points set up the equations:

\[
\begin{align*}
    f_x &= y - 3x^2 = 0 \\
    f_y &= x - 2y = 0
\end{align*}
\]

We have \( x = 2y, \ y - 12y^2 = 0 \), and so \( y = 0 \) or \( y = \frac{1}{12} \).

This gives two stationary points \((0, 0)\) and \((\frac{1}{6}, \frac{1}{12})\).

We will need second partials in our analysis:

\[
\begin{align*}
    f_{xx} &= -6x, \quad f_{yy} = -2, \quad f_{xy} = f_{yx} = 1.
\end{align*}
\]

Examine \((0, 0)\) first. The Hessian matrix of partial derivatives is

\[
\begin{pmatrix}
    f_{xx} & f_{yx} \\
    f_{xy} & f_{yy}
\end{pmatrix}
\bigg|_{(0,0)} = \begin{pmatrix}
    0 & 1 \\
    1 & -2
\end{pmatrix}
\]

Since \( 0 \cdot (-2) - 1^2 = -1 < 0 \), \((0, 0)\) is a saddle point according to the Second Partial Tests.

Note that, if \( y = 0 \), then \( f(x, 0) = -x^3 \) and so \( f(x, y) \) cannot possibly have a maximum or a minimum at \((0, 0)\). On the other hand, \( x = 0 \) trace is \( f(0, y) = -y^2 \), which has a maximum at the origin. Also \( x = 2t, \ y = t \) trace is \( f(2t, t) = t^2 - 8t^3 \) and has a local minimum at the origin. This situation is typical for a saddle point. The contour map around \((0, 0)\) resembles that of a hyperbolic paraboloid.

At \((\frac{1}{6}, \frac{1}{12})\) we have:

\[
\begin{pmatrix}
    f_{xx} & f_{yx} \\
    f_{xy} & f_{yy}
\end{pmatrix}
\bigg|_{\left(\frac{1}{6}, \frac{1}{12}\right)} = \begin{pmatrix}
    -1 & 1 \\
    1 & -2
\end{pmatrix}
\]

Since \(-1 < 0\) and \((-1) \cdot (-2) - 1^2 = 1 > 0\), \((\frac{1}{6}, \frac{1}{12})\) is a point of local maximum.

The local maximum value is \( f(\frac{1}{6}, \frac{1}{12}) = \frac{1}{432} \).

Locally, around \((\frac{1}{6}, \frac{1}{12})\), the graph of \( f \) looks like an elliptic paraboloid opening down.

This is confirmed by examining profiles \( f(x, \frac{1}{12}) \approx \frac{1}{432} - \frac{1}{2} (x - \frac{1}{6})^2 \) and \( f(\frac{1}{6}, y) \approx \frac{1}{432} - (y - \frac{1}{12})^2 \), where \( x \approx \frac{1}{6} \) and \( y \approx \frac{1}{12} \) and cubic terms have been discarded.

Here is the contour map in the neighborhood of stationary points.
Absolute maximum and absolute minimum

Consider the square region $\Delta$ with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

What are the largest and the smallest values of $f(x, y) = xy - x^3 - y^2$ on $\Delta$?

The largest/smallest values may be attained at interior stationary points or on the edge.

There is only one stationary point in the interior of $\Delta$. The value $f\left(\frac{1}{6}, \frac{1}{12}\right) = \frac{1}{432}$ is a candidate for the absolute maximum value on $\Delta$.

How does $f$ behave as we walk around the boundary?

Bottom edge. If $y = 0$ and $x$ changes from 0 to 1, $f(x, 0) = -x^3$ decreases from $f(0, 0) = 0$ to $f(1, 0) = -1$.

Right edge. If $x = 1$ and $y$ changes from 0 to 1, the quadratic $f(1, y) = y - 1 - y^2$ first increases from $f(1, 0) = -1$ to its maximum value $f(1, \frac{1}{2}) = -\frac{3}{4}$ and then decreases back to $f(1, 1) = -1$.

Top edge. If $y = 1$ and $x$ changes from 0 to 1, the cubic $f(x, 1) = x - x^3 - 1$ is concave down. It goes up from $f(0, 1) = -1$ to its local maximum value $f\left(\frac{1}{\sqrt{3}}, 1\right) = -1 + \frac{2}{3}\sqrt{3}$ and then back down to $f(1, 1) = -1$.

Left edge. If $x = 0$ and $y$ changes from 0 to 1, $f(0, y) = -y^2$ decreases from $f(0, 0) = 0$ to $f(0, 1) = -1$.

All in all, we have the following diagram:
Observe that, on the boundary of $\Delta$, $f(x, y) \leq 0$. Hence the value $\frac{1}{4\sqrt{2}}$ remains undefeated, it is the absolute maximum on $\Delta$. By inspection, the value $-1$ is the smallest on $\Delta$. It is attained at three different points of the boundary.

Superimposing two maps we gain more insight: the behavior of $f$ along the sides of the square is completely determined by the geometry of contour curves.