A recurrence for the Stirling numbers

Let m and n be nonnegative integers.

Let $S(m, n)$ be the number of surjections from an m-element set to an n-element set.

We have $S(m, m) = m!$ and $S(m, n) = 0$, if $m < n$.

Also, $S(0, 0) = 1$ and $S(0, n) = S(m, 0) = 0$.

The inclusion-exclusion principle yields a complex formula,

$$S(m, n) = n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m + \dots .$$

However, a simple recurrence relation takes place:

$$(*) \qquad S(m+1, n) = nS(m, n) + nS(m, n-1)$$

Proof of recurrence.

Let X be a set of $(m + 1)$ elements and let Y be a set of n elements.

Write $X = X' \cup \{a\}$, so that X' consists of m elements.

For every mapping f of X *onto* Y, there are two mutually exclusive possibilities:

(1) f maps X' onto Y

(2) f does not map X' onto Y

The number of mappings f obeying condition (1) is $S(m, n) \times n$, because a can be mapped to any element of Y. This gives the first term on the right of $(*)$.

If f obeys condition (2), then $f(X')$ is all of Y except for a single element, $b = f(a)$. The number of such mappings is $S(m, n-1) \times n$, since there are n ways to pick the excepted element b. This gives the second term on the right of $(*)$.

The numbers $S(m, n)/n!$ are known as Stirling's numbers of the second kind.