The likelihood ratio test for the mean of a normal distribution

Let \( X_1, \ldots, X_n \) be a random sample from a normal distribution with unknown mean \( \mu \) and known variance \( \sigma^2 \). Suggested are two simple hypotheses, \( H_0 : \mu = \mu_0 \) vs \( H_1 : \mu = \mu_1 \).

Given \( 0 < \alpha < 1 \), what would the likelihood ratio test at significance level \( \alpha \) be? The answer turns out to be directly related to the sample mean \( \bar{X} \).

Let us write \( f_0 \) and \( f_1 \) to indicate the density functions under \( H_0 \) and \( H_1 \), respectively. By independence, the joint density function of the sample under \( H_0 \) is

\[
    f_0(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_0(x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi} \sigma} e^{-(x_i-\mu_0)^2/2\sigma^2} = (\sqrt{2\pi} \sigma)^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 \right\}.
\]

Similarly,

\[
    f_1(x_1, \ldots, x_n) = (\sqrt{2\pi} \sigma)^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_1)^2 \right\}.
\]

So the likelihood ratio is given by

\[
    \ell(x_1, \ldots, x_n) = \frac{f_0(x_1, \ldots, x_n)}{f_1(x_1, \ldots, x_n)} = \exp \left\{ \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_1)^2 - (x_i - \mu_0)^2 \right\} = \exp \left\{ \frac{n(\mu_0 - \mu_1)^2}{\sigma^2} \left( x - \frac{\mu_0 + \mu_1}{2} \right) \right\}.
\]

Observe that the condition \( \ell(x_1, \ldots, x_n) < c \) is equivalent to

\[
    \bar{x} > \tilde{c}, \quad \text{if} \quad \mu_0 < \mu_1, \quad \text{or} \quad \bar{x} < \tilde{c}, \quad \text{if} \quad \mu_0 > \mu_1,
\]

where the critical value \( \tilde{c} \) is given by

\[
    \tilde{c} = \frac{\mu_0 + \mu_1}{2} + \sqrt{\frac{\ln c}{n} \frac{\sigma^2}{\mu_0 - \mu_1} }.
\]

Hence the rejection criterion has a simple form in terms of the statistic \( \bar{X} \):

\[
    \bar{X} > \tilde{c} \quad (\mu_0 < \mu_1) \quad \text{or} \quad \bar{X} < \tilde{c} \quad (\mu_0 > \mu_1).
\]

If the hypotheses have equal priors, \( c = 1 \), then, of course, \( \tilde{c} = \frac{\mu_0 + \mu_1}{2} \).

Also, if \( n \) is sufficiently large, then \( \tilde{c} \approx \frac{\mu_0 + \mu_1}{2} \).

The exact value of \( \tilde{c} \) can be determined from \( \alpha \).
Assume that $\mu_0 < \mu_1$.

Under $H_0$, we have $\bar{X} \sim N(\mu_0, \sigma^2/n)$, and so

$$\alpha = P_0(\ell(X_1, \ldots, X_n) < c) = P_0(\bar{X} > \bar{c}) = P \left( Z > \frac{\bar{c} - \mu_0}{\sigma / \sqrt{n}} \right).$$

This gives

$$\bar{c} = \mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha}$ is the upper $\alpha$-quantile for the standard normal distribution. Note that $\bar{c}$ does not depend on $\mu_1$.

Under $H_1$, we have $\bar{X} \sim N(\mu_1, \sigma^2/n)$, and so the type II error probability is

$$\beta = P_1(\bar{X} < \bar{c}) = P \left( Z < \frac{\bar{c} - \mu_1}{\sigma / \sqrt{n}} \right) = P \left( Z \leq z_{\beta} + (\mu_0 - \mu_1) \frac{\sqrt{n}}{\sigma} \right).$$

This gives $z_{\beta} = -z_{\alpha} + (\mu_1 - \mu_0) \frac{\sqrt{n}}{\sigma}$. Note that $\beta$ is a monotone decreasing function of $n$.

It follows that

$$n = \left( \frac{z_{\alpha} + z_{\beta}}{\mu_1 - \mu_0} \frac{\sqrt{n}}{\sigma} \right)^2$$

is the minimum sample size needed for the probabilities of type I and type II errors not to exceed the values of $\alpha$ and $\beta$, respectively. In fact, $\lceil n \rceil$ should be taken.

**Exercise** Let $n = 25, \mu_0 = 0, \mu_1 = 1, \sigma = 1$, and $\alpha = 0.05$. Determine the critical value $\bar{c}$, the probability of a type II error $\beta$, and the power of the test.

**Solution.** We have $\bar{c} = \mu_0 + z_{0.05} \frac{\sigma}{\sqrt{25}} = 0 + 1.645/5 = 0.329$. So the hypothesis $\mu = 0$ is to be rejected whenever $\bar{X} > 0.329$. The chance of a type II error is

$$\beta = P(Z \leq 1.645 + (0 - 1) \cdot 5) = P(Z \leq -3.355) = 0.0004.$$ 

The power is $1 - \beta = 0.9996$.

**Exercise** Let $\mu_0 = 0, \mu_1 = 1, \sigma = 1$, and $\alpha = 0.05$. How many observations are required to have the probability of a type II error below 0.1?

**Solution.** Since $z_{0.05} = 1.645$ and $z_{0.1} = 1.282$,

$$n = \left( \frac{1.645 + 1.282}{1 - 0} \cdot 1 \right)^2 \approx 8.6.$$ 

Hence 9 observations are required.

**Exercise** Suppose we want to test an unknown coin,

$$H_0 : p = 0.5 \quad \text{vs} \quad H_1 : p = 0.51,$$

with $\alpha \leq 0.05$ and $\beta \leq 0.1$. Estimate the number of flips required.

**Solution.** Using normal approximation with

$$\mu = p, \quad \sigma = \sqrt{p(1-p)} \approx 0.5, \quad z_{\alpha} \geq 1.654, \quad z_{\beta} \geq 1.282,$$

we have

$$n \sim \left( \frac{1.645 + 1.282}{0.51 - 0.5} \cdot 0.5 \right)^2 \approx 21418.$$