## Cusps and vertical tangents

**Example 1.** The function $f(x) = x^{1/3}$ has a vertical tangent at the critical point $x = 0$ :

$$\text{as } x \to 0, \quad f'(x) = \frac{1}{3x^{2/3}} \to \infty.$$

**Example 2.** The function $f(x) = x^{2/3}$ has a cusp at the critical point $x = 0$ :

$$\text{as } x \to 0^+, \quad f'(x) = \frac{2}{3x^{1/3}} \to +\infty$$

and

$$\text{as } x \to 0^-, \quad f'(x) = \frac{2}{3x^{1/3}} \to -\infty.$$

**Example 3.** Verify that the function $f(x) = 3x^{1/3}(x+2)$ has the following properties.

Defined and continuous for all $x$.
Roots: $x = 0$, $x = -2$. Intercepts: $(-2, 0)$, $(0, 0)$.
No horizontal or vertical asymptotes.
Tends to $+\infty$ like $y = 3x^{4/3}$ as $x \to \pm\infty$ (curvilinear asymptote at infinity).
Critical points: $x = -\frac{1}{2}$ (stationary), $x = 0$ (infinite slope).
Vertical tangent line at the origin.
Decreasing for $x < -\frac{1}{2}$, increasing for $x > -\frac{1}{2}$.
Absolute minimum at $x = -\frac{1}{2}$.
Concave up for $x < 0$ and for $x > 1$. Concave down for $0 < x < 1$.
Points of inflection: $(0, 0)$, $(1, 9)$.