Approximation Order Equivalence Properties of Manifold-Valued Data Subdivision Schemes

Gang Xie *   Thomas P.-Y. Yu †

April 27, 2008

Abstract:

There has been an emerging interest in developing an approximation theory for manifold-valued functions. In this paper, we address the following fundamental problem:

Let M be a manifold with a metric d. For each smoothness factor $r > 0$ and approximation order $R > 0$, is there an approximation operator $\mathcal{A}_h = \mathcal{A}_{h;r,R}$ that maps samples of any $f : \mathbb{R} \to M$ on a grid of size h to an approximant $f_h = \mathcal{A}_h(f|_{h\mathbb{Z}}) : \mathbb{R} \to M$ with the properties that a) $\sup_x d(f_h(x), f(x)) = O(h^R)$ whenever f is a bounded C^R function, and b) f_h is C^r smooth ?

The case of $M = \mathbb{R}$ is of course well-studied. In the recent paper [14], the authors show that subdivision methods can be used to create arbitrarily smooth interpolants for M-valued data, addressing b) above. In this paper we further show that interpolatory subdivision schemes can be used to solve a) above. So, altogether, we establish the fact that if a linear interpolatory subdivision scheme possesses a smoothness order r and an approximation order R, then a nonlinear interpolatroy subdivision scheme for M-valued data constructed based on this linear scheme has the same smoothness and approximation orders. In other words, subdivision schemes furnish a constructive approximation method for solving the open problem posted above.

We discuss the construction of quasi-interpolants of manifold-valued data based on general (not necessarily interpolatory) subdivision schemes.

Acknowledgments. The work of this research was partially supported by the National Science Foundation grant DMS 0542237. The empirical observation on approximation equivalence (Figure 1) was first presented in the Sixth International Conference on Curves and Surfaces (Avignon, France, June 29-July 5, 2006.) and further discussed in a mini-symposium in the 12th International Conference in Approximation Theory (San Antonio, Texas, March 4-8, 2007) and the MAIA 2007 conference (Ålesund, Norway, August 22-26, 2007.) A preliminary effort on this problem was made in [5]. The second named author thanks Nira Dyn for inviting him to a workshop on "Subdivision and Refinability" (Pontignano, Siena, Italy, May 1-4, 2008), at which the main result of this paper was first presented.

Keywords. Approximation order, Subdivision scheme, Nonlinear subdivision scheme, Interpolation, Quasi-Interpolation, Manifold

*Email: gangxie2006@gmail.com.
†Email: yut@drexel.edu

1 Introduction

There has been an emerging interest in developing an approximation theory for manifold-valued data motivated by the numerical solution of ODEs evolving on manifolds (e.g., [6]), and efficient representation of manifold-valued data arising from areas such as diffusion tensor imaging, motion capturing, etc. (see, e.g. [7].) In this paper, we address the following natural fundamental problem:

Let M be a smooth manifold equipped with a metric d. For each smoothness factor $r > 0$ and approximation order $R > 0$, is there an approximation operator $\mathcal{A}_h = \mathcal{A}_{h;r,R}$ that maps samples of any $f : \mathbb{R} \to M$ on a grid of size h to an approximant $f_h = \mathcal{A}_h(f|_{h\mathbb{Z}}) : \mathbb{R} \to M$ with the properties that

a) $\sup_x d(f_h(x), f(x)) = O(h^R)$ whenever f is a bounded C^R function, and

b) f_h is C^r smooth?

The case of $M = \mathbb{R}$ is of course well-known, and such an approximation operator $\mathcal{A}_h$ can be furnished by, for example, spline methods.

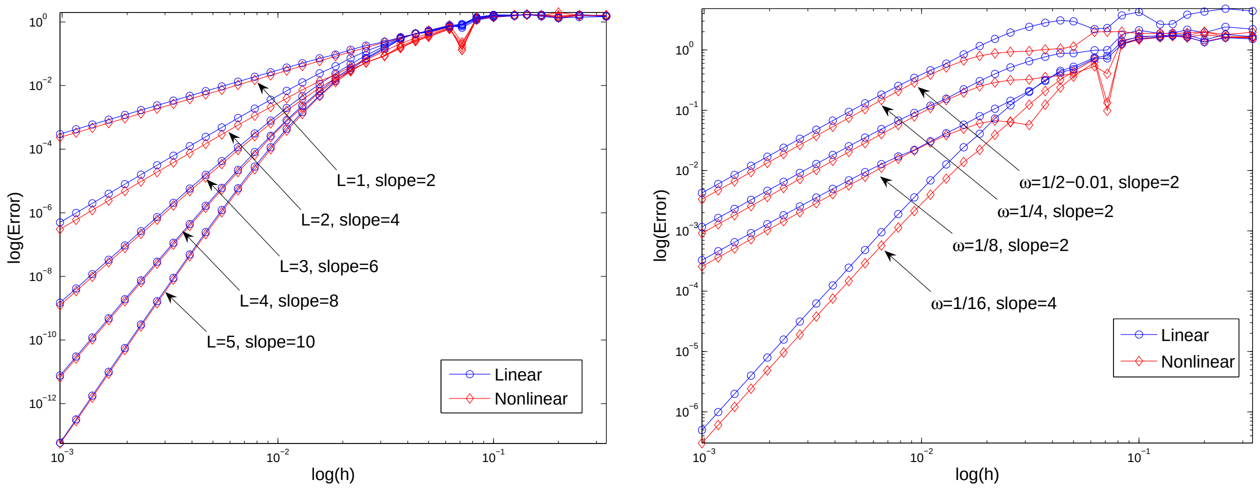
We had already shown in [14] that b) can be solved using subdivision schemes. In this paper we further show that interpolatory subdivision schemes can be used to solve a), provided that the metric is appropriately chosen. Our solution of a) in this paper relies on the key proximity inequality proven in [14].

As an example of our main result in this paper, we sample the following smooth 1-periodic function

$$F : \mathbb{R} \to S^2 = \{x \in \mathbb{R}^3 : \|x\|_{\mathbb{R}^3} = 1\}, \quad F(x) = \begin{bmatrix} \sin(\cos(2\pi x))\cos(\sin(2\pi x) + \cos(6\pi x)) \\ \sin(\cos(2\pi x))\sin(\sin(2\pi x) + \cos(6\pi x)) \\ \cos(\cos(2\pi x)) \end{bmatrix} \tag{1.1}$$

on a grid of size h and we plot $\max_{x\in\mathbb{R}} \|F(x) - T^\infty F|_{h\mathbb{Z}}(x/h)\|_{\mathbb{R}^3}$ against h and $\max_{x\in\mathbb{R}} \|F - S^\infty F|_{h\mathbb{Z}}(\cdot/h)\|_{\mathbb{R}^3}$ against h. Here T is either a Deslauriers-Dubuc scheme [3] or the 4-point scheme with parameter ω, and $S = P \circ T$ is the extrinsic projection scheme proposed in [11] and analyzed in [13]. From the linear theory it is known that $T^\infty F|_{h\mathbb{Z}}(\cdot/h)$, a function that does not necessarily take values on S^2, approximates the S^2-valued function F with the rate $\|F - T^\infty F|_{h\mathbb{Z}}(\cdot/h)\|_\infty = O(h^{R+1})$ if T reproduces Π_R. Our computational experiment strongly indicates that $\|F - S^\infty F|_{h\mathbb{Z}}(\cdot/h)\|_\infty = O(h^{R+1})$ as well, with a different hidden constant.

In fact the plots also indicate that the hidden constant for S is usually slightly smaller than that for the corresponding T. This we will not explore mathematically but it comes with no surprise: to approximate a S^2-valued function, it seems better to use a function that takes values on S^2 than one that does not.



(a) $2L$-point Deslauriers-Dubuc schemes (b) 4-point scheme, different ω's

Figure 1: Approximation Order Equivalence

The same equivalence holds if S is constructed from T based on the log-exp scheme in [7]. It is the goal of the next section to prove the observed approximation order equivalence in a general context.

2 Main Result

Let M be a smooth manifold and T be a linear interpolatory subdivision scheme defined by

$$(Ty)_{2i} = y_i, \quad (Ty)_{2i+1} = \sum_{\ell} w_\ell \, y_{i-\ell}. \tag{2.1}$$

Following [14], a nonlinear subdivision scheme S for M-valued data is defined as follows. Let $\pi : \hat{V} \to M$ be a vector bundle over M, $g_x : U_x \to V(x) =: \pi^{-1}(x)$ where U_x is an open neighborhood of x. For a technical reason explained in [14], we assume also that $U := \{(x, y) : x \in M, y \in U_x\}$ is open in $M \times M$, and then we also assume that $g : U \to V$, $g(x, y) := g_x(y)$, is C^∞ smooth. Next, we assume we have a smooth $f : E \to M$ where E is an open set in the vector bundle V. We write $E_x := \{(x, v) \in \pi^{-1}(x) : (x, v) \in E\}$, and $f_x : E_x \to M$, $f_x(v) := f(x, v)$. Moreover we assume that f is a left inverse of g in the following sense:

$$f_x(g_x(y)) = y, \quad \forall\, x \in M, \;\; y \in U_x.$$

Then, the M-valued subdivision scheme S based on T (and (V, f, g)) is defined as:

$$(Sy)_{2i} = y_i, \quad (Sy)_{2i+1} = f_{y_i}\left(\sum_{\ell} w_\ell \, g_{y_i}(y_{i-\ell}) \right). \tag{2.2}$$

It is shown in [14] that whenever the underlying linear scheme T is convergent, then this subdivision scheme S is well-defined and also convergent for dense enough initial data.

Recall that if T is a linear convergent Π_R-reproducing interpolatory subdivision scheme, then for any C^{R+1} smooth $F : \mathbb{R} \to \mathbb{R}$, the approximant

$$F_h := (T^\infty y_h)(\cdot/h), \quad \text{where } (y_h)_k = F(hk), \; k \in \mathbb{Z},$$

satisfies

$$\max_{x \in I} |F(x) - F_h(x)| = O(h^{R+1}),$$

for any fixed compact interval I. This can be viewed as a very special case of the approximation theory of shift invariant spaces; a simple proof can be obtained using the argument in Dubuc [4, Theorem 20]. Dubuc's argument relies quite heavily on linearity of T.

Remark. The smoothness of T, not to be confused with the smoothness of the target function F, plays no role in this result. The approximation order R can be arbitrarily high while the smoothness of T can be arbitrarily low. For example, the four point scheme, with mask $(w_{-1}, w_0, w_1, w_2) = (-\omega, \frac{1}{2} + \omega, \frac{1}{2} + \omega, -\omega)$ has the highest critical Hölder regularity equals to 2 at $\omega = 1/16$, and the regularity approaches 0 when ω approaches the boundary of the interval $(-1/2, 1/2)$. (In the interior of the interval, the scheme is always convergent and has a positive Hölder regularity.) Yet the approximation order is 2 for all $\omega \in (-\frac{1}{2}, \frac{1}{2}) \backslash \{\frac{1}{16}\}$ and 4 at $\omega = 1/16$. See Figure 1(b).

We first develop our main result in the Euclidean setting. This can also be viewed as a "Proximity $\Rightarrow$ Approximation Order" result, in parallel to the earlier "Proximity $\Rightarrow$ Smoothness" results in [14, 10].

Theorem 2.1. Let S and T be two interpolatory subdivision schemes on real-valued data. Assume T is linear, convergent and reproduces Π_R for some $R \geqslant 1$. Moreover, assume that S and T satisfy an order R proximity condition [14], i.e. there exists a constant $C > 0$ such that for any bounded sequence x with $|\Delta x|_\infty$ sufficiently small,

$$\|Sx - Tx\|_\infty \leqslant C \, \Omega_R(x), \tag{2.3}$$

where

$$\Omega_R(x) := \sum_{\gamma \in \Gamma_R} \prod_{i=1}^{R} |\Delta^i x|_\infty^{\gamma_i}, \quad \Gamma_R := \left\{ \gamma = (\gamma_1, \cdots, \gamma_R) \;\middle|\; \gamma_i \in \mathbb{Z}^+, \sum_{i=1}^{R} i\, \gamma_i = R + 1 \right\}. \tag{2.4}$$

Then S possesses the same approximation order as T, i.e. for any C^{R+1} smooth $F : \mathbb{R} \to \mathbb{R}$, the approximant

$$F_h := (S^\infty y_h)(\cdot/h), \quad \text{where } (y_h)_k = F(hk), \; k \in \mathbb{Z},$$

satisfies

$$\max_{x\in I} |F(x) - F_h(x)| = O(h^{R+1}), \tag{2.5}$$

for any compact interval I.

Proof: 1° Since T is convergent and reproduces Π_R, by the linear theory (see, e.g., [8]) there are constants $1 > \rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_R > 0$, $C_1, C_2, \ldots, C_R > 0$ such that

$$|\Delta^i T^n x|_\infty \leqslant C_i\, \rho_i^n |\Delta^i x|_\infty, \quad \forall\, n = 0, 1, 2, \ldots, \quad i = 1, \ldots, R. \tag{2.6}$$

(The contracting factors ρ_i can be written down explicitly in terms of the smoothness of T, but this is irrelevant to the proof here. All we need is $\rho_i < 1$.) Choose a $k \in \mathbb{N}$ large enough such that

$$\max(C_1\rho_1^k, \ldots, C_R\rho_R^k) =: \alpha < 1.$$

Then if we write $\mathcal{T} := T^k$, we have

$$|\Delta^i \mathcal{T} x|_\infty \leqslant \alpha |\Delta^i x|_\infty, \quad \forall\, i = 1, \ldots, R. \tag{2.7}$$

Let $\mathcal{S} := S^k$. By [14, Lemma A.3], there exists $B > 0$ (independent of x) such that

$$|\mathcal{S}x - \mathcal{T}x|_\infty \leqslant B\, \Omega_R(x), \tag{2.8}$$

when $|\Delta x|_\infty$ is sufficient small, say,

$$|\Delta x|_\infty < \delta. \tag{2.9}$$

Below, all sequences x are bounded and satisfy (2.9).

Using $|\Delta^i x|_\infty \leqslant 2^i |x|_\infty$, we have

$$|\Delta^i (\mathcal{S} - \mathcal{T})x|_\infty \leqslant 2^i |(\mathcal{S} - \mathcal{T})x|_\infty \leqslant 2^i B\, \Omega_R(x).$$

Thus,

$$|\Delta^i \mathcal{S}x|_\infty \leqslant |\Delta^i \mathcal{T}x|_\infty + 2^i B\, \Omega_R(x) \overset{(2.7)}{\leqslant} \alpha |\Delta^i x|_\infty + 2^i B\, \Omega_R(x). \tag{2.10}$$

It then follows that

$$\begin{aligned}
\Omega_R(\mathcal{S}x) &= \sum_{\gamma\in\Gamma_R} \prod_{i=1}^R |\Delta^i \mathcal{S}x|_\infty^{\gamma_i} \\
&\leqslant \sum_{\gamma\in\Gamma_R} \prod_{i=1}^R \big(\alpha |\Delta^i x|_\infty + 2^i B\, \Omega_R(x)\big)^{\gamma_i} \\
&= \sum_{\gamma\in\Gamma_R} \alpha^{\gamma_1+\cdots+\gamma_R} \prod_{i=1}^R |\Delta^i x|_\infty^{\gamma_i} + \sum_{\gamma\in\Gamma_R} \sum_{\tau} T_{\gamma,\tau},
\end{aligned} \tag{2.11}$$

where $\tau = (\tau_1, \ldots, \tau_R) \leqslant \gamma = (\gamma_1, \ldots, \gamma_R)$ componentwise, $\tau \neq \gamma$, and $T_{\gamma,\tau}$ is of the form

$$T_{\gamma,\tau} = (\text{Constant})_{\gamma,\tau}\, |\Delta^1 x|_\infty^{\tau_1} \cdots |\Delta^R x|_\infty^{\tau_R}\, \Omega_R(x)^{\gamma_1+\cdots+\gamma_R-\tau_1-\cdots-\tau_R}.$$

Note that the definition of Γ_R implies $\sum \gamma_i \geqslant 2$. As such, the first term of the right-hand of (2.11) can be bounded as follows:

$$\sum_{\gamma\in\Gamma_R} \alpha^{\gamma_1+\cdots+\gamma_R} \prod_{i=1}^R |\Delta^i x|_\infty^{\gamma_i} \leqslant \sum_{\gamma\in\Gamma_R} \alpha^2 \prod_{i=1}^R |\Delta^i x|_\infty^{\gamma_i} = \alpha^2\, \Omega_R(x). \tag{2.12}$$

To bound the second term, notice that when $|\Delta x|_\infty < 1$,

$$\Omega_R(x) = \sum_{\gamma\in\Gamma_R} \prod_{i=1}^R |\Delta^i x|_\infty^{\gamma_i} \leqslant \sum_{\gamma\in\Gamma_R} \prod_{i=1}^R 2^{(i-1)\gamma_i} |\Delta x|_\infty^{\gamma_i} = \sum_{\gamma\in\Gamma_R} \prod_{i=1}^R 2^{(i-1)\gamma_i}\, |\Delta x|_\infty^{\sum \gamma_i} \leqslant \sum_{\gamma\in\Gamma_R} \prod_{i=1}^R 2^{(i-1)\gamma_i}\, |\Delta x|_\infty^2.$$

Let $B' := \sum_{\gamma\in\Gamma_R}\prod_{i=1}^{R} 2^{(i-1)\gamma_i}$, then

$$\Omega_R(x) < B'|\Delta x|_\infty^2 < 1, \quad \text{if } |\Delta x|_\infty < 1/\sqrt{B'}. \tag{2.13}$$

Therefore, when $|\Delta x|_\infty < 1/\sqrt{B'}$,

$$T_{\gamma,\tau} \leqslant \begin{cases} (\text{Constant})\, |\Delta x|_\infty\, \Omega_R(x), & \text{if } 0 < \tau \\ (\text{Constant})\, [\Omega_R(x)]^2, & \text{if } 0 = \tau \end{cases}. \tag{2.14}$$

So putting (2.11)-(2.14) together, there are constants $B'', B''' > 0$ independent of x such that

$$\begin{aligned} \Omega_R(\mathcal{S}x) &\leqslant (\alpha^2 + B''|\Delta x|_\infty + B'''\Omega_R(x))\, \Omega_R(x) \\ &< (\alpha^2 + B''|\Delta x|_\infty + B'''B'|\Delta x|_\infty)\, \Omega_R(x), \end{aligned}$$

when $|\Delta x|_\infty < 1/\sqrt{B'}$. But this implies that

$$\Omega_R(\mathcal{S}x) < \alpha\, \Omega_R(x) \quad \text{ when } |\Delta x|_\infty < \min\Big(\frac{1}{\sqrt{B'}}, \frac{\alpha-\alpha^2}{B''+B'B'''}\Big). \tag{2.15}$$

This is almost enough to allow us to conclude that

$$\Omega_R(\mathcal{S}^n x) < \alpha^n\, \Omega_R(x), \tag{2.16}$$

provided that we can guarantee $|\Delta\mathcal{S}^n x|_\infty$, $n = 1, 2, \ldots$, stay small enough for (2.15) to apply iteratively. By (2.10) and (2.15),

$$\begin{aligned} |\Delta\mathcal{S}x|_\infty &\leqslant \alpha|\Delta x|_\infty + 2B\Omega_R(x) \\ &< (\alpha + 2BB'|\Delta x|_\infty)|\Delta x|_\infty, \quad \text{when } |\Delta x|_\infty < \tfrac{1}{\sqrt{B'}}\ . \end{aligned}$$

So

$$|\Delta\mathcal{S}x|_\infty < |\Delta x|_\infty, \quad \text{when } |\Delta x|_\infty < \min\Big(\tfrac{1}{\sqrt{B'}}, \tfrac{1-\alpha}{2BB'}\Big). \tag{2.17}$$

Putting (2.9), (2.15) and (2.17) together, the final 'dense enough condition' we need for (2.16) to hold is

$$|\Delta x|_\infty < \min\Big(\delta, \frac{1}{\sqrt{B'}}, \frac{1-\alpha}{2BB'}, \frac{\alpha-\alpha^2}{B''+B'B'''}\Big). \tag{2.18}$$

2° For dense enough bounded sequences x, we have for any $n = 1, 2, \ldots$,

$$\begin{aligned} &|S^{kn}x - T^{kn}x|_\infty = |\mathcal{S}^n x - \mathcal{T}^n x|_\infty \\ =&|\mathcal{S}^n x - \mathcal{T}\mathcal{S}^{n-1}x + \mathcal{T}\mathcal{S}^{n-1}x - \cdots - \mathcal{T}^{n-1}\mathcal{S}x + \mathcal{T}^{n-1}\mathcal{S}x - \mathcal{T}^n x|_\infty \\ \leqslant&|(\mathcal{S}-\mathcal{T})(\mathcal{S}^{n-1}x)|_\infty + |\mathcal{T}(\mathcal{S}-\mathcal{T})(\mathcal{S}^{n-2}x)|_\infty + \cdots + |\mathcal{T}^{n-1}(\mathcal{S}-\mathcal{T})x)|_\infty \\ \leqslant&\sum_{i=0}^{n-1}|\mathcal{T}^i|_\infty|(\mathcal{S}-\mathcal{T})(\mathcal{S}^{n-1-i}x)|_\infty \\ \leqslant&C_T B\sum_{i=0}^{n-1}\Omega_R(\mathcal{S}^{n-1-i}x) \leqslant C_T B\sum_{i=0}^{n-1}\alpha^{n-1-i}\Omega_R(x) \leqslant \frac{C_T B}{1-\alpha}\,\Omega_R(x). \end{aligned} \tag{2.19}$$

In above, we use the fact that T is convergent and consequently, by the uniform boundedness principle, the operator norm $|T^i|_\infty$ is uniformly bounded by a constant denoted by C_T.

Since n is arbitrary in (2.19),

$$\|S^\infty x - T^\infty x\|_\infty \leqslant \frac{C_T B}{1-\alpha}\,\Omega_R(x).$$

If $x_i = f(ih)$ for some C^{R+1} smooth function f with $\max_{0\leqslant r\leqslant R+1}\|f^{(r)}\|_{L^\infty} < \infty$, then

$$|\Delta^i x|_\infty = O(h^i), \quad i = 1, 2, \ldots, R,$$

and

$$\Omega_R(x) = O(h^{R+1}).$$

Hence

$$\begin{aligned}\|(S^\infty x)(\cdot/h) - f\|_{L^\infty} &\leqslant \|(S^\infty x)(\cdot/h) - (T^\infty x)(\cdot/h)\|_{L^\infty} + \|(T^\infty x)(\cdot/h) - f\|_{L^\infty} \\ &= O(h^{R+1}) + O(h^{R+1}).\end{aligned}$$

Since any C^{R+1} smooth function f has bounded derivatives up to order $R+1$ when restricted to a compact interval, (2.5) is proved. ∎

Remark. The condition $\sum_i \gamma_i \geqslant 2$ plays a key role in the proof above; this condition is a easy consequence of the definition of Γ_R, and is related to the fact that a term of the form $|\Delta^{R+1}x|_\infty$ is forbidden in the proximity condition. Put differently, if we change the definition of Γ_R from that in (2.4) to $\{\gamma = (\gamma_1, \cdots, \gamma_{R+1}) : \gamma_i \in \mathbb{Z}^+, \sum_{i=1}^R i\gamma_i = R+1\}$, then the term $|\Delta^{R+1}x|_\infty$ shows up in the proximity condition and $\sum_i \gamma_i \geqslant 2$ does not hold; consequently, the whole proof of our "Proximity ⇒ Approximation Order" theorem falls apart.

Open question: is the absence of $|\Delta^{R+1}x|_\infty$ *truely* necessary for the theorem to hold ?

There is a similar issue in the "Proximity ⇒ Smoothness" theorems of [10, 14]. For smoothness, the absence of $|\Delta^{R+1}x|_\infty$ is absolutely necessary as shown by the following example: Let $(Sx)_{2i} = x_i$ and

$$(Sx)_{2i+1} = (Tx)_{2i+1} + \alpha\Delta^{2L}([x_{i-L}, \ldots, x_{i+L}]) \tag{2.20}$$

where α is a constant and T is the $2L$-point Deslauriers-Dubuc scheme. Note that S is actually a linear scheme, and $|Sx - Tx|_\infty \leqslant \alpha|\Delta^{2L}x|_\infty$. However, S can be made arbitrarily non-smooth by tweaking the constant α, meaning that there is no chance S shares the same smoothness as T. An example of a similar nature (but a bit more complicated) is due to Wallner [10, Section 3.2].

However, we suspect that the absence of $|\Delta^{R+1}x|_\infty$ is *not* necessary for Theorem 2.1 to hold. (In other words, we conjecture that the theorem is true with Γ_R in (2.4) relaxed to $\{\gamma = (\gamma_1, \cdots, \gamma_{R+1}) : \gamma_i \in \mathbb{Z}^+, \sum_{i=1}^R i\gamma_i = R+1\}$.) Firstly, notice that the linear S in (2.20) reproduces Π_{2L-1} no matter what the constant α is. Therefore, S shares the same approximation order with T. Secondly, a series of numerical experiments based on changing the constant α in (2.20) to a non-constant (to even a random variable i.i.d. in *scale* and *position*) strongly suggests that S continues to share the same approximation order with T.

Choice of metric. In order to transfer our approximation order result in the Euclidean setting to a manifold M, we need a metric on M for measuring approximation error. While the topology of any reasonable manifold is metrizable, we cannot simply choose an arbitrary metric compatible with the topology of M. This is because even if two metrics on M yield the same notion of convergence (i.e. the same topology), they may not yield the same notion of *rate of convergence*. For example, if $M = (-1, 1)$, $d(x, y) := |x - y|$ and $d'(x, y) = |x^3 - y^3|$, then (M, d) and (M, d') are homeomorphic but the sequence $(1/n)$ converges to 0 at different rates in the two metrics.

A manifold is by definition locally Euclidean and we need a metric d on M that locally provides the same sense of rate of convergence as the standard metric of $\mathbb{R}^n$ provides for $\mathbb{R}^n$. Precisely, we require that whenever d is restricted to any chart (U, ϕ), then the induced metric $d_\phi : \phi(U) \times \phi(U) \to \mathbb{R}$, $d_\phi(x, y) := d(\phi^{-1}(x), \phi^{-1}(y))$, is equivalent to the usual metric in $\mathbb{R}^n$ on any compact subset, i.e. for any compact $K \subset \phi(U)$, there exist constants $m_K, M_K > 0$ such that

$$m_K\|x - y\| \leqslant d_\phi(x, y) \leqslant M_K\|x - y\|, \quad \forall\, x, y \in K. \tag{2.21}$$

Such a metric can *always* be found for a manifold with a reasonable topology. Every reasonable manifold can be endowed with a Riemannian metric [9, Theorem 4, Chapter 9], and the (topological) metric induced by a Riemannian metric is guaranteed to possess our desired property [9, Lemma 7′, Chapter 9]. An alternate way is to regularly embed M to an Euclidean space and then restrict the usual metric from the ambient space to the surface.

Theorem 2.2. Let d be any metric on M with the property (2.21), S be a M-valued subdivision scheme based on a linear T as defined at the beginning of the section. Assume that T is convergent and reproduces Π_R for some $R \geqslant 1$. For any C^{R+1} smooth $F : \mathbb{R} \to M$ and any compact interval I,

$$F_h := (S^\infty y_h)(\cdot/h), \quad \text{where } (y_h)_k = F(hk),\ k \in \mathbb{Z},$$

is well-defined for any small enough $h > 0$ and satisfies

$$\max_{x \in I} d(F(x), F_h(x)) = O(h^{R+1}).$$

Proof: Since $F(I)$ can be covered by a finite number of charts, it suffices to prove the theorem for the case when $F(I)$ lies within a single chart (U, ϕ). Then by the proximity inequality in [14, Theorem 4.1], if $(y_k)_k$ is any dense enough sequence in a compact subset K of U, then

$$\|S_\phi(\phi(y)) - T(\phi(y))\|_\infty \leqslant C_K \Omega_R(\phi(x))$$

for a constant C_K that may depend on K but independent of y; here S_ϕ is S written in local coordinates and $\phi(y)$ simply means the sequence $(\phi(y_k))_k$ in $\phi(K) \subset \phi(U) \subset \mathbb{R}^n$.

Applying Theorem 2.1 componentwise, if $(y_k)_k$ consists of the samples of the C^{R+1} smooth function $\phi \circ F$ at kh for those $k \in \mathbb{Z}$ such that $kh \in I$, then

$$\max_{x \in I} \|\phi \circ F(x) - S_\phi^\infty(\phi(y))(x/h)\| = O(h^{R+1}).$$

In virtue of (2.21), as $h \downarrow 0$,

$$\max_{x \in I} d(F(x), S^\infty(y)(x/h))\| \asymp \max_{x \in I} \|\phi \circ F(x) - S_\phi^\infty(\phi(y))(x/h)\| = O(h^{R+1}).$$

■

Remark. Indeed we need only the upper bound in (2.21) for the proof to go through. Without the lower bound, all that it means is that the metric on M may make the approximation rate looks higher than that observed in local coordinates and the Euclidean metric.

3 Quasi-Interpolants ?

Stimulated by our main result in this paper, it is natural to ask if **quasi-interpolants for manifold-valued data** can be constructed based on general, not necessarily interpolatory, subdivision schemes.

We review the basics of quasi-interpolation by following the expositions in [2] and [1, Chapter 4]. Let ϕ be a compactly supported function such that $S(\phi) := \operatorname{span}(\phi(\cdot - k) : k \in \mathbb{Z}) \supseteq \Pi_R$. Let σ_h be the dilation operator $\sigma_h(f) := f(\cdot/h)$. In general, the best approximation of a $F \in C^{R+1}(\mathbb{R})$ by the scaled space $S_h(\phi) := \{\sigma_h(f) : f \in S(\phi)\}$ satisfies:

$$\operatorname{dist}(F, S_h(\phi)) := \inf_{F_h \in S_h(\phi)} \|F - F_h\| = O(h^{R+1}).$$

The basic theory of quasi-interpolation states that if a quasi-interpolant Q of the form

$$Q(F) = \sum_k \lambda(F(\cdot + k))\phi(\cdot - k),$$

where λ is a suitable local bounded linear functional, is exact on polynomials of degree up to R, i.e.

$$Q(\pi) = \pi, \quad \forall\, \pi \in \Pi_R,$$

then

$$\|F - \sigma_h Q \sigma_{1/h}(F)\|_\infty = O(h^{R+1}),$$

for any $F \in C^{R+1}(\mathbb{R})$. This is an appealing result because the best approximant of F from $S_h(\phi)$ is usually not easy to solve, on the other hand $Q(F)$ is easy to compute as it only involves local information of F.

For example, if ϕ is the cubic B-spline, then

$$Q(F) := \sum_k \Big(-\frac{1}{6}F(k-1) + \frac{4}{3}F(k) - \frac{1}{6}F(k+1)\Big)\, \phi(\cdot - k)$$

is exact for all cubics and hence $\|F-\sigma_h Q\sigma_{1/h}(F)\|_\infty = O(h^4)$ for F with bounded and continuous derivatives up to order 4. Of course, B-Splines are refinable, so $Q(F)$ can be efficiently computed by

$$Q(F) = T^\infty(F|_{\mathbb{Z}} * \lambda), \tag{3.1}$$

where λ is the discrete sequence $[-1/6, 4/3, -1/6]$, $*$ standards for discrete convolution, and T is the linear subdivision schemes of the cubic B-Spline, with mask $= [1, 4, 6, 4, 1]/8$.

In practice, we only subdivide a finite number of times. Since T is non-interpolatory, $Q(F)|_{2^{-j}\mathbb{Z}} \neq T^j(F|_{\mathbb{Z}} * \lambda)$. In merit of the fact that T is linear and stationary, an easy eigen-analysis shows

$$Q(F)|_{2^{-j}\mathbb{Z}} = \mu * T^j(F|_{\mathbb{Z}} * \lambda), \tag{3.2}$$

where $\mu = [1/6, 2/3, 1/6]$ is a left eigenvector of a subdivision matrix derived from T, with eigenvalue 1. The same formula (3.2) holds for higher degree B-splines, with longer filters in T, λ and μ. For interpolatory schemes, $\lambda = \mu = \delta$.

3.1 Nonlinear quasi-interpolants

We now describe generalizations of (3.1)-(3.2) to manifold-valued functions. Under our abstract vector bundle framework, these generalizations are all based on extending the usual discrete convolution $*$ between two real-valued sequences to a nonlinear convolution $\circledast$ between a manifold-valued sequences $y : \mathbb{Z} \to M$ and a real-valued sequence. Let V, f, g be as defined at the beginning of Section 2, then $\circledast = \circledast_{V,f,g}$ is formally defined by

$$(y \circledast a)_k = f_{y_k}\Big(\sum_\ell a_\ell\, g_{y_k}(y_{k-\ell})\Big).$$

Using this notation, the interpolatory scheme (2.2) can be rewritten as $(Sy)_{2\cdot} = y$, $(Sy)_{2\cdot+1} = y \circledast w$. Given a general linear subdivision scheme $(Ty)_{2\cdot} = y * a$, $(Ty)_{2\cdot+1} = y * b$, one can generalize it to M-valued data by

$$(Sy)_{2\cdot} = y \circledast a, \quad (Sy)_{2\cdot+1} = y \circledast b. \tag{3.3}$$

Special cases of this abstract scheme include:

- **[P]** $V = M \times \mathbb{R}^n$ (trivial bundle), $g_x(y) = (x, i(y)$ where $i : M \to \mathbb{R}^n$ is an embedding of M into $\mathbb{R}^n$, and $f_x(v) = i^{-1}$(the point in $i(M)$ closest to v), then S is a so-called **projection scheme** [11, 13];
- **[D]** $V = TM$ (tangent bundle), $g_x(y) = \log_x(M)$ and $f_x(v) = \exp_x(v)$, then S is **Donoho's** log-exp **scheme** [7].

In our study of the smoothness equivalence problem [14], we discovered that, in general, the S above does not share the same smoothness as the underlying linear scheme T; however the following modified scheme does:

$$(Sy)_{2\cdot} = y \circledast a, \quad (Sy)_{2\cdot+1} = y \boxtimes_w b, \tag{3.4}$$

where $(w_\ell)_\ell$ is an interpolatory subdivision mask with a high enough approximation order and

$$(y \boxtimes_w b)_k := f_{z_k}(\sum_\ell b_\ell g_{z_k}(y_{k-\ell}) \quad \text{with } z = y \circledast w.$$

Deciphering these abstract notations will tell us that this modified scheme is identical to the original scheme (3.3) in the case of the projection scheme **[P]**, but is genuinely different from Donoho's log-exp scheme in the case of $(V, g, f) = (TM, \log, \exp)$; in the latter case, we refer to (3.4) as the **modified** log-exp **scheme**. It was proved in [14] that the modified log-exp scheme satisfies smoothness equivalence, while Donoho's log-exp scheme was shown experimentally to fail smoothness equivalence.

Our nonlinear quasi-interpolants are then defined by taking (3.1) and replacing the linear T by any of the nonlinear S's in (3.3) or (3.4) and the linear $*$ by the nonlinear $\circledast = \circledast_{V,f,g}$, i.e.

$$QF := S^\infty(y \circledast \lambda). \tag{3.5}$$

For our computational experiments below, we use $Q_j F := (S^j(F|_{\mathbb{Z}} \circledast \lambda)) \circledast \mu$ as an approximation to $(QF)|_{2^{-j}\mathbb{Z}}$. (Note: unlike the linear case, $(QF)|_{2^{-j}\mathbb{Z}} \neq Q_j F$.)

The natural theoretical question, then, is:

$$\max_{x \in I} d(F(x), \sigma_h Q \sigma_{1/h} F(x)) \stackrel{?}{=} O(h^{R+1}). \tag{3.6}$$

3.2 Empirical breakdown of approximation order equivalence

While Theorem 2.2 answers the question affirmatively in the interpolatory case, the answer to (3.6) in general appears to be *negative* according to our computational experiments. In our experiments,

- the 1-periodic function $F : \mathbb{R} \to S^2$ in (1.1) is used as a target function for approximation,
- S is based on either the projection scheme, Donoho's log-exp scheme or the modified log-exp scheme,
- the underlying linear scheme T is either the degree 3 or 5 or 7 B-spline scheme,
- F is sampled on a grid of size $h = (2^{-1/4})^k$, $k = 10, \ldots, 40$, and then $\|F|_{2^{-5}h\mathbb{Z}} - Q_5 \sigma_{1/h} F\|_\infty$ is plotted against h in log scale.
- For comparison, we also plot in each case $\|F|_{2^{-5}h\mathbb{Z}} - (\sigma_h Q_{\text{Linear}} \sigma_{1/h} F)|_{2^{-5}h\mathbb{Z}} * \mu\|_\infty$ against h, where Q_{Linear} is the linear quasi-interpolant (3.1) applied componentwise to F. This serves as a basis of comparison because from the theory of quasi-interpolation we know that $\|F - \sigma_h Q_{\text{Linear}} \sigma_{1/h} F\|_\infty = O(h^4)$, $O(h^6)$, $O(h^8)$ when T is the B-spline scheme with degree 3,5,7, respectively.

The resulted plots are shown in Figure 2. By just looking at the degree 3 case it seems like that all three strategies of adapting T to the nonlinear setting satisfy approximation equivalence: all three nonlinear schemes occur to yield $O(h^4)$ accurate quasi-interpolants. This is encouraging if we take the point of view that the cubic B-spline is a defacto industrial standard. In the degree 5 case, only the modified log-exp scheme can live up to the $O(h^6)$ accuracy of the underlying linear scheme. The other two quasi-interpolants only appear to be $O(h^4)$ accurate. In the degree 7 case, modified log-exp continues to defeat Donoho's log-exp and projection, but falls two orders short of the hope-for $O(h^8)$ accuracy.

Conclusion: All three nonlinear methods (projection scheme, log-exp and modified log-exp), and also a few others not documented here, fail to produce quasi-interpolants that satisfy approximation order equivalence.

This kind of breakdown is not unfamiliar from our earlier study of smoothness equivalence [13, 12, 14]: several schemes that fist appear to be natural fail to satisfy smoothness equivalence.

Probably a better scheme is waiting to be discovered.

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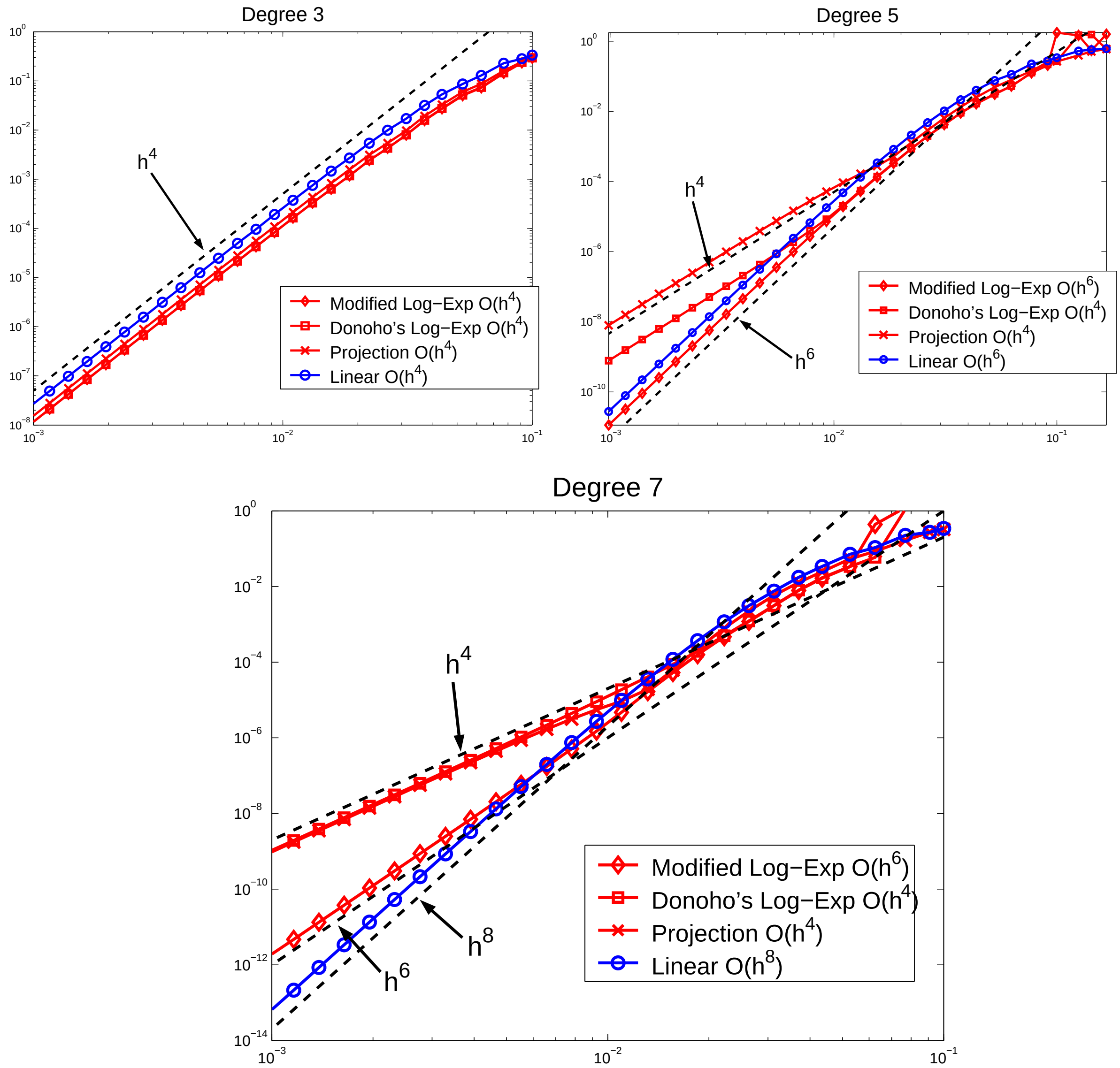

Figure 2: Breakdown of Approximation Order Equivalence for our proposed nonlinear quasi-interpolants

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