INVARIANCE PROPERTY OF PROXIMITY CONDITIONS IN NONLINEAR SUBDIVISION

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ABSTRACT. Proximity conditions are used extensively in the analysis of smoothness and approximation order properties of subdivision schemes for manifold-valued data. While these properties under question are independent of choice of coordinates on the manifold, it is not known whether the proximity condition itself is invariant under arbitrary change of coordinates. In this note, we answer this question to the affirmative, i.e. we prove that the proximity condition is satisfied in one coordinate system if and only if it is satisfied in any other coordinate system. In passing, we prove a connection between the general proximity condition and an alternate proximity condition used in the interpolatory case. This interpolatory proximity condition also enjoys the same invariance under change of coordinates.

1. Introduction

Subdivision algorithms and multiscale representations for manifold-valued data have been a subject of recent interest. While many different subdivision schemes proposed for manifold-valued data are relatively easy to implement, they are nontrivial to analyze. So far, the only analysis tool is the so-called *proximity condition* first introduced in [9, 8] and later generalized in [14]. It is used extensively in the smoothness analysis of subdivision schemes for manifold-valued data [13, 12, 14, 17, 2, 5, 4, 10, 3, 11], and also in the analysis of approximation order [15, 16, 6].

If a parametric curve $c : [0,1] \to M$ on a differentiable manifold M is smooth when expressed in one coordinate system on M, then by the very definition of a differentiable structure c has to be smooth when viewed in any other coordinate system. Similarly, if $c_h : [0,1] \to M$ is an approximation to c and we observe a certain approximation rate, say $O(h^R)$, in one coordinate system, then the same rate must be observed in any other coordinate system. This is simply because any change of coordinate map $\chi = \phi \circ \varphi^{-1} : \varphi(U) \to \phi(V)$ (see Figure 1) from one bounded domain to another must have a derivative uniformly bounded above and below, and therefore $(\sup \|\chi'\|)^{-1}\|p-q\| \leq \|\chi(p)-\chi(q)\| \leq (\sup \|\chi'\|)\|p-q\|$, for all $p, q \in \varphi(U)$. Since the proximity condition is used to infer smoothness and approximation order properties of subdivision curves on manifolds, a natural open question is whether the proximity condition itself is also independent of choice of coordinates.

In the first version appeared in the literature [9], the proximity condition between a nonlinear subdivision scheme S and a linear scheme S_{lin} reads:

$$\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty \leq C\|\Delta\mathbf{x}\|_\infty^2. \tag{1.1}$$

Date: August 28, 2011.

2000 *Mathematics Subject Classification.* 41A25, 26B05, 22E05, 68U05.

Key words and phrases. Nonlinear subdivision, Proximity condition, Manifold, Coordinate independence, Hermite-Biehler theorem.

Gang Xie's research was sponsored by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, Ministry of Education, P. R. China.

Thomas Yu's research was partially supported by the National Science Foundation grants DMS 0542237 and DMS 1115915. We thank Tom Duchamp for extensive discussions.

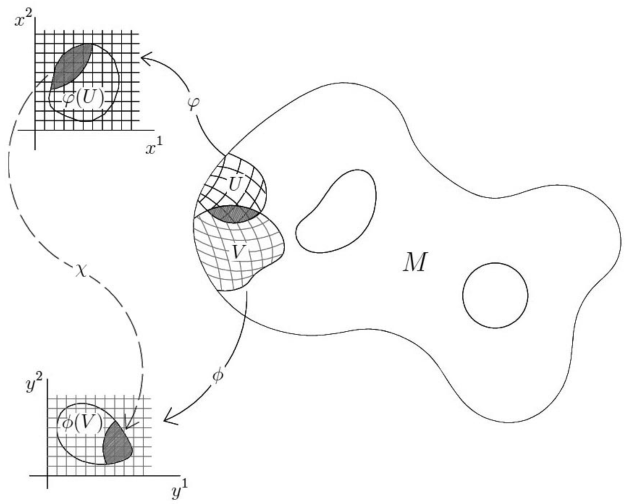

FIGURE 1. Change of coordinates

This condition guarantees that S is both convergent (for dense enough initial data) and C^1 whenever S_{lin} is C^1, also known as the "C^1 equivalence" property. A more general proximity condition which guarantees C^k equivalence is given in [8, Definition 3]; this condition is further improved to the following form [14]:

$$\|\Delta^{j-1}S\mathbf{x} - \Delta^{j-1}S_{\text{lin}}\mathbf{x}\|_\infty \le C\,\Omega_j(\mathbf{x}), \quad j = 1, \dots, k, \tag{1.2}$$

where

$$\Omega_j(\mathbf{x}) := \sum_{\gamma\in\Gamma_j}\prod_{i=1}^{j}\|\Delta^i\mathbf{x}\|_\infty^{\gamma_i}, \quad \Gamma_j := \left\{\gamma = (\gamma_1, \cdots, \gamma_j) \,\middle|\, \gamma_i \in \mathbb{Z}^+, \sum_{i=1}^{j} i\,\gamma_i = j+1\right\}. \tag{1.3}$$

It is proved in [14, Theorem 2.4] that the proximity condition (1.2)-(1.3) guarantees a C^k smoothness equivalence property of S and S_{lin}. We refer to (1.2) as the **order k proximity condition**.

Some examples of proximity condition are shown below:

Order k	Order k proximity condition
1	$\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\| \lesssim \|\Delta\mathbf{x}\|^2$
k	Order k proximity condition = Order $k-1$ condition +
2	$\|\Delta^1 S\mathbf{x} - \Delta^1 S_{\text{lin}}\mathbf{x}\| \lesssim \|\Delta\mathbf{x}\|^3 + \|\Delta\mathbf{x}\|\|\Delta^2\mathbf{x}\|$
3	$\|\Delta^2 S\mathbf{x} - \Delta^2 S_{\text{lin}}\mathbf{x}\| \lesssim \|\Delta\mathbf{x}\|^4 + \|\Delta\mathbf{x}\|\|\Delta^3\mathbf{x}\| + \|\Delta^2\mathbf{x}\|^2$
4	$\|\Delta^3 S\mathbf{x} - \Delta^3 S_{\text{lin}}\mathbf{x}\| \lesssim \|\Delta\mathbf{x}\|^5 + \|\Delta\mathbf{x}\|\|\Delta^4\mathbf{x}\| + \|\Delta\mathbf{x}\|^2\|\Delta^3\mathbf{x}\| + \|\Delta^2\mathbf{x}\|\|\Delta^3\mathbf{x}\| + \|\Delta\mathbf{x}\|\|\Delta^2\mathbf{x}\|^2$
5	$\|\Delta^4 S\mathbf{x} - \Delta^4 S_{\text{lin}}\mathbf{x}\| \lesssim \|\Delta\mathbf{x}\|^6 + \|\Delta\mathbf{x}\|^4\|\Delta^2\mathbf{x}\| + \|\Delta\mathbf{x}\|\|\Delta^5\mathbf{x}\| + \|\Delta\mathbf{x}\|^2\|\Delta^2\mathbf{x}\|^2 + \|\Delta\mathbf{x}\|^2\|\Delta^4\mathbf{x}\| +$ $\|\Delta\mathbf{x}\|^3\|\Delta^3\mathbf{x}\| + \|\Delta\mathbf{x}\|\|\Delta^2\mathbf{x}\|\|\Delta^3\mathbf{x}\| + \|\Delta^2\mathbf{x}\|\|\Delta^4\mathbf{x}\| + \|\Delta^3\mathbf{x}\|^2$

Notice the differencing order $j-1$ on the left-hand side of (1.2) and the 'combined degree' $j+1$ on the right-hand side of (1.3). This subtle feature of the proximity condition accounts for a number of technicalities in the analysis. Notice also that the term $\|\Delta^{j+1}\mathbf{x}\|_\infty$ is forbidden on the right-hand side.

The main goal of this paper is to show that the proximity condition is invariant under change of coordinates. In Section 2, we study a stronger version of (1.2) used for interpolatory schemes, and prove a connection between the two conditions. Our final result is that both proximity conditions enjoy the same invariance under change of coordinates.

Generality. In formulating our result for the coordinate independence of proximity condition, we assume only that $\mathcal{S}$ is a general intrinsically defined subdivision scheme on M, see the definition at the beginning of Section 3. Besides the basic subdivision structure, we do *not* assume that $\mathcal{S}$ is a smooth perturbation of an underlying linear scheme S_{lin} in any sense. In particular, our analysis does not rely on any Taylor expansion of $\mathcal{S}$ typically seen in our previous papers. The proof uses only the fact that the intrinsically defined scheme $\mathcal{S}$, when written in two different charts, result in two nonlinear subdivision schemes S and $\overline{S}$ related by $\overline{S} = \chi \circ S \circ \chi^{-1}$; and the proof only uses the Taylor expansion of the change of coordinate map χ.

Discussion. On the one hand, there is a strong indication from previous work that the invariance result ought to be true. In [17], we obtain a sufficient condition pertaining to a retraction map $f : TM \to M$ in order for the single basepoint scheme, constructed based on f, to satisfy the order 3 proximity condition with the underlying linear scheme. This condition has the form '$P_f = 0$' where P_f is a certain differential expression derived from f; and we have

$$P_f = 0 \Rightarrow \text{Order 3 proximity condition} \Rightarrow C^3 \text{ equivalence.} \tag{1.4}$$

It is further shown in [17] that in fact P_f is a type $(1, 3)$ tensor on the manifold, which in particular means that the $P_f = 0$ condition is coordinate independent. Moreover, a geometric interpretation for this tensor is found by Tom Duchamp; see [2]. Since both the first and the third conditions in the trilogy (1.4) are coordinate independent, it seems unlikely that the proximity condition sandwiched in between is coordinate dependent.

On the other hand, in the general setting we consider here it is unclear if the claimed invariance result should hold even in the simplest case (1.1). When using the proximity condition in the intrinsic setup (see Section 3), we study the difference between S and S_{lin} applied to an $\mathbb{R}^n$-valued sequence $\mathbf{x}$ representing points on a manifold expressed in terms of an arbitrarily chosen coordinate chart; here n is the dimension of the manifold. If we choose a different chart, we then study the difference between $\overline{S} = \chi \circ S \circ \chi^{-1}$ again with the *same* S_{lin} — ***not*** $\chi \circ S_{\text{lin}} \circ \chi^{-1}$ — applied to the sequence $\overline{\mathbf{x}} = \chi(\mathbf{x})$, where χ is the change of coordinate map. From this consideration, it seems at first glance that the proximity calculations carried out in the two charts are not comparable, suggesting that the invariance result may not hold.

This paradox is dissolved by the fact that S_{lin} and $\chi \circ S_{\text{lin}} \circ \chi^{-1}$ in turn satisfy the order k proximity condition. See result (II) in the proof of our main result. Note that the nonlinear $\chi \circ S_{\text{lin}} \circ \chi^{-1}$ is just the linear scheme S_{lin} disguised by a nonlinear change of coordinates, and clearly shares the same smoothness as S_{lin}; so in hindsight one may think that this intermediate result is not surprising.However, beware of the situation that there is no known converse result to the "Proximity $\Rightarrow$ Smoothness equivalence" theorem.[1]

The invariance result developed in this paper is directly used in the recent work [2]. In virtue of this result, we are free to choose any coordinate system to carry out the proximity calculation. In part of the analysis carried out in [2], we show that the use of geodesic polar coordinates can substantially simplify the analysis.

[1]It was pointed out by one of the anonymous referees that (II) can be viewed as a special case of the main result in [14] or that in [5]. The idea is to simply view S_{lin} as the kind of log-exp scheme in [14] or [5] on the flat manifold $\mathbb{R}^n$. We still provide a proof of (II) for the sake of self-contained-ness.

2. Proximity condition for interpolatory schemes

There is a simpler – but stronger – proximity condition, used only in the interpolatory case [13, 12, 14, 4, 15], which reads:

$$\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty = O(\Omega_k(\mathbf{x})). \tag{2.1}$$

Since $\|\Delta^k \mathbf{y}\|_\infty \leq 2^k \|\mathbf{y}\|_\infty$ and $\Omega_k(\mathbf{x}) = O(\Omega_{k'}(\mathbf{x}))$ if $k' > k$, we have

$$\text{Proximity condition (2.1)} \Rightarrow \text{Proximity condition (1.2)}.$$

Therefore the C^k smoothness equivalence property of S and S_{lin} is also guaranteed by this stronger proximity condition. Moreover, it is proved in [15] that, in the interpolatory setting, the same condition (2.1) implies an approximation order equivalence property. We refer to (2.1) as the **interpolatory order k proximity condition**.

In this section, we show that, under the interpolatory assumption, the two proximity conditions are actually equivalent. This explains at a higher level of generality why, in the cited papers above, the seemingly much stronger proximity condition (2.1) can always be established.

Notice that when S and S_{lin} are interpolatory, the sequence $S\mathbf{x} - S_{\text{lin}}\mathbf{x}$ is of the form

$$\cdots 0, \mathbf{d}_{-1}, 0, \mathbf{d}_0, 0, \mathbf{d}_1, 0, \mathbf{d}_2, \cdots, \tag{2.2}$$

with $\mathbf{d}_i = (S\mathbf{x} - S_{\text{lin}}\mathbf{x})_{2i+1}$. It is then easy to check that $\|\Delta^k(S\mathbf{x} - S_{\text{lin}}\mathbf{x})\|_\infty = k\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty$, for $k = 1, 2$. The situation for $k > 2$ is trickier, and we have the following result:

Theorem 1. *Under the interpolatory assumption, for any differencing order $k \geq 1$, there exists a constant $C_k > 0$ such that*

$$C_k \|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty \leq \|\Delta^k(S\mathbf{x} - S_{\text{lin}}\mathbf{x})\|_\infty \leq 2^{k-1}\|S\mathbf{x} - S_{\text{lin}}\mathbf{x}\|_\infty. \tag{2.3}$$

Consequently, the two proximity conditions (2.1) *and* (1.2) *are equivalent under the interpolatory assumption.*

The upper bound in (2.3) is trivial. The lower bound follows immediately from the following lemma:

Lemma 2. *While the operator*

$$\Delta^k : \ell^\infty \to \ell^\infty$$

is far from invertible, it has a stable inverse when restricted to the (closed) subspace of bounded sequences of the form (2.2).

This lemma manifests the principle that "an ill-posed inverse problem maybe solved by exploiting apriori information". Its proof relies on the Hermite-Biehler theorem; see Appendix A.

3. Main Result

Let M be a differentiable manifold of dimension n. And d is a metric on M such that whenever d is restricted to any chart (U, ϕ), then the induced metric $d_\phi : \phi(U) \times \phi(U) \to \mathbb{R}$, $d_\phi(x, y) := d(\phi^{-1}(x), \phi^{-1}(y))$, is equivalent to the usual metric in $\mathbb{R}^n$ on any compact set, i.e. for any compact $K \subset \phi(U)$, there exist constants $m_K, M_K > 0$ such that

$$m_K \|x - y\|_2 \leq d_\phi(x, y) \leq M_K \|x - y\|_2, \quad \forall x, y \in K.$$

Such a metric can always be found for a manifold with a reasonable topology.

For every $\delta > 0$ and $L \in \mathbb{N},\ L > 1$, write $M_\delta^L := \{(x_1, x_2, \cdots, x_L) \mid x_i \in M,\ \max_i d(x_i, x_{i+1}) < \delta\}$, and $M_\delta^\infty := \{(x_i)_{i=-\infty}^{+\infty} \mid x_i \in M,\ \sup_i d(x_i, x_{i+1}) < \delta\}$. A map $\mathcal{S} : M_\delta^\infty \to \ell(M)$ is called a subdivision scheme defined on M if there exist $L, m \in \mathbb{Z},\ L > 1$ and continuous maps $Q_0, Q_1 :\ M_\delta^L \to M$ such that

$$(\mathcal{S}x)_{2i+\sigma} = Q_\sigma(x_{i-m+1}, \cdots, x_{i-m+L}), \quad \sigma = 0, 1,\ i \in \mathbb{Z}.$$

Recall from [13, 12, 14, 17, 2, 5, 4, 10, 3, 15, 9, 8] that there are two ways to use the proximity condition for a subdivision scheme $\mathcal{S}$ defined on a manifold M:

(1) Pick a coordinate chart, and express $\mathcal{S}$ in local coordinates. The resulted nonlinear scheme S and the associated linear scheme S_{lin} both act on $\mathbb{R}^n$-valued data, where n is the intrinsic dimension of M.
(2) Pick an embedding of M into some $\mathbb{R}^N$, $N > n$, and express $\mathcal{S}$ in the ambient coordinates. The resulted nonlinear scheme S and the associated S_{lin} both act on $\mathbb{R}^N$-valued data.

Although the extrinsic approach (2) is quite useful, particularly in the symmetric space or Lie group settings [17, 13, 12, 4, 10], the general consensus is that it is more natural to analyze an intrinsically defined scheme using the intrinsic approach (1).

Now assume that we pick two coordinate charts φ and ϕ on M, express $\mathcal{S}$ in these two charts, and we call the resulting nonlinear subdivision operators S and $\overline{S}$. Let $\chi = \phi \circ \varphi^{-1}$ be a change of coordinate map, since $\mathcal{S}$ is invariantly defined on the manifold, it is necessarily true that

$$\overline{S} = \chi \circ S \circ \chi^{-1}. \tag{3.1}$$

We abuse notation and use χ and χ^{-1} to refer to the corresponding maps that transform a sequence of n-vectors in one coordinate system to a sequence in the other coordinate system.

We now state and prove our main result.

Theorem 3. *Assume that the underlying linear subdivision scheme S_{lin} reproduces Π_k, then both the order k proximity conditions (1.2) and* (2.1) *are invariant under change of coordinates.*

Proof. We first show that the proof of coordinate independence of the order k proximity condition (1.2) can be based on the following three intermediate results:

(I) There are constants $C_1,\ C_2 > 0$ such that $C_1\ \Omega_k(\overline{\mathbf{x}}) \le \Omega_k(\mathbf{x}) \le C_2\ \Omega_k(\overline{\mathbf{x}})$ for any dense enough sequence $\mathbf{x}$ and $\overline{\mathbf{x}} := \chi\mathbf{x}$.
(II) If S_{lin} reproduces Π_k, then S_{lin} and $\chi \circ S_{\text{lin}} \circ \chi^{-1}$ satisfy the order k proximity condition.
(III) If S and S_{lin} satisfy the order k proximity condition, then so do $\chi \circ S \circ \chi^{-1}$ and $\chi \circ S_{\text{lin}} \circ \chi^{-1}$.

(III) has the longest proof, (II) is the least expected initially – see the discussion in Section 1.

Assume that $\chi = \phi \circ \varphi^{-1}$ is a diffeomorphism between two bounded domains, and all the sequences $\mathbf{x}$, $\overline{\mathbf{x}}$ considered stay within the corresponding bounded domains. All the hidden constants in (I)-(III) above depend only on the size of the derivatives of χ and χ^{-1}. To see why (I)-(III) imply the invariance result, assume that S and S_{lin} satisfy the order k proximity condition, then

$$\begin{aligned}
\|\Delta^{k-1}(\overline{S}\overline{\mathbf{x}} - S_{\text{lin}}\overline{\mathbf{x}})\| &\overset{(3.1)}{=} \|\Delta^{k-1}(\chi \circ S \circ \chi^{-1}\, \overline{\mathbf{x}} - S_{\text{lin}}\overline{\mathbf{x}})\| \\
&\overset{(\text{II})}{\leq} \|\Delta^{k-1}(\chi \circ S \circ \chi^{-1}\overline{\mathbf{x}} - \chi \circ S_{\text{lin}} \circ \chi^{-1}\overline{\mathbf{x}})\| + O(\Omega_k(\overline{\mathbf{x}})) \\
&\overset{(\text{III})}{=} O(\Omega_k(\mathbf{x})) + O(\Omega_k(\overline{\mathbf{x}})) \\
&\overset{(\text{I})}{=} O(\Omega_k(\overline{\mathbf{x}})),
\end{aligned} \tag{3.2}$$

so $\overline{S}$ and S_{lin} also satisfy the order k proximity condition.

In order to prove the coordinate independence of the interpolatory proximity condition (2.1), adapt (3.2) to see that it suffices to show:

- (II′) If S and S_{lin} are interpolatory and S_{lin} reproduces Π_k, then S_{lin} and $\chi \circ S_{\text{lin}} \circ \chi^{-1}$ satisfy the interpolatory order k proximity condition (2.1).
- (III′) If S and S_{lin} satisfy the interpolatory order k proximity condition (2.1), then so do $\chi \circ S \circ \chi^{-1}$ and $\chi \circ S_{\text{lin}} \circ \chi^{-1}$.

Below, let $\chi^{(m)} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ be the m-th derivative of χ at the point $\mathbf{x}_0$ divided by $m!$. The fact that $\chi^{(m)}$ is an m-linear map will be used repeatedly. Write $A^x_j := \binom{x}{j}$; think of it as a degree j polynomial in the variable x with roots at 0, 1, ..., $j-1$. For $h \geq 0$, define $D_h := (\Delta^h \mathbf{x})_0$, so that $\mathbf{x}_h = \sum_{j=0}^h \binom{h}{j} D_j$. Then we have

$$\mathbf{x}_h - \mathbf{x}_0 = \sum_{j=1}^{h} \binom{h}{j} D_j = \sum_{j \geq 1} A^h_j D_j. \tag{3.3}$$

Proof of (I). For $h \geq 0$,

$$\begin{aligned}
\overline{\mathbf{x}}_h = \chi(\mathbf{x}_h) &= \chi(\mathbf{x}_0) + \chi^{(1)}(\mathbf{x}_h - \mathbf{x}_0) + \sum_{m=2}^{k} \chi^{(m)}(\mathbf{x}_h - \mathbf{x}_0)^m + O(\|\mathbf{x}_h - \mathbf{x}_0\|^{k+1}) \\
&= \chi(\mathbf{x}_0) + \chi^{(1)}(\mathbf{x}_h - \mathbf{x}_0) + \sum_{m=2}^{k} \chi^{(m)}\Big(\sum_{j\geq 1} \binom{h}{j} D_h\Big)^m + O(\|\mathbf{x}_h - \mathbf{x}_0\|^{k+1}) \\
&= \chi(\mathbf{x}_0) + \chi^{(1)}(\mathbf{x}_h - \mathbf{x}_0) + \sum_{m=2}^{k} \sum_{J=(j_1,\ldots,j_m)} A^h_J\, \chi^{(m)}(D_J) + O(\|\mathbf{x}_h - \mathbf{x}_0\|^{k+1}),
\end{aligned} \tag{3.4}$$

where $D_J := (D_{j_1}, \ldots, D_{j_m})$ and $A^x_J := A^x_{j_1} \cdots A^x_{j_m}$ is a polynomial in x with degree $|J| = j_1 + \cdots + j_m$. Since A^h_J is annihilated by Δ^k for any J with $|J| < k$, we have

$$(\Delta^k \overline{\mathbf{x}})_0 = \chi^{(1)}((\Delta^k \mathbf{x})_0) + \sum_{m=2}^{k} \sum_{|J| \geq k} (\Delta^k A_J)_0 \chi^{(m)}(D_J) + O(\|\Delta \mathbf{x}\|^{k+1}_\infty). \tag{3.5}$$

Therefore, we can choose $C_{1,0}$, dependent on the size of $\chi^{(1)}$ at $\mathbf{x}_0$, and $C_{k,m,J,0}$, dependent on k, m, J and the size of $\chi^{(m)}$ at $\mathbf{x}_0$, so that

$$\|(\Delta^k \overline{\mathbf{x}})_0\| \leq C_{1,0}\|((\Delta^k \mathbf{x})_0)\| + \sum_{m=2}^{k} \sum_{|J| \geq k} C_{k,m,J,0} \|(\Delta^{j_1} \mathbf{x})_0\| \cdots \|(\Delta^{j_m} \mathbf{x})_0\| + O(\|\Delta \mathbf{x}\|^{k+1}_\infty).$$

But since $\overline{\mathbf{x}}$ lives on a bounded domain on which χ has uniformly bounded derivatives, we have

$$\begin{aligned}\|\Delta^k\overline{\mathbf{x}}\|_\infty &\le C_1\|\Delta^k\mathbf{x}\|_\infty + \sum_{m=2}^{k}\sum_{|J|\ge k} C_{k,m,J}\|\Delta^{j_1}\mathbf{x}\|_\infty\cdots\|\Delta^{j_m}\mathbf{x}\|_\infty + O(\|\Delta\mathbf{x}\|_\infty^{k+1})\\ &= C_1\|\Delta^k\mathbf{x}\|_\infty + O(\Omega_{k-1}(\mathbf{x})).\end{aligned}$$

But then we also have $\|\Delta^{k'}\overline{\mathbf{x}}\|_\infty \le C_1\|\Delta^{k'}\mathbf{x}\|_\infty + O(\Omega_{k'-1}(\mathbf{x}))$ for any k' between 1 and k. Combining these estimates, we have

$$\Omega_k(\overline{\mathbf{x}}) = O(\Omega_k(\mathbf{x})).$$

By reversing the roles of $\mathbf{x}$ and $\overline{\mathbf{x}}$ and those of χ and χ^{-1} in the above argument, we get $\Omega_k(\mathbf{x}) = O(\Omega_k(\overline{\mathbf{x}}))$.

Proof of (II) and (II′). For (II), we estimate the difference between $(\Delta^{k-1}S_{\text{lin}}\overline{\mathbf{x}})_i$ with $(\Delta^{k-1}\chi\circ S_{\text{lin}}\circ\chi^{-1}\overline{\mathbf{x}})_i$ for each i. We can first assume that the index i is such that the two terms are determined only by $\mathbf{x}_0,\dots,\mathbf{x}_L$ (where L depends only on the support of mask of S_{lin}), and subsequently, as in the previous proof, extend the estimate to an arbitrary index i by shift invariance and the uniform boundedness of $\chi^{(m)}$.

We first note that

$$\begin{aligned}(S_{\text{lin}}\mathbf{x})_{2h+\sigma} &= \mathbf{x}_0 + \sum_\ell a_{2\ell+\sigma}(\mathbf{x}_{h-\ell}-\mathbf{x}_0)\\ &= \mathbf{x}_0 + \sum_\ell a_{2\ell+\sigma}\sum_{j\ge1}A_j^{h-\ell}D_j = \mathbf{x}_0 + \sum_{j\ge1}\left(\sum_\ell a_{2\ell+\sigma}A_j^{h-\ell}\right)D_j\end{aligned} \tag{3.6}$$

Now, rewrite $(S_{\text{lin}}\overline{\mathbf{x}})_{2h+\sigma}$ and $(\chi\circ S_{\text{lin}}\circ\chi^{-1}\overline{\mathbf{x}})_{2h+\sigma}$ as follows.

$$\begin{aligned}&(\chi\circ S_{\text{lin}}\circ\chi^{-1}\overline{\mathbf{x}})_{2h+\sigma} = \chi((S_{\text{lin}}\mathbf{x})_{2h+\sigma})\\ =&\chi(\mathbf{x}_0) + \chi^{(1)}\left((S_{\text{lin}}\mathbf{x})_{2h+\sigma}-\mathbf{x}_0\right) + \sum_{m=2}^{k}\chi^{(m)}\left(\left((S_{\text{lin}}\mathbf{x})_{2h+\sigma}-\mathbf{x}_0\right)^m\right) + O(\|(S_{\text{lin}}\mathbf{x})_{2h+\sigma}-\mathbf{x}_0\|^{k+1})\\ \overset{(3.6)}{=}&\chi(\mathbf{x}_0) + \chi^{(1)}\left(\sum_\ell a_{2\ell+\sigma}(\mathbf{x}_{h-\ell}-\mathbf{x}_0)\right) + \sum_{m=2}^{k}\chi^{(m)}\left(\sum_{j\ge1}\left(\sum_\ell a_{2\ell+\sigma}A_j^{h-\ell}\right)D_j\right)^m + O(\|\Delta\mathbf{x}\|_\infty^{k+1})\\ =&\chi(\mathbf{x}_0) + \sum_\ell a_{2\ell+\sigma}\chi^{(1)}(\mathbf{x}_{h-\ell}-\mathbf{x}_0) + \sum_{m=2}^{k}\sum_{\substack{J=(j_1,\cdots,j_m)\\|J|\le k}}\prod_{i=1}^{m}\left(\sum_\ell a_{2\ell+\sigma}A_{j_i}^{h-\ell}\right)\chi^{(m)}(D_J) + O(\Omega_k(\mathbf{x}))\end{aligned}$$

$$\begin{aligned}&(S_{\text{lin}}\overline{\mathbf{x}})_{2h+\sigma} = \sum_\ell a_{2\ell+\sigma}\chi(\mathbf{x}_{h-\ell})\\ =&\sum_\ell a_{2\ell+\sigma}\left[\chi(\mathbf{x}_0) + \chi^{(1)}(\mathbf{x}_{h-\ell}-\mathbf{x}_0) + \sum_{m=2}^{k}\chi^{(m)}((\mathbf{x}_{h-\ell}-\mathbf{x}_0)^m)\right] + O(\|\Delta\mathbf{x}\|_\infty^{k+1})\\ =&\chi(\mathbf{x}_0) + \sum_\ell a_{2\ell+\sigma}\chi^{(1)}(\mathbf{x}_{h-\ell}-\mathbf{x}_0) + \sum_\ell a_{2\ell+\sigma}\sum_{m=2}^{k}\chi^{(m)}\left(\left[\sum_{j=1}^{k}A_j^{h-\ell}D_j\right]^m\right) + O(\|\Delta\mathbf{x}\|_\infty^{k+1})\\ =&\chi(\mathbf{x}_0) + \sum_\ell a_{2\ell+\sigma}\chi^{(1)}(\mathbf{x}_{h-\ell}-\mathbf{x}_0) + \sum_{m=2}^{k}\sum_{\substack{J=(j_1,\cdots,j_m)\\|J|\le k}}\sum_\ell\left(a_{2\ell+\sigma}\prod_{i=1}^{m}A_{j_i}^{h-\ell}\right)\chi^{(m)}(D_J) + O(\Omega_k(\mathbf{x}))\end{aligned}$$

Hence,

$$(\chi \circ S_{\text{lin}} \circ \chi^{-1}\overline{\mathbf{x}})_{2h+\sigma} - (S_{\text{lin}}\overline{\mathbf{x}})_{2h+\sigma} = \sum_{m=2}^{k} \sum_{\text{length}(J)=m, |J|\le k} c_J^{2h+\sigma} \chi^{(m)}(D_J) + O(\Omega_k(\mathbf{x})), \tag{3.7}$$

where

$$c_J^{h,\sigma} := \prod_{i=1}^{m} \sum_{\ell} a_{2\ell+\sigma} A_{j_i}^{h-\ell} - \sum_{\ell} a_{2\ell+\sigma} \prod_{i=1}^{m} A_{j_i}^{h-\ell}. \tag{3.8}$$

Interpret $c_J^{h,\sigma}$ as a sequence c_J for which the $(2h+\sigma)$-th entry is $c_J^{h,\sigma}$; similarly interpret A_j^h as a (polynomial) sequence A_j for which the h-th entry is A_j^h. Then

$$c_J = \prod_{i=1}^{m} S_{\text{lin}} A_{j_i} - S_{\text{lin}} \prod_{i=1}^{m} A_{j_i}. \tag{3.9}$$

By assumption, S_{lin} reproduces Π_k, so when $|J| \le k$, c_J is also a polynomial of degree not exceeding $|J|$. Moreover, it is not hard to see that the two highest degree terms in $\prod_{i=1}^{m} S_{\text{lin}} A_{j_i}$ and $S_{\text{lin}} \prod_{i=1}^{m} A_{j_i}$ must cancel each other (see [14, 2]), so degree$(c_J) \le |J| - 2 \le k-2$, which also means $\Delta^{k-1} c_J = 0$. Therefore,

$$\|\Delta^{k-1}(\chi \circ S_{\text{lin}} \circ \chi^{-1}\overline{\mathbf{x}} - S_{\text{lin}}\overline{\mathbf{x}})\|_\infty = O(\Omega_k(\mathbf{x})) = O(\Omega_k(\overline{\mathbf{x}})). \tag{3.10}$$

The argument above can easily be adapted to prove (II$'$): If S_{lin} is interpolatory, the sequence c_J in (3.9) vanishes for $|J| \le k$, hence (3.10) holds even without the difference operator Δ^{k-1}, in other words $\chi \circ S_{\text{lin}} \circ \chi^{-1}$ and S_{lin} satisfy the interpolatory proximity condition.

Proof of (III). By assumption $\|\Delta^{\ell-1}(S\mathbf{x} - S_{\text{lin}}\mathbf{x})\|_\infty = O(\Omega_\ell(\mathbf{x}))$ for $\ell = 1, \ldots, k$. We now estimate the size of $\Delta^{\ell-1}(\chi S \chi^{-1}\overline{\mathbf{x}} - \chi S_{\text{lin}} \chi^{-1}\overline{\mathbf{x}}) = \Delta^{\ell-1}\chi S\mathbf{x} - \Delta^{\ell-1}\chi S_{\text{lin}}\mathbf{x}$.

For $0 \le h < \ell$,

$$\chi((S\mathbf{x})_h) = \chi(\mathbf{x}_0) + \chi^{(1)}((S\mathbf{x})_h - \mathbf{x}_0) + \sum_{m=2}^{\ell} \chi^{(m)}\big((S\mathbf{x})_h - \mathbf{x}_0\big)^m + O(\|(S\mathbf{x})_h - \mathbf{x}_0\|_\infty^{\ell+1}).$$

The same expression holds with S replaced by S_{lin}. We then have

$$(\chi(S\mathbf{x}))_h - (\chi(S_{\text{lin}}\mathbf{x}))_h = \Xi_h^1 + \Xi_h^2 + \Xi_h^3 \tag{3.11}$$

where

$$\begin{aligned} \Xi_h^1 &= \chi^{(1)}((S\mathbf{x})_h - (S_{\text{lin}}\mathbf{x})_h) \\ \Xi_h^2 &= \sum_{m=2}^{\ell} \Big\{ \chi^{(m)}\big((S\mathbf{x})_h - \mathbf{x}_0\big)^m - \chi^{(m)}\big((S_{\text{lin}}\mathbf{x})_h - \mathbf{x}_0\big)^m \Big\} \\ \Xi_h^3 &= O(\|(S\mathbf{x})_h - \mathbf{x}_0\|_\infty^{\ell+1}) + O(\|(S_{\text{lin}}\mathbf{x})_h - \mathbf{x}_0\|_\infty^{\ell+1}). \end{aligned} \tag{3.12}$$

Since $(\Delta^{\ell-1}\chi(S\mathbf{x}))_0 - (\Delta^{\ell-1}\chi(S_{\text{lin}}\mathbf{x}))_0 = (\Delta^{\ell-1}\Xi^1)_0 + (\Delta^{\ell-1}\Xi^2)_0 + (\Delta^{\ell-1}\Xi^3)_0$, it suffices to show that the size of each $(\Delta^{\ell-1}\Xi^i)_0$, $i = 1, 2, 3$, is bounded by $O(\Omega_\ell(\mathbf{x}))$.

By the assumption that S and S_{lin} satisfy the order k proximity condition, when $\ell \le k$,

$$\|(\Delta^{\ell-1}\Xi^1)_0\| = \|\chi^{(1)}((\Delta^{\ell-1}S\mathbf{x})_0 - (\Delta^{\ell-1}S_{\text{lin}}\mathbf{x})_0)\| = O(\Omega_\ell(\mathbf{x})).$$

Next, we estimate the size of $\Delta^{\ell-1}\Xi^3$. Since S_{lin} reproduces constants, for $0 \le h < \ell$, $\|(S_{\text{lin}}\mathbf{x})_h - \mathbf{x}_0\| = \|S_{\text{lin}}(\mathbf{x}_h - \mathbf{x}_0)\| = O(\|\Delta\mathbf{x}\|_\infty)$. Since, by assumption, S and S_{lin} satisfy the first proximity condition (1.1),

$\|(S\mathbf{x})_h - \mathbf{x}_0\| \le \|(S\mathbf{x})_h - (S_{\text{lin}}\mathbf{x})_h\| + \|(S_{\text{lin}}\mathbf{x})_h - \mathbf{x}_0\| = O(\|\Delta\mathbf{x}\|_\infty^2) + O(\|\Delta\mathbf{x}\|_\infty) = O(\|\Delta\mathbf{x}\|_\infty)$. Therefore,

$$\max_{0\le h<\ell} \|\Xi_h^3\| = O(\|\Delta\mathbf{x}\|^{\ell+1}) = O(\Omega_\ell(\mathbf{x})),$$

and of course we also have $\|(\Delta^{\ell-1}\Xi^3)_0\| = O(\Omega_\ell(\mathbf{x}))$.

We now deal with the more challenging term Ξ^2. For this part, not only do we have to use the assumed order k proximity condition between S and S_{lin} in its full power, we also need to use the estimate

$$\|\Delta^j S_{\text{lin}}\mathbf{x}\|_\infty = O(\|\Delta^j\mathbf{x}\|_\infty), \quad j \le k+1,$$

which follows from the linear theory. Notice also that

$$O(\Omega_{j_1}(\mathbf{x}))O(\|\Delta^{j_2}\mathbf{x}\|_\infty) = O(\Omega_{j_1+j_2}(\mathbf{x})), \tag{3.13}$$

$$O(\Omega_{j_1}(\mathbf{x}))O(\Omega_{j_2}(\mathbf{x})) = O(\Omega_{j_1+j_2+1}(\mathbf{x})), \tag{3.14}$$

$$\Omega_{j_2}(\mathbf{x}) = O(\Omega_{j_1}(\mathbf{x})) \text{ if } j_2 > j_1. \tag{3.15}$$

We rewrite Ξ_h^2 in the following way:

$$\begin{aligned}
\Xi_h^2 &= \sum_{m=2}^{\ell} \Big\{\chi^{(m)}\big((S\mathbf{x})_h - (S\mathbf{x})_0 + (S\mathbf{x})_0 - \mathbf{x}_0\big)^m - \chi^{(m)}\big((S_{\text{lin}}\mathbf{x})_h - (S_{\text{lin}}\mathbf{x})_0 + (S_{\text{lin}}\mathbf{x})_0 - \mathbf{x}_0\big)^m\Big\} \\
&= \sum_{m=2}^{\ell}\sum_{n=0}^{m}\binom{m}{n}\Big\{\chi^{(m)}\Big([(S\mathbf{x})_h - (S\mathbf{x})_0]^n, [(S\mathbf{x})_0 - \mathbf{x}_0]^n\Big) - \\
&\qquad\qquad \chi^{(m)}\Big([(S_{\text{lin}}\mathbf{x})_h - (S_{\text{lin}}\mathbf{x})_0]^n, [(S_{\text{lin}}\mathbf{x})_0 - \mathbf{x}_0]^n\Big)\Big\} \\
&= \sum_{m=2}^{\ell}\sum_{n=0}^{m}\binom{m}{n}\Big\{\chi^{(m)}\Big(\Big[\sum_{j=1}^{h} A_j^h(\Delta^j S\mathbf{x})_0\Big]^n, \Big[(S\mathbf{x})_0 - \mathbf{x}_0\Big]^{m-n}\Big) - \\
&\qquad\qquad \chi^{(m)}\Big(\Big[\sum_{j=1}^{h} A_j^h(\Delta^j S_{\text{lin}}\mathbf{x})_0\Big]^n, \Big[(S_{\text{lin}}\mathbf{x})_0 - \mathbf{x}_0\Big]^{m-n}\Big)\Big\} \\
&= \sum_{m=2}^{\ell}\sum_{n=0}^{m}\binom{m}{n}\sum_{J=(j_1,\dots,j_n)} A_J^j\, T_{m,n,J},
\end{aligned}$$

where

$$\begin{aligned}
T_{m,n,J} := \chi^{(m)}\Big((\Delta^{j_1}S\mathbf{x})_0, \dots, (\Delta^{j_n}S\mathbf{x})_0, \big[(S\mathbf{x})_0 - \mathbf{x}_0\big]^{m-n}\Big) - \\
\chi^{(m)}\Big((\Delta^{j_1}S_{\text{lin}}\mathbf{x})_0, \dots, (\Delta^{j_n}S_{\text{lin}}\mathbf{x})_0, \big[(S_{\text{lin}}\mathbf{x})_0 - \mathbf{x}_0\big]^{m-n}\Big).
\end{aligned} \tag{3.16}$$

Note that each $T_{m,n,J}$ is independent of h, and hence

$$(\Delta^{\ell-l}\Xi^2)_0 = \sum_{m=2}^{\ell}\sum_{n=0}^{m}\binom{m}{n}\sum_{J=(j_1,\dots,j_n)} (\Delta^{\ell-l}A_J)_0\, T_{m,n,J}.$$

Since $\Delta^{\ell-l}A_J \equiv 0$ for $|J| < \ell-1$, we only need to analyze those $T_{m,n,J}$ with $|J| \ge \ell-1$. The proof will be completed if we can show that every such $T_{m,n,J}$ can be bounded by $\Omega_\ell(\mathbf{x})$.

Notice that

$$(\Delta^j S\mathbf{x})_0 = \overbrace{(\Delta^j S_{\text{lin}}\mathbf{x})_0}^{=O(\|\Delta^j\mathbf{x}\|_\infty)} + \overbrace{(\Delta^j S\mathbf{x})_0 - (\Delta^j S_{\text{lin}}\mathbf{x})_0}^{=O(\Omega_{j+1}(\mathbf{x}))}, \quad \text{and}$$
$$(S\mathbf{x})_0 - \mathbf{x}_0 = \underbrace{(S_{\text{lin}}\mathbf{x})_0 - \mathbf{x}_0}_{=O(\|\Delta\mathbf{x}\|_\infty)} + \underbrace{(S\mathbf{x})_0 - (S_{\text{lin}}\mathbf{x})_0}_{=O(\|\Delta\mathbf{x}\|_\infty^2)}$$

Based on these splittings, we can then use the multi-linearity of $\chi^{(m)}$ to expand the first term on the right-hand side of (3.16) into 2^m terms; exactly one of these terms will cancel with the second term on the right-hand side of (3.16). Therefore, $T_{m,n,J}$ can be written as the sum of $2^m - 1$ terms each of the form

$$\chi^{(m)}(*_1, \ldots, *_n, *_{n+1}, \ldots, *_m) \tag{3.17}$$

where

$$*_i = \begin{cases} O(\|\Delta^j\mathbf{x}\|_\infty) \text{ or } O(\Omega_{j+1}(\mathbf{x})), & i \le n \\ O(\|\Delta\mathbf{x}\|_\infty) \text{ or } O(\|\Delta\mathbf{x}\|_\infty^2), & i > n \end{cases},$$

and, moreover, an argument of the *latter type*, i.e. an $O(\Omega_{j+1}(\mathbf{x}))$ term for $i \le n$ or an $O(\|\Delta\mathbf{x}\|_\infty^2)$ term for $i > n$, must show up ***at least once*** on the argument list $(*_1, \ldots, *_m)$ of (3.17). This, together with the the condition $j_1 + \cdots + j_n \ge \ell - 1$ and (3.13)-(3.15), imply that we can bound (3.17) by

$$O(\Omega_{j_1+\cdots+j_n+1}(\mathbf{x})) = O(\Omega_\ell(\mathbf{x})).$$

Proof of (III′). We use (3.11) and (3.12) with $\ell = k$ in the proof of (III). (Note that $\Xi_h^2 = 0$ if $k = 1$.) Assume S and S_{lin} satisfy the interpolatory order k proximity condition, we immediately have $\|\Xi^1\|_\infty = O(\Omega_k(\mathbf{x}))$ and $\|\Xi^3\|_\infty = O(\Omega_k(\mathbf{x}))$. For Ξ^2, we write

$$(S\mathbf{x})_h - \mathbf{x}_0 = \underbrace{(S_{\text{lin}}\mathbf{x})_h - \mathbf{x}_0}_{=O(\|\Delta\mathbf{x}\|_\infty)} + \underbrace{(S\mathbf{x})_h - (S_{\text{lin}}\mathbf{x})_h}_{O(\Omega_k(\mathbf{x})))},$$

and based on this splitting we decompose $\chi^{(m)}((S\mathbf{x})_h - \mathbf{x}_0)^m$ into 2^m terms, with exactly one term equal to $\chi^{(m)}((S_{\text{lin}}\mathbf{x})_h - \mathbf{x}_0)^m$. Therefore, each expression inside $\{\}$ in the definition of Ξ_h^2 can be written as a sum of $2^m - 1$ of the form $\chi^{(m)}(*_1, \ldots, *_m)$ where at least one $*_i$ is bounded by $O(\Omega_k(\mathbf{x}))$ and the remaining arguments are bounded by $O(\|\Delta\mathbf{x}\|_\infty)$ or $O(\Omega_k(\mathbf{x}))$, so each $\chi^{(m)}(*_1, \ldots, *_m)$ can be bounded by $O(\Omega_k(\mathbf{x}))$; and so is $\|\Xi^2\|_\infty$. □

Equivalence of extrinsic and intrinsic proximity conditions. Theorem 3 pertains to the proximity conditions formulated in an intrinsic way. What about the extrinsic formulation? We now argue that the intrinsic and extrinsic formulations are actually equivalent. Let $\phi : U \subset M \to \mathbb{R}^n$ be a coordinate chart on M and let $\Phi : M \to \mathbb{R}^N$ be a smooth embedding. Consider the regular surface $\Phi(U)$ in $\mathbb{R}^N$. We can find (with the proviso of possibly trimming down the size of U) an open set W in $\mathbb{R}^N$ so that $W \cap \Phi(M) = \Phi(U)$ and a so-called *preferred coordinate system on W relative to* $\Phi(U)$ [1], $\chi : W \subset \mathbb{R}^N \to \mathbb{R}^n \times (-1,1)^{N-n}$ so that $\chi(W)$ is the 'cylinder' $\phi(U) \times (-1,1)^{N-n}$, and $\chi(\Phi(U))$ is the cross-section $\phi(U) \times \{\mathbf{0}\}$ of the cylinder. Intuitively, χ^{-1} is a 'bulked-up' version of the parametrization $\Phi \circ \phi^{-1}$ of the surface $\Phi(U)$. See Figure 2.

Now $\chi : W \to \phi(U) \times (-1,1)^{N-n}$ is a diffeomorphism between two open sets in $\mathbb{R}^N$, and the manifold subdivision scheme $\mathcal{S}$ expressed in extrinsic coordinates (S) and in local coordinates ($\overline{S}$) are again related by (3.1), with the caveat that $S\mathbf{x}$ is only defined for input sequences $\mathbf{x}$ residing in $\Phi(U)$ and $\overline{S}\overline{\mathbf{x}}$ is defined only for sequences $\overline{\mathbf{x}}$ residing in the cross-section $\phi(U) \times \{\mathbf{0}\}$. Armed with this setup, the same argument for proving Theorem 3 can be used to prove that S and S_{lin} satisfy the extrinsic proximity condition if and only if $\overline{S}$ and S_{lin} satisfy the intrinsic proximity condition.

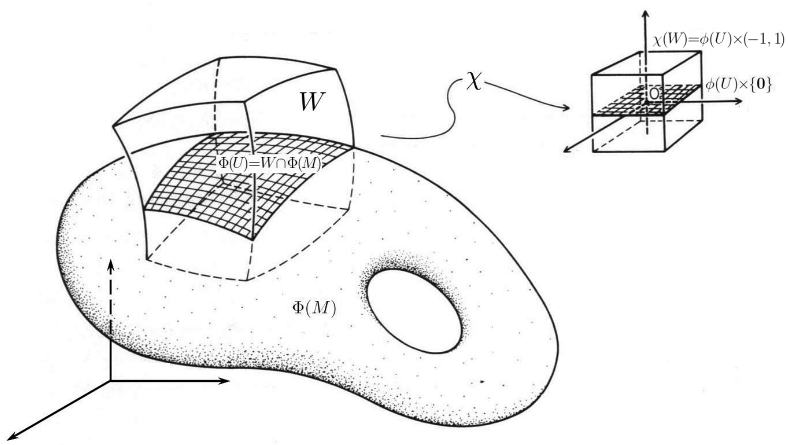

FIGURE 2. A 'bulk-up' of the parametrization map $\Phi \circ \phi^{-1}$ becomes a diffeomorphism χ. In this illustration, $n = 2$ and $N = 3$.

APPENDIX A. PROOF OF LEMMA 2

The linear shift-invariant operator Δ^k can be expressed as $(\Delta^k \mathbf{x})_n = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \mathbf{x}_{n+\ell}$. When $\mathbf{x}$ has the form of (2.2), i.e. $\mathbf{x}_{2n} = 0$ and $\mathbf{x}_{2n+1} = \mathbf{d}_n$, then Δ^k behaves exactly like *two* linear shift-invariant operators applied to the subsequence $\mathbf{d}$, as the following shows:

$$(\text{A.1}) \quad \begin{aligned} (\Delta^k \mathbf{x})_{2n} &= \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \mathbf{x}_{2n+\ell} = -\sum_{\ell=1,3,\dots} \binom{k}{\ell} \mathbf{x}_{2n+\ell} = -\sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m+1} \mathbf{d}_{n+m} =: (C_1 \mathbf{d})_n, \\ (\Delta^k \mathbf{x})_{2n+1} &= \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \mathbf{x}_{2n+1+\ell} = \sum_{\ell=0,2,\dots} \binom{k}{\ell} \mathbf{x}_{2n+1+\ell} = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} \mathbf{d}_{n+m} =: (C_0 \mathbf{d})_n. \end{aligned}$$

To prove Lemma 2, it suffices to show that one of the two convolution operators $C_0, C_1 : \ell^\infty \to \ell^\infty$ has a bounded inverse.

A convolution operator with a finitely supported impulse response has a stable inverse (i.e. the inverse exists as a bounded operator on ℓ^∞) if and only if the discrete-time Fourier transform of the impulse response does not change sign for all frequency or, equivalently, the z-transform of the impulse response has no zero on the unit circle. Therefore, it suffices to show that at least one of

$$P(z) := \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m} z^m, \quad Q(z) := \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k}{2m+1} z^m$$

has no root on the unit circle. Since

$$(\text{A.2}) \qquad (1+z)^k = P(z^2) + zQ(z^2),$$

by the Hermite-Biehler theorem[2] the roots of P are Q are simple, real, interlacing, and are of the same sign. (See Figure 3.) Since the binomial coefficients are positive, all the roots of P and Q must be negative. Therefore, it remains to show that at least one of P and Q does not have -1 as a root. But this is obvious from (A.2), as $P(-1)$ and $Q(-1)$ are the real and imaginary parts of $(1+i)^k$. □

[2]See [7, Theorem 1] for a self-contained treatment of this classical result.

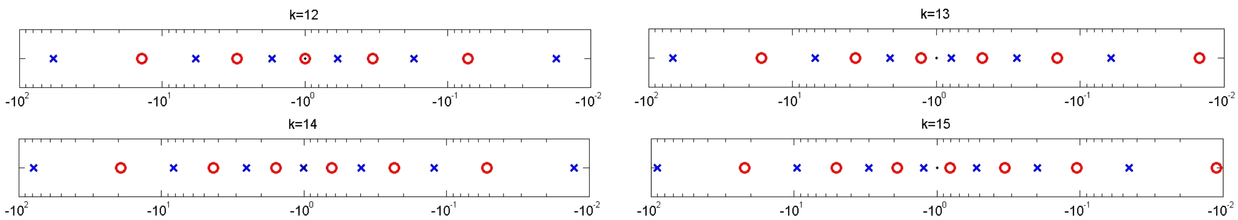

FIGURE 3. Interlacing roots of $P(z)$ ('$\times$') and $Q(z)$ ('$\circ$')

For pedagogical purpose, we give an alternative argument which does not require the Hermite-Biehler theorem or even Fourier analysis; however the argument can only be used to prove the lemma for $k \leq 8$. Using the contraction mapping theorem, it can be easily shown that a bounded convolution operator has a bounded inverse if its impulse response (a_n) is dominant at one entry, i.e. there exists an entry a_{n^*} such that $|a_{n^*}| > \sum_{n \neq n^*} |a_n|$. This furnishes an easy-to-check sufficient condition for stable deconvolution. In order to use this argument to show that one of C_0 or C_1 has a bounded inverse, we must have

$$\binom{k}{\lfloor k/2 \rfloor} > \sum_{m=\pm 1, \pm 2, \ldots} \binom{k}{\lfloor k/2 \rfloor + 2m}.$$

Since $\sum_m \binom{k}{2m} = \sum_m \binom{k}{2m+1} = 2^{k-1}$, the above inequality is equivalent to

$$\binom{k}{\lfloor k/2 \rfloor} > 2^{k-2},$$

which is only true for $k \leq 8$.

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