Proof of Proposition 1 in [2]: minimizing W_{Bobenko} would not give an approximation to a Clifford torus

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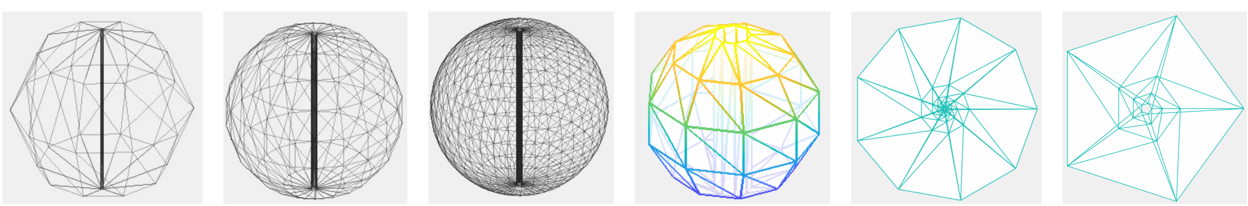
For every grid size m and n, we define a family of triangulated torus, denoted by $T_{m,n,\varepsilon}$, $\varepsilon \in (0, \pi/2)$, with the following properties:

1. It has mn vertices, $2mn$ triangles and all vertices have valence 6.
2. All vertices lie on a single sphere.
3. The diameter of the 'tunnel' of the torus goes to 0 as $\varepsilon \downarrow 0$.
4. The 'tunnel' of the torus is triangulated by $2n$ (long and skinny) triangles.

In details, it is defined as follows. Let $\Delta u = 2\pi/m$, $\Delta v = 2\pi/n$. For $i = 0, \ldots, m-1$, $j = 0, \ldots, n-1$, let

$$V_{i,j} = \begin{bmatrix} \sin(\tau(i\Delta u))\cos(j\Delta v) \\ \sin(\tau(i\Delta u))\sin(j\Delta v) \\ \cos(\tau(i\Delta u)) \end{bmatrix}, \quad \text{where } \tau(u) = \frac{(\pi - 2\varepsilon)u}{2\pi(1-1/m)} + \varepsilon. \tag{0.1}$$

Note that τ maps $[0, 2\pi(1-1/m)]$ to $[\varepsilon, \pi - \varepsilon]$. These $m \cdot n$ points on the unit sphere are the vertices of $T_{m,n,\varepsilon}$. For the triangulation, each $V_{i,j}$ is connected to the six neighbors $V_{i+1,j}$, $V_{i+1,j+1}$, $V_{i,j+1}$, $V_{i-1,j}$, $V_{i-1,j-1}$, $V_{i,j-1}$, forming also six triangular faces incident on the vertex. In above, the $+/-$ in the first and second indices are modulo m and modulo n, respectively.



$T_{5,10,.01}$ $T_{10,20,.05}$ $T_{20,40,.1}$ $T_{6,9,.2}$ $T'_{6,9,.2}$ $T'_{4,5,.3}$

Figure 1: Various spherical tori and a flattened torus based on a sphere inversion

Note that the $2n$ vertices $V_{0,j}$ and $V_{m-1,j}$, $j = 0, \ldots n-1$ are the vertices closest to the north and south pole, respectively. While these two group of vertices are the furthest apart geometrically, they are connected and form a vertical tunnel of the torus with $2n$ long skinny triangles.

Proposition 0.1. For every grid size (m, n), $W_{\text{Bobenko}}(T_{m,n,\varepsilon})$ decreases monotonically to 4π as $\varepsilon \downarrow 0$.

Proof. We divide the proof into three steps.
1° For any triangulated torus $T_{m,n,\varepsilon}$ defined above,

$$W_{\text{Bobenko}}(T_{m,n,\varepsilon}) = \frac{1}{2}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} W_{i,j}, \quad \text{where } W_{i,j} = \sum_{(i',j')\sim(i,j)} \beta(i',j') - 2\pi \tag{0.2}$$

and $\beta(i',j')$ is an angle formed by the circumscribed circles of the two triangles sharing the edge connecting (i,j) to (i',j') [1, Definition 1]. We then notice that

$$W_{i,j} = 0, \; \forall\, i \neq 0, \, m-1,$$

since the six vertices around (i,j) form a convex neighborhood lying on a common sphere [1, Proposition 1]. By symmetry, $W_{0,j}$ and $W_{m-1,j}$ share the same value for all $j = 0, \ldots, n-1$. So we have

$$W_{\text{Bobenko}}(T_{m,n,\varepsilon}) = n\, W_{0,0}.$$

It remains to show that $W_{0,0}$ decreases monotonicity to $4\pi/n$ as $\varepsilon \downarrow 0$.

2° To simplify computation, we take advantage of the Möbius invariance of W_{Bobenko} by applying a sphere inversion that maps the unit sphere to the $z = 0$ plane.[1] Specifically, we invert about the sphere with radius $\sqrt{2}$ and centered at the north pole $[0,0,1]^T$ of the unit sphere. This maps the south pole of the unit sphere to the origin, and the north pole to infinity; and it turns our spherical torus $T_{m,n,\varepsilon}$ to a flattened torus. See Figure 1.

Among the 7 vertices $V_{0,0}$, $V_{0,1}$, $V_{-1,0}$, $V_{-1,-1}$, $V_{0,-1}$, $V_{1,0}$, $V_{1,1}$ contributing to $W_{0,0}$, $V_{0,0}$, $V_{0,1}$, $V_{0,-1}$ are close to the north pole, they are sphere inverted to points far away from the origin; $V_{-1,0}$, $V_{-1,-1}$ are close to the south pole, they are mapped to points close to the origin. The last two neighbors $V_{1,0}$, $V_{1,1}$ are at an approximately constant distance from the north pole: their common polar angle is uniformly larger than, and approaches, $\pi/(m-1)$ as $\varepsilon \to 0$. Therefore, these 7 vertices are mapped to $V'_i \in \mathbb{R}^2$, $i = 0, 1, \ldots, 6$ (in the same cyclic order) with the form

$$V'_0 = \rho_1(\varepsilon)[1,0], \quad V'_1 = \rho_1(\varepsilon)[c,s], \quad V'_2 = \rho_2(\varepsilon)[1,0], \quad V'_3 = \rho_2(\varepsilon)[c,-s],$$

$$V'_4 = \rho_1(\varepsilon)[c,-s], \quad V'_5 = \rho_3(\varepsilon)[1,0], \quad V'_6 = \rho_3(\varepsilon)[c,s], \quad c = \cos(2\pi/n), \quad s = \sin(2\pi/n),$$

where $\underbrace{\rho_1(\varepsilon)}_{=\omega(1)} \gg \underbrace{\rho_3(\varepsilon)}_{=\Theta(1)} \gg \underbrace{\rho_2(\varepsilon)}_{=o(1)}$, as $\varepsilon \to 0$. By scale invariance, $W_{0,0}$ depends on ε through

$$\epsilon_1 := \rho_3(\varepsilon)/\rho_1(\varepsilon), \text{ and } \epsilon_2 := \rho_2(\varepsilon)/\rho_1(\varepsilon). \tag{0.3}$$

Clearly, ρ_1 (resp. ρ_2) increases (resp. decreases) as ε decreases. So $\epsilon_2(\varepsilon)$ is monotonic increasing for $\varepsilon \in (0, \pi/2)$.

Below, we only need the facts that $1 > \epsilon_1 > \epsilon_2 > 0$, ϵ_2 is monotone in ε and $\epsilon_1, \epsilon_2 \to 0$ as $\varepsilon \to 0$.

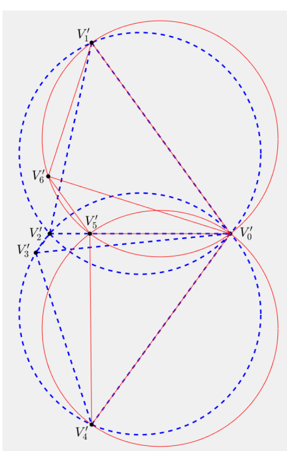


Figure 2: Computation of $W_{0,0}$ based on $V'_0 = [1,0]$, $V'_1 = [c,s]$, $V'_2 = \epsilon_2[1,0]$, $V'_3 = \epsilon_2[c,-s]$, $V'_4 = [c,-s]$, $V'_5 = \epsilon_1[1,0]$, $V'_6 = \epsilon_1[c,s]$. Among the six circumcircles, two pairs coincide due to the isosceles quadrilaterals $V'_0V'_2V'_3V'_4$ and $V'_0V'_5V'_6V'_1$, thus only 4 circles are seen; they are also divided into two groups of three (or two rather), displayed in solid and dashed line styles, which are oriented differently when viewed from the outside of the plane. This is caused by the 'folding' of the neighborhood of V'_0, and is also why $W_{0,0}$ does not vanish despite all the vertices are coplanar. In this figure, $(c,s) = (\cos(2\pi/5), \sin(2\pi/5))$, i.e. $n = 5$, $(\epsilon_1, \epsilon_2) = (.3, .1)$ and it does not come from an ε via (0.3). The proof also does not rely explicitly on (0.3).

3° To calculate $W_{0,0} = \sum_{i=1}^{6} \beta_i - 2\pi$ we analyze each angle β_i between the circumcircles of the two triangles sharing the edge $e_i = \overline{V'_0V'_i}$, now thought of as functions of (ϵ_1, ϵ_2).

[1]The approach also introduces a family of planar tori needed in another result.

We first notice that the two triangles $V_0'V_3'V_2'$, $V_0'V_3'V_4'$ sharing e_3 are co-cyclic because $V_0'V_2'V_3'V_4'$ form an isosceles quadrilateral. Therefore β_3 is either 0 or π. A closer inspection based on orientation (or simply applying the formula below) tells us that $\beta_3 = 0$. Similarly, $\beta_6 = 0$. Next, we show that

$$\lim_{\epsilon_1,\epsilon_2\to 0} \beta_1 = \pi = \lim_{\epsilon_1,\epsilon_2\to 0} \beta_4, \quad \lim_{\epsilon_1,\epsilon_2\to 0} \beta_2 = \frac{2\pi}{n} = \lim_{\epsilon_1,\epsilon_2\to 0} \beta_5. \tag{0.4}$$

These limits are not hard to see geometrically based on Figure 2; but to get the finer monotonicity result we resort to algebra and use the formula

$$\cos(\beta_i) = \frac{\langle A,C\rangle\langle B,D\rangle - \langle A,B\rangle\langle C,D\rangle - \langle B,C\rangle\langle D,A\rangle}{\|A\|\|B\|\|C\|\|D\|},$$

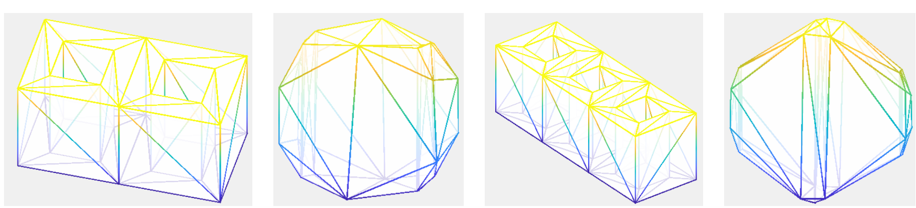
where $A = V_0' - V_{i\oplus 1}'$, $B = V_{i\ominus 1}' - V_0'$, $C = V_i' - V_{i\ominus 1}'$, $D = V_{i\oplus 1}' - V_i'$. Here $i\oplus 1 = i+1$ if $i<6$, and 1 if $i=6$, $i\ominus 1 = i-1$ if $i>1$, and 6 if $i=1$. By computation, we get $\cos(\beta_1) = \cos(\beta_4) = -\frac{1-\epsilon_1 c-\epsilon_2 c+\epsilon_1\epsilon_2}{\sqrt{1-2\epsilon_1 c+\epsilon_1^2}\sqrt{1-\epsilon_2 c+\epsilon_2^2}}$, $\cos(\beta_2) = \frac{c-2\epsilon_1+c\epsilon_1^2}{1-2\epsilon_1 c+\epsilon_1^2}$, $\cos(\beta_5) = \frac{c-2\epsilon_2+c\epsilon_2^2}{1-\epsilon_2 c+\epsilon_2^2}$. The limits (0.4) then follow immediately and we have proved that $\lim_{\epsilon_1,\epsilon_2\to 0} W_{0,0}(\epsilon_1,\epsilon_2) = 4\pi/n$. To see that the convergence is monotone in the original ε, write

$$W_{0,0}(\epsilon_1,\epsilon_2) = \cos^{-1}\left(\frac{c-2\epsilon_1+c\epsilon_1^2}{1-2\epsilon_1 c+\epsilon_1^2}\right) + \cos^{-1}\left(\frac{c-2\epsilon_2+c\epsilon_2^2}{1-2\epsilon_2 c+\epsilon_2^2}\right) - 2\cos^{-1}\left(\frac{1-\epsilon_1 c-\epsilon_2 c+\epsilon_1\epsilon_2}{\sqrt{1-2\epsilon_1 c+\epsilon_1^2}\sqrt{1-2\epsilon_2 c+\epsilon_2^2}}\right).$$

From this, we get $\frac{\partial W_{0,0}}{\partial \epsilon_1} = 0$, $\frac{\partial W_{0,0}}{\partial \epsilon_2} = \frac{4s}{1-2\epsilon_2 c+\epsilon_2^2} > 0$, when $1 \geqslant \epsilon_1 \geqslant \epsilon_2 \geqslant 0$. This shows $W_{0,0}(\epsilon_1,\epsilon_2)$ is independent of ϵ_1, which also means $W_{0,0} = 2\cos^{-1}\left(\frac{c-2\epsilon_2+c\epsilon_2^2}{1-2\epsilon_2 c+\epsilon_2^2}\right)$, and is monotone in ϵ_2. Combined with the monotonicity of $\epsilon_2(\varepsilon)$, the proof is completed. ■

In the genus $g=0$ case, as long as the connectivity is reasonable (see [1, Proposition 9]), the simplicial sphere is *inscribable* in a sphere and hence has a minimum W_{Bobenko}-energy 4π (0 in Bobenko's definition of W = our definition of $W_{\text{Bobenko}} - 4\pi(1-g)$), the same as what one expects from the smooth setting.

Our result here shows that in the genus 1 case with even the perfectly regular connectivity, the infimum is never what one expects from the smooth setting. It also suggests, but does not prove, that the infimum is 4π regardless of m and n, and that a minimizer does *not* exist as what we typically call a genus 1 PL surface. The latter, if proved, is another contrast to the smooth setting; cf. [3]. It is also possible to formulate and prove a generalization of this result for genus $g>1$.



(a) (b) (c) (d)

Figure 3: The minimization process rounds the 2- and 3-hole tori in (a) and (c) and closes up the holes, resulting in the empirical 'infimumizers' in (b) & (d), both with $W_{\text{Bobenko}} \approx 4\pi$.

References

[1] A. I. Bobenko. A conformal energy for simplicial surfaces. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 135–145. Cambridge Univ. Press, Cambridge, 2005.

[2] J. P. Brogan, Y. Yang, and T. P.-Y. Yu. Numerical methods for biomembranes based on PL surfaces. Submitted to the conference proceedings of ENUMATH 2017, December 2017.

[3] L. Simon. Existence of surfaces minimizing the Willmore functional. *Communications in Analysis and Geometry*, 1(2):281–326, 1993.