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MULTIVARIATE REFINABLE HERMITE INTERPOLANT

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ABSTRACT. We introduce a general definition of refinable Hermite interpolants and investigate their general properties. We also study a notion of symmetry of these refinable interpolants. Results and ideas from the extensive theory of general refinement equations are applied to obtain results on refinable Hermite interpolants. The theory developed here is constructive and yields an easy-to-use construction method for multivariate refinable Hermite interpolants. Using this method, several new refinable Hermite interpolants with respect to different dilation matrices and symmetry groups are constructed and analyzed.

Some of the Hermite interpolants constructed here are related to well-known spline interpolation schemes developed in the computer-aided geometric design community (e.g., the Powell-Sabin scheme). We make some of these connections precise. A spline connection allows us to determine critical Hölder regularity in a trivial way (as opposed to the case of general refinable functions, whose critical Hölder regularity exponents are often difficult to compute).

While it is often mentioned in published articles that "refinable functions are important for subdivision surfaces in CAGD applications", it is rather unclear whether an arbitrary refinable function vector can be meaningfully applied to build free-form subdivision surfaces. The bivariate symmetric refinable Hermite interpolants constructed in this article, along with algorithmic developments elsewhere, give an application of vector refinability to subdivision surfaces. We briefly discuss several potential advantages offered by such Hermite subdivision surfaces.

1. MOTIVATION AND INTRODUCTION

Let $\phi = (\phi_i)_{i=1}^m$ be a column vector of functions defined on $\mathbb{R}^s$. We say that ϕ is refinable with respect to the dilation matrix M if there exists a finitely supported sequence $\mathbf{a}$ of $m \times m$ matrices such that

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} \mathbf{a}(\alpha)\phi(M \cdot -\alpha). \tag{1.1}$$

Refinability is important for at least two—not immediately related—reasons. On the one hand, it allows for the definition of a nested sequence of shift-invariant spaces $V_j := \text{span}\{\phi(M^j \cdot -\alpha) : \alpha \in \mathbb{Z}^s\}$. This so-called multi-resolution analysis (MRA) is the key to wavelet constructions and their associated fast filter-bank algorithms [3]. On the other hand, functions composed from linear combinations

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of shifts of a refinable ϕ can be computed using a simple subdivision algorithm. Consequently, such functions can be computed at any desired resolution and any desired position. Such an adaptive "zoom-in" property makes subdivision curves and surfaces very attractive for interactive geometric modelling applications.

It is probably hard to perceive how functions like (1.1), constructed on a regular grid in "flat-space" $\mathbb{R}^s$, can actually be used to model the kind of "free-form" surfaces of arbitrary topological type encountered in geometric modelling and computer graphics applications. This is an advance first made in the 1970's in the computer graphics community by Catmull-Clark [9] and Doo-Sabin [14]. We cannot afford the space to review even the basic idea of *subdivision surfaces* here, but simply mention that the construction of a subdivision surface is typically based on first constructing what we now call a refinable function in the regular grid setting followed by designing additional subdivision rules at the so-called *extraordinary vertices*. We refer to the recent book [35] for this subject.

This paper focuses on the first step mentioned above: construction of refinable functions. But instead of *scalar* refinable functions (i.e., $m = 1$ in (1.1)), which underlie all existing methods of subdivision surfaces (e.g., Catmull-Clark, Loop, Butterfly), we are interested here in the more general refinable function vectors. While the concept of vector refinability is extensively studied by a number of researchers—mainly in conjunction with the idea of *multiwavelets* (see, e.g., [2], [5]), to our knowledge there is no formal publication on any attempts of applying refinable function vectors to the setting of subdivision surfaces. Some initial attempts in this direction, based on the results of this paper, can be found in the recent conference paper [36].

It is well known that smoothness of a (scalar) refinable function and shortness of the support size of its mask are two contradicting requirements. For example, it was proved in [21] that no C^2 interpolating dyadic refinable function can be supported on $[-3,3]^s$ and no C^1 dyadic refinable function can be supported on $[-1,1]^s$ in any dimension s. Therefore, to obtain smoother refinable interpolants we have to make certain tradeoffs somewhere; vector refinability, which is based on increasing the number of generating functions, is one interesting approach. This idea seems particularly appealing in the setting of subdivision surfaces where, as mentioned earlier, one needs to be concerned about the construction of special subdivision rules at extraordinary vertices and (consequently) one would like to keep the support size of the underlying refinable functions as small as possible.

Another motivation of our study of multivariate refinable Hermite interpolants is perhaps more far fetched: existing interpolating subdivision surfaces typically suffer from inferior surface *fairness* [34] when compared to noninterpolating schemes. In this setting, one is given a coarse mesh and the goal is to fit a smooth yet "fair" surface that goes through all the points in the coarse mesh. Typically gradient values or tangent plane conditions are not prescribed at most or all of the points of the coarse mesh, meaning that a user of Hermite subdivision schemes is free to assign any gradient values at the base mesh. Our theory guarantees that one gets the same smoothness of the underlying parametrization of the surface regardless of the assigned gradient values. This flexibility offers a possibility for optimizing the "fairness" of the resulting subdivision surface.

In the univariate case, refinable Hermite interpolants were first considered by Merrien in [30]. Also see [16], [20], [22], [36] for discussion of univariate refinable

Hermite interpolants. However, the situation in higher dimensions is more complicated. One can of course use a tensor-product construction based on a univariate refinable Hermite interpolant of multiplicity r, which leads to a Hermite interpolant of multiplicity r^s in dimension s. Such an explosion in multiplicity will likely make the resulting interpolants impractical. We advocate here nontensor-product constructions based on interpolating Hermite data of *total degree* $\leq r$, leading to multiplicity $\binom{r+s}{s} \ll r^s$. This yields more practical constructions in the subdivision surface applications we have in mind.

1.1. **This paper. Contributions.** Since there are by now hundreds—if not more—of published articles on the mathematical structures and applications of the refinement equation (1.1), we find it necessary to invest a major effort in explaining the key contributions of this article. We hope that the readers (especially the more mathematically inclined ones) will excuse the wordiness of this section.

• *Vector refinability and free-form subdivision surfaces.* the refinable Hermite interpolants constructed in this paper have been found to be applicable to constructing subdivision surfaces of arbitrary topology [36], [37]. While one very frequently finds claims in published articles like "*refinable functions are important in CAGD applications*", the general concept of vector refinability seems to be of unknown applicability to the construction of free-form subdivision surfaces. At least the authors are not aware of any methods or attempts in this direction or any proposed method for free-form subdivision surfaces based on refinable function vectors; the second author, however, had heard talks in conferences (by, for example, Nira Dyn in 1998) speculating such possibilities. Consequently, the refinable Hermite interpolants constructed in this article, together with the algorithmic developments [37] in geometric settings, provide genuinely new tools for exploring several new possibilities (two mentioned in Section 1) offered by subdivision surfaces based on refinable function vectors.

• *Spline connections.* One of the hallmarks in the spline literature is the de Boor knot insertion algorithm for efficient computation of B-splines [1], related to this is the connection of B-spline with subdivision due to Lane and Risenfeld [29]. We connect two of the bivariate refinable Hermite interpolants constructed in this paper to two spline schemes proposed in the literature (Powell-Sabin's and Sibson's.) In fact, after the completion of this paper, we became aware of references [31], [15], [18]. The papers [31] and [15] essentially contain a proof for Propositions 3.3 and 3.7, respectively. In [18] the authors derived a Hermite subdivision scheme that reproduces a version of Powell-Sabin spline different from the one considered in Theorem 3.2 and Proposition 3.3.

Our method of proof, however, differs from those in [31], [15], [18] and relies on the use of Bernstein-Bézier form [19], [13], an important tool developed in the 1980's for the study of multivariate splines.

• *Theoretical issues.* One of the analysis difficulties we had faced was to find a computable necessary and sufficient condition for the existence of a smooth solution which satisfies **both** (2.4) and (2.1). This problem was first addressed in the brief conference paper [16] in which the authors develop a "tailor-made" factorization method based on what numerical analysts call divided differences of Hermite data. Unfortunately, it is well known from the study of refinement equations that in general the factorization method is not applicable to multivariate settings; moreover, there is simply no known definition of multivariate divided difference suitable for our purpose [12]. A different approach to this problem in the univariate setting is given in [38]. We are not aware of any publication dealing with this problem in the generality pursued in this article.

We settle this issue with the help of the strong convergence theory of a cascade algorithm [11]. The latter was originally studied for purposes unrelated to the Hermite interpolatory properties concerned in this paper. Once this connection is made, a recent result by Jia and Jiang [28] gives a simple computational procedure for checking the technical conditions we desire.

• *Appeals to optimization.* The smoothness optimization problem (4.1) that arises from our construction method had stimulated Michael Overton and his coworkers to develop efficient numerical methods for solving such nonsmooth, nonconvex optimization problems. Overton *et al.* had found that a modification of a novel gradient bundle method [7], originally designed for solving the spectral abscissa optimization problem in control theory, can be used to solve our problem (4.1).

Organization. In Section 2, we define refinable Hermite interpolants in a general setting and address various issues pertaining to their construction and analysis. In Section 2.6 we give a definition of symmetry of refinable Hermite interpolants based on the notion of a *symmetry group w.r.t. a given dilation matrix* [25]. In Section 3 we show how a simple computational procedure can be used to construct smooth Hermite interpolants. Since our application is geared toward applications in subdivision surfaces, we shall focus on 2-dimensional examples (although the results in Section 2 apply to general dimensions). We also relate some of the bivariate interpolants constructed using our method to some known schemes proposed in the spline literature. Section 4 concludes this paper and also outlines the contents of a companion paper.

2. Symmetric refinable hermite interpolants

2.1. **General notations.** Let $\mathbb{N}_0$ denote all the nonnegative integers. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, $\mu! := \mu_1! \cdots \mu_s!$, $|\mu| := \mu_1 + \cdots + \mu_s$, and $\xi^\mu := \xi_1^{\mu_1} \cdots \xi_s^{\mu_s}$ for $\xi = (\xi_1, \dots, \xi_s) \in \mathbb{R}^s$. The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by $\partial_j f$, $j = 1, \dots, s$, and for $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, ∂^μ is the differential operator $\partial_1^{\mu_1} \cdots \partial_s^{\mu_s}$.

Let $\Lambda_r := \{\mu \in \mathbb{N}_0^s \ : \ |\mu| \le r\}$ and by $\#\Lambda_r$ we denote the cardinality of the set Λ_r. Now the elements in Λ_r can be ordered according to the lexicographic order. That is, $(\nu_1, \dots, \nu_s)$ is less than $(\mu_1, \dots, \mu_s)$ in lexicographic order if $|\nu| < |\mu|$ or $\nu_j = \mu_j$ for $j = 1, \dots, i-1$ and $\nu_i < \mu_i$ for some i. The set Λ_r is always ordered in the lexicographic order in this paper with the default first element being 0.

For a vector space B, denote by $[B]^{m \times n}$ the vector space of all matrices of elements in B, equipped with the cartesian vector space structure. Define $[B]^m := [B]^{m \times 1}$. If B is a normed space with norm $||\cdot||_B$, then $[B]^{m \times n}$ is a normed space with norm $||v||_{[B]^{m\times n}} := ||(||v_{i,j}||_B)_{1 \le i \le m, 1 \le j \le n}||$, where the outer $||\cdot||$ is any norm in $\mathbb{R}^{m \times n}$. If $(B, ||\cdot||_B)$ is a Banach space, then so is $([B]^{m \times n}, ||\cdot||_{[B]^{m \times n}})$.

We denote by $l(\mathbb{Z}^s)$ and $L(\mathbb{R}^s)$ the vector spaces of all complex valued sequences and functions defined on $\mathbb{Z}^s$ and $\mathbb{R}^s$. The space of finitely supported sequences is denoted by $l^0(\mathbb{Z}^s)$.

2.2. **Refinable Hermite interpolant.** Let $\phi = (\phi_\mu)_{\mu \in \Lambda_r}$ be a column vector of functions on $\mathbb{R}^s$. We say that ϕ is a **Hermite interpolant** of order r if $\partial^\nu \phi_\mu$ are well-defined continuous functions for all $\mu, \nu \in \Lambda_r$ such that

$$[\partial^\nu \phi_\mu](\alpha) = \delta(\mu - \nu)\delta(\alpha) \qquad \forall\, \mu, \nu \in \Lambda_r,\ \alpha \in \mathbb{Z}^s, \tag{2.1}$$

where δ is the **Dirac sequence** such that $\delta(0) = 1$ and $\delta(\alpha) = 0$ for all $\alpha \neq 0$. Any Hermite interpolant ϕ defines an interpolation operator to data v via

$$\mathcal{I}_\phi v := \sum_\alpha v(\alpha)^T \phi(\cdot - \alpha), \quad v \in [l(\mathbb{Z}^s)]^{\#\Lambda_r}. \tag{2.2}$$

Conversely, any interpolation operator $I : [l(\mathbb{Z}^s)]^m \to C^r(\mathbb{R}^s)$ which is **linear**, **local** (i.e., v locally supported implies Iv is compactly supported) and **shift invariant** (i.e., $(Iv)(\cdot - \alpha) = Iv(\cdot - \alpha)$) defines a Hermite interpolant. (We shall need this comment in the proof of Lemma 2.6.)

In this paper, we are particularly interested in Hermite interpolants which are *refinable*. An $s \times s$ integer matrix M is called a **dilation matrix** if $\lim_{n\to\infty} M^{-n} = 0$. That is, all the eigenvalues of a dilation matrix M are greater than one in modulus. Although some of our auxiliary results apply to general dilation matrices, we assume exclusively that M is **isotropic**, i.e., there exists an invertible $s \times s$ matrix Λ such that

$$\Lambda M \Lambda^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_s), \tag{2.3}$$

where $|\sigma_1| = \cdots = |\sigma_s| = \rho$.

We are concerned with the following **refinement equation**:

$$\phi = \sum_{\alpha\in\mathbb{Z}^s} \mathbf{a}(\alpha)\phi(M \cdot - \alpha), \tag{2.4}$$

where $\phi = (\phi_1, \ldots, \phi_m)^T$ is an $m \times 1$ column vector of functions, called a **refinable function vector**, and $\mathbf{a}$ is a finitely supported sequence of $m \times m$ matrices on $\mathbb{Z}^s$, called the **mask**. When needed we write

$$\phi_{\mathbf{a},M} := \phi$$

to emphasize the dependence of ϕ on $\mathbf{a}$ and M.

For an $s \times s$ matrix E, define another $(\#\Lambda_r) \times (\#\Lambda_r)$ matrix $S(E)$ ([25]) by

$$[S(E)]_{\mu,\nu} := \frac{[\partial^\nu (E\cdot)^\mu](0)}{\mu!}, \quad \text{or equivalently,} \quad \frac{(Ex)^\mu}{\mu!} = \sum_{\nu\in\Lambda_r} [S(E)]_{\mu,\nu} \frac{x^\nu}{\nu!}, \quad \mu \in \Lambda_r. \tag{2.5}$$

This matrix measures how Hermite data change under a linear change of variables:

$$g = f(E\cdot) \quad \text{implies} \quad \partial^{\le r} g = \partial^{\le r} f(E\cdot) S(E), \tag{2.6}$$

where $\partial^{\le r} f(x)$ is the row vector of length $\#\Lambda_r$ with entries $\partial^\mu f(x)$, $\mu \in \Lambda_r$. From this interpretation, we get $S(E_1E_2) = S(E_1)S(E_2)$ and $S(E^{-1}) = [S(E)]^{-1}$, provided that E is invertible.

For latter convenience, we also define

$$D_n^{\le r} f(\alpha) := \partial^{\le r} f(M^{-n}\alpha) S(M^{-n}). \tag{2.7}$$

The vector on the right-hand side consists precisely of all the mixed directional derivatives of f of order up to r at the point $M^{-n}\alpha$ and in directions $M^{-n}e_j$, $j = 1, \ldots, s$.

It is well known that refinement equation (2.4) has interesting structures vis-a-vis

• the subdivision operator $S = S_{\mathbf{a}} : [l(\mathbb{Z}^s)]^{m' \times m} \to [l(\mathbb{Z}^s)]^{m' \times m}$ ($m = \#\Lambda_r$) defined by

$$(Sv)(\alpha) = \sum_{\beta \in \mathbb{Z}^s} v(\beta)\, \mathbf{a}(\alpha - M\beta), \quad \text{and} \tag{2.8}$$

• the cascade operator $Q = Q_{\mathbf{a}} : [L(\mathbb{R}^s)]^{m \times m'} \to [L(\mathbb{R}^s)]^{m \times m'}$ defined by

$$Qf = \sum_{\alpha \in \mathbb{Z}^s} \mathbf{a}(\alpha) f(M \cdot - \alpha). \tag{2.9}$$

Note that m' can be arbitrary, although for our applications it is either 1 or $m = \#\Lambda_r$. The finite-supportedness of mask $\mathbf{a}$ allows one to think of S (Q) as a bounded operator on $[B]^{m' \times m}$ ($[B]^{m \times m'}$), where B is any normed subspace of $l(\mathbb{Z}^s)$ ($L(\mathbb{R}^s)$).

Let $a_n := S_{\mathbf{a}}^n(\delta I_m)$, where $\delta I_m \in [l(\mathbb{Z}^s)]^{m \times m}$ is the matrix sequence which is equal to the identity matrix I_m at $\alpha = 0$ and the zero matrix elsewhere. We have

$$Q_{\mathbf{a}}^n f = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) f(M^n \cdot - \alpha). \tag{2.10}$$

Hence, if ϕ satisfies (2.4), then

$$\phi = Q_{\mathbf{a}}^n \phi = \sum_{\alpha \in \mathbb{Z}^s} a_n(\alpha) \phi(M^n \cdot - \alpha) \quad \forall n. \tag{2.11}$$

From (2.11), (2.6) and (2.1), we deduce

$$D_n^{\leq r} \phi(\alpha) =: \begin{bmatrix} D_n^{\leq r} \phi_1(\alpha) \\ \vdots \\ D_n^{\leq r} \phi_m(\alpha) \end{bmatrix} = a_n(\alpha) \quad \forall n, \alpha. \tag{2.12}$$

We also write $\partial^{\leq r} \phi := \begin{bmatrix} \partial^{\leq r} \phi_1 \\ \vdots \\ \partial^{\leq r} \phi_m \end{bmatrix}$.

Proposition 2.1. *For a finitely supported sequence* $\mathbf{a}$ *of* $(\#\Lambda_r) \times (\#\Lambda_r)$ *matrices on* $\mathbb{Z}^s$ *and a dilation matrix* M*, the following are equivalent:*

(a) *there exists a compactly supported Hermite interpolant* ϕ *which satisfies* (2.4)

(b) $\mathbf{a}$ *satisfies*

$$\mathbf{a}(0) = S(M^{-1}), \quad \mathbf{a}(Mk) = 0, \qquad k \in \mathbb{Z}^s \backslash \{0\} \tag{2.13}$$

and for any $v \in [l(\mathbb{Z}^s)]^{1 \times m}$ *there exists a unique* r*-times continuously differentiable function* f *such that*

$$D_n^{\leq r} f = S^n v \quad \forall n. \tag{2.14}$$

Proof. (a) ⇒ (b): For (2.13), differentiate both sides of (2.4) and use (2.6). Next, we check that $f := \sum_\alpha v(\alpha)\phi(\cdot - \alpha)$ satisfies (2.14):

$$\partial^{\leq r} f(M^{-n}\beta) = \sum_\alpha v(\alpha) \partial^{\leq r} \phi(M^{-n}\beta - \alpha),$$

therefore

$$D_n^{\leq r} f(\beta) = \sum_\alpha v(\alpha) D_n^{\leq r} \phi(\beta - M^n \alpha) = \sum_\alpha v(\alpha) a_n(\beta - M^n\alpha) = S_{\mathbf{a}}^n v(\beta).$$

If $\tilde{f}$ is C^r and satisfies (2.14), then $\tilde{f} = f$, since $\bigcup_n M^{-n}\mathbb{Z}^s$ is dense in $\mathbb{R}^s$.

(b) $\Rightarrow$ (a): By assumption, there exists a compactly supported C^r function vector ϕ such that $D_n^{\le r}\phi = a_n$ for all n. We claim that $D_n^{\le r}(Q_{\mathbf{a}}\phi) = D_n^{\le r}\phi$ for all n, consequently $\phi = Q_{\mathbf{a}}\phi$.

To prove the claim, observe that

$$\begin{aligned}
(Q_{\mathbf{a}}\phi)(x) &= \sum_\alpha \mathbf{a}(\alpha)\ \phi(Mx-\alpha),\\
\partial^{\le r}(Q_{\mathbf{a}}\phi)(x) &= \sum_\alpha \mathbf{a}(\alpha)\ (\partial^{\le r}\phi)(Mx-\alpha)S(M),\\
D_n^{\le r}(Q_{\mathbf{a}}\phi)(\beta) &= \sum_\alpha \mathbf{a}(\alpha)\ (\partial^{\le r}\phi)(M^{-(n-1)}\beta-\alpha)S(M)S(M^{-n})\\
&= \sum_\alpha \mathbf{a}(\alpha)(D_{n-1}^{\le r}\phi)(\beta - M^{n-1}\alpha)\\
&= \sum_\alpha \mathbf{a}(\alpha)\ a_{n-1}(\beta - M^{n-1}\alpha)\\
&= a_n(\beta) = D_n^{\le r}\phi(\beta).
\end{aligned}$$

The second to last equality can be easily checked in the Fourier domain: $\widehat{a_n}(\xi) = \widehat{\mathbf{a}}((M^T)^{n-1}\xi)\cdots\widehat{\mathbf{a}}(M^T\xi)\widehat{\mathbf{a}}(\xi) = \widehat{\mathbf{a}}((M^T)^{n-1}\xi)\widehat{a_{n-1}}(\xi)$. $\square$

Motivated by Proposition 2.1, we say that $\mathbf{a}$ is a **Hermite interpolatory mask** of order r with respect to the dilation matrix M if (2.13) is satisfied.

2.3. **Smoothness $\Rightarrow$ polynomial reproduction.** Writing informally, it is well known that:

smoothness+refinability $\Rightarrow$ approximation order $\Rightarrow$ accuracy order $\Rightarrow$ sum rules.

For the first implication, we refer to Ron [33] and also Meyer ("A Remarkable Identity", page 32 [4]). For the second and third implications, see e.g., [27]. (Note that for accuracy order $\Rightarrow$ sum rules, one needs some mild additional condition on the refinable function vector.) These references are by no means exhaustive. For instance, related results are independently discovered (even earlier) in the subdivision literature [10].

Sum rule conditions for general multivariate vector refinement equations have been proposed in [8], [22]. For our purposes, we prefer the more intuitive

Definition 2.2. A Hermite interpolatory mask **reproduces** Π_k if

$$S_{\mathbf{a}}\left(D_0^{\le r}p\right) = D_1^{\le r}p \quad \forall p \in \Pi_k. \tag{2.15}$$

Note. Condition (2.15) imposes a finite set of linear conditions on mask $\mathbf{a}$, similar to usual sum rule conditions. A simple rescaling shows that (2.15) is equivalent to saying that $S_{\mathbf{a}}\left(D_n^{\le r}p\right) = D_{n+1}^{\le r}p$, for all $p \in \Pi_k$ and $n \ge 0$.

Theorem 2.3. *Let ϕ be a refinable Hermite interpolant of order r. Let $1 \le p \le \infty$. If $\phi \in [W_p^k(\mathbb{R}^s)]^m$, then its mask reproduces Π_k.*

Note. Since a refinable Hermite interpolant of order r must, by definition, be in $[C^r(\mathbb{R}^s)]^m \subset [W_\infty^r(\mathbb{R}^s)]^m$, any Hermite interpolatory mask must at least reproduce Π_r.

Definition 2.4. The mask $\mathbf{a}$ of a general refinement equation (2.4) (not necessarily of Hermite interpolatory type) satisfies the **eigenvalue condition** of order k if the $m \times m$ matrix $J := \sum_\alpha \mathbf{a}(\alpha)/|\det(M)|$ has 1 as a simple eigenvalue and the other eigenvalues are of modulus less than $|\det(M)|^{-k/s}$.

Remark. It is known that for a refinable Hermite interpolant ϕ in $W_p^k(\mathbb{R}^s)$, the above is a necessary condition [11], [24]. Theorem 2.3 gives yet another set of necessary conditions on the refinement mask $\mathbf{a}$. In the case of Hermite interpolatory mask $\mathbf{a}$ with $m = 1$ (i.e., $r = 0$), if $\mathbf{a}$ reproduces Π_0, then $\mathbf{a}$ obviously satisfies the eigenvalue condition (of any order). When $r > 0$, there seems to be no obvious connection between these two sets of necessary conditions.

To facilitate the proof of Theorem 2.3, we mention a basic result in Section 2 of [11] related to polynomial reproduction. If (i) ϕ is a refinement function vector which satisfies (2.4), (ii) $\phi \in [W_p^k(\mathbb{R}^s)]^m$, and (iii) $\mathbf{a}$ satisfies the eigenvalue condition of order k, then ϕ has **accuracy order** $k+1$:

$$\operatorname{span}\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\} \supset \Pi_k. \tag{2.16}$$

Proof of Theorem 2.3. By assumptions of the theorem, ϕ is a refinable Hermite interpolant with membership in $[W_p^k(\mathbb{R}^s)]^m$. Observe that the integer shifts of a Hermite interpolant ϕ must be linearly independent, hence L^p-stable. By Lemma 2.1 of [11], the refinement mask $\mathbf{a}$ must satisfy the eigenvalue condition of order k. Consequently, we have (2.16). Let $p \in \Pi_k$. By the proof of Proposition 2.1, $f := \sum_\alpha D_0^{\le r} p(\alpha)\ \phi(\cdot - \alpha)$ satisfies (2.14). By (2.16) and the assumption that ϕ is a Hermite interpolant, $f = p$. Summarizing, we have $D_n^{\le r} p = S_a^n D_0^{\le r} p$, $\forall n$, as desired. □

For latter purposes, we mention finer results in the paper [11] pertaining to polynomial reproduction. These results are intimately related to the classical Strang-Fix theorem and Poisson summation formula.

In the case when conditions (i)–(iii) leading to (2.16) are satisfied, Lemma 2.3 of [11] illustrates *how* $\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ reproduces Π_k: there exist $1 \times m$ vectors B_μ, $|\mu| \le k$, such that

$$\sum_{\alpha \in \mathbb{Z}^s} \sum_{\nu \le \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} \phi(x - \alpha) = (\Lambda x)^\mu \quad \text{for a.e. } x. \tag{2.17}$$

Moreover, a specific choice of B_μ (in the form of a set of formulas in a Fourier domain) achieving (2.17) is given. For refinable Hermite interpolants, it is clear that these row vectors are uniquely determined by (2.17):

$$(B_\mu)_\nu = \partial^\nu (\Lambda x)^\mu |_{x=0}. \tag{2.18}$$

Following [11], define Y_k to be the space of compactly supported $F \in [W_p^k(\mathbb{R}^s)]^m$ such that

$$\sum_{\alpha \in \mathbb{Z}^s} \sum_{\nu \le \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} F(\cdot - \alpha) - (\Lambda \cdot)^\mu \in \Pi_{|\mu|-1} \ \ \forall |\mu| \le k. \tag{2.19}$$

Under appropriate conditions, $\phi_0 \in Y_k$ is a necessary condition for the strong convergence of $(Q_{\mathbf{a}}^n \phi_0)_n$. We say that a Hermite subdivision scheme associated with a Hermite interpolatory mask a converges in $W_p^k(\mathbb{R}^s)$ if there exists $\phi \in [W_p^k(\mathbb{R}^s)]^m$ such that $\lim_{n\to\infty} ||\phi - Q_{\mathbf{a}}^n \phi_0||_{[W_p^k(\mathbb{R}^s)]^m} = 0$ for every $\phi_0 \in Y_k$. A relatively comprehensive study of vector subdivision schemes and cascade algorithms has been established in a recent paper [24]. A simple description of the set Y_k of the initial function vectors for a mask a and a dilation matrix M has been given and also many related papers on subdivision schemes and cascade algorithms can be found in [24].

For any smooth enough Hermite interpolant F with accuracy order $k+1$, the left-hand side of (2.19) is zero, and hence belongs to Y_k. We shall need this fact in the next section.

2.4. **Strong convergence $\Rightarrow$ smoothness + interpolation property.** Solutions of (2.4) and (2.1), if they exist, must be unique. For the existence of a smooth solution, we have

Lemma 2.5. *Let* $\mathbf{a}$ *be a Hermite interpolatory mask of order* r. *Let* B *be any Banach space continuously embedded in* $C^r(\mathbb{R}^s)$. *There exists a refinable Hermite interpolant* $\phi \in [B]^m$, $m = \#\Lambda_r$, *with mask* $\mathbf{a}$ *if and only if for some Hermite interpolant* $\phi_0 \in [B]^m$ *the sequence* $(Q_{\mathbf{a}}^n \phi_0)_{n\in\mathbb{N}}$ *converges in the Banach space* $[B]^m$.

Proof. The forward direction is trivial: simply choose $\phi_0 = \phi$. For the converse direction, we claim that the limit function vector ϕ, necessarily in $[B]^m$, is a refinable Hermite interpolant. It is refinable since it is a fixed point of $Q_{\mathbf{a}}$, and hence satisfies (2.4). Since the norm of B is stronger than that of $C^r(\mathbb{R}^s)$, we have $\lim_{n\to\infty} |(\partial^\mu Q_{\mathbf{a}}^n \phi_0)(x) - (\partial^\mu \phi)(x)| = 0$ for each $\mu \in \Lambda_r$ and $x \in \mathbb{R}^s$. But mask property (2.13) guarantees that each $Q_{\mathbf{a}}^n \phi_0$ satisfies (2.1), hence so does ϕ. ☐

Lemma 2.6. *For any* $s \ge 1$, $r \ge 0$, $k \ge 0$, $r' \ge r$, *there exists a Hermite interpolant of order* r *in* $[C^{r'}(\mathbb{R}^s)]^m$ ($m = \binom{r+s}{s} = \#\Lambda_r$) *with accuracy order* k.

Proof. See Appendix. ☐

While strong convergence of subdivision schemes is originally studied for reasons not directly related to the kind of interpolation property addressed in this paper, Lemma 2.5 suggests that one can use strong convergence to establish the Hermite interpolatory property of a refinable function vector.

Theorem 2.7. *Let* $\mathbf{a}$ *be a Hermite interpolatory mask of order* $r \ge 0$. *For* $1 \le p < \infty$, *let* $k > r + \frac{s}{p}$, *the following are equivalent:*

1) *There exists a unique refinable Hermite interpolant in* $[W_p^k(\mathbb{R}^s)]^m$ *with mask* $\mathbf{a}$.
2) *There exists a Hermite interpolant* $\phi_0 \in [W_p^k(\mathbb{R}^s)]^m$ *with accuracy order* $k+1$ *such that* $(Q_{\mathbf{a}}^n \phi_0)_n$ *converges in* $[W_p^k(\mathbb{R}^s)]^m$.
3) *Mask* $\mathbf{a}$ *satisfies the eigenvalue condition of order* k, *and the subdivision scheme associated with* $\mathbf{a}$ *converges in* $W_p^k(\mathbb{R}^s)$; *that is,* $(Q_{\mathbf{a}}^n \phi_0)_n$ *converges in* $[W_p^k(\mathbb{R}^s)]^m$ *for every* $\phi_0 \in Y_k$, *where* Y_k *is defined as in* (2.19) *with the particular* $(B_\mu)_\nu$ *given in* (2.18).
4) *Mask* $\mathbf{a}$ *reproduces* Π_k *in the sense of* (2.15), *satisfies the eigenvalue condition of order* k, *and*

$$\max\left\{\lim_{n\to\infty} ||a_n * v||_p^{1/n} : v \in H_k\right\} < |\det(M)|^{1/p-k/s}, \tag{2.20}$$

where

$$H_k := \left\{ v \in [l^0(\mathbb{Z}^s)]^m : \sum_{\alpha\in\mathbb{Z}^s} (D_0^{\le r} p)(-\alpha) v(\alpha) = 0, \ \forall p \in \Pi_k(\mathbb{R}^s) \right\}. \tag{2.21}$$

The same result holds when $p = \infty$, $k \geq r$, *but with* $W^k_\infty(\mathbb{R}^s)$ *replaced by the smaller* $C^k(\mathbb{R}^s)$.

5) *Conditions in 4) but with* (2.20) *replaced by*

$$\max\left\{\lim_{n\to\infty} ||\nabla^\nu a_n||_p^{1/n} : |\nu| = k+1\right\} < |\det(M)|^{1/p-k/s}. \tag{2.22}$$

Proof. The proof of this result is really a putting-together of several pieces.

1) $\Leftrightarrow$ 2) is due to Lemma 2.5 and a Sobolev embedding theorem. 3) $\Leftrightarrow$ 4) is just a restatement of the main result of [11]. 3) $\Rightarrow$ 2) follows from Lemma 2.6 and our earlier comment that every smooth enough Hermite interpolant with accuracy order $k+1$ belongs to Y_k.

The implication 1) $\Rightarrow$ 4) is perhaps the most technical, and is a direct consequence of Theorem 4.1 of [28] in the $p = 2$ case. Oral communication with R. Q. Jia confirmed that similar techniques can be applied to establish the same result for other p values and for details about the case of general $1 \leq p \leq \infty$, see the recent paper [24]. To help appreciate the implication 1) $\Rightarrow$ 4), we first mention that condition 1) can be shown, by straightforward generalization of arguments (in, e.g., [23]) to be equivalent to 5).

Since each ∇^ν is a translation invariant operator, for each ν with $|\nu| = k+1$ there exists a v_ν such that $\nabla^\nu a_n = a_n * v_\nu$. Moreover, it is obvious that $v_\nu \in H_k$. Thus we have

$$\max\left\{\lim_{n\to\infty} ||\nabla^\nu a_n||_p^{1/n} : |\nu| = k+1\right\} \leq \max\left\{\lim_{n\to\infty} ||a_n * v||_p^{1/n} : v \in H_k\right\}. \tag{2.23}$$

In other words, 4) $\Rightarrow$ 5). Another way to interpret Theorem 4.1 of [28] is that, in fact, equality holds in (2.23). Summarizing, 1)–5) are all equivalent; proving 1) $\Rightarrow$ 5) is relatively easy, but proving 1) $\Rightarrow$ 4) requires the finer techniques in [28]. □

2.5. **Critical smoothness and computation.** We use membership in (fractional order) Sobolev spaces to measure critical regularity of a Hermite interpolant:

$$\nu_p(\phi) := \sup\{\nu : \phi \in [W^\nu_p(\mathbb{R}^s)]^m\}.$$

Obviously, when (2.20) is satisfied we have $\phi \in [W^k_p(\mathbb{R}^s)]^m$ and $\nu_p(\phi) \geq k$. For a Hermite interpolatory mask $\mathbf{a}$ reproducing Π_k but not Π_{k+1} in the sense of (2.15), as in [24], [25], for $1 \leq p \leq \infty$, we denote

$$\nu_p(\mathbf{a}, M) := s/p - s\log_{|\det M|} \max\left\{\lim_{n\to\infty} ||a_n * v||_p^{1/n} : v \in H_k\right\}. \tag{2.24}$$

The quantity $\nu_p(\mathbf{a}, M)$ plays a very important role in the study of the convergence of cascade algorithms (or subdivision schemes) and smoothness of refinable function vectors (see [24] as well as many other references cited therein). For example, the cascade algorithm associated with mask $\mathbf{a}$ and dilation matrix M converges in W^k_p ($1 \leq p \leq \infty$) if and only if $\nu_p(\mathbf{a}, M) > k$. Let ϕ be the refinable function vector with the mask $\mathbf{a}$ and the dilation matrix M. Then one always has $\nu_p(\phi) \geq \nu_p(\mathbf{a}, M)$. If in addition the shifts of ϕ are stable (for example, in our case ϕ is a Hermite interpolant and therefore its shifts are stable), then one has $\nu_p(\phi) = \nu_p(\mathbf{a}, M)$. For detailed discussion on these issues, see [24].

The right hand side of (2.24) can be easily computed in case of $p = 2$ and $\operatorname{supp}(\mathbf{a})$ not too large, thanks to a recent result by Jia and Jiang [28]. Let $b(\alpha) := \sum_\beta \overline{\mathbf{a}(\beta)} \otimes \mathbf{a}(\alpha+\beta)/|\det(M)|$ and $K := \mathbb{Z}^s \bigcap \sum_{n=1}^\infty M^{-n}(\operatorname{supp} b)$. Consider the size

$m^2|K|$ matrix $F := (b(M\alpha-\beta))_{\alpha,\beta\in K}$. If Hermite interpolatory mask $\mathbf{a}$ reproduces Π_{k-1} but not Π_k, dilation matrix M satisfies (2.3), and $J := \sum \mathbf{a}(\alpha)/|\det M|$ satisfies $\operatorname{spec}(J) = \{\eta_1 = 1, \eta_2, \dots, \eta_m\}$, $|\eta_j| < 1$ for $j > 1$, then

$$\nu_2(\mathbf{a}, M) = -(\log_{|\det M|} \rho_k)s/2, \tag{2.25}$$

where $\rho_k = \max\{|\nu| : \nu \in \operatorname{spec}(F)\backslash E_k\}$, $E_k = \{\eta_j\overline{\sigma^{-\mu}}, \overline{\eta_j}\sigma^{-\mu} : |\mu| < k, j = 2, \dots, m\} \cup \{\sigma^{-\mu} : |\mu| < 2k\}$.

This method will be employed to compute the smoothness of some of the refinable Hermite interpolants constructed in the next section.

2.6. **Symmetry.** Let G be a finite subset of integer matrices whose determinants are ± 1. We say that G is a **symmetry group with respect to a dilation matrix** M [25] if G forms a group under matrix multiplication and

$$MEM^{-1} \in G \quad \text{for all } E \in G. \tag{2.26}$$

This definition is essentially for the proof of Proposition 2.8. Obviously, each element in a symmetry group induces a linear isomorphism on $\mathbb{Z}^s$. In this paper, we are particularly interested in the following two-dimensional cases:

- the **hexagonal symmetry group** (a.k.a. dihedral group D_6) with respect to $M = 2I_2$

$$D_6 = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}. \tag{2.27}$$

- the **square symmetry group** (a.k.a. dihedral group D_4) with respect to $M = 2I_2$

$$D_4 = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}. \tag{2.28}$$

- D_6 with respect to the $\sqrt{3}$-dilation matrix $M = M_{\sqrt{3}} := \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$.
- D_4 with respect to the quincunx dilation matrix $M = M_{\text{Quincunx}} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

These are natural settings for constructing refinable Hermite interpolants on the regular quadrilateral and triangular meshes; note that D_4 and D_6 are the largest groups of linear maps leaving the 2-directional and 3-directional meshes invariant.

Given a Hermite interpolant $\phi = (\phi_\mu)_{\mu\in\Lambda_r}$ and a symmetry group G, we say that ϕ is G**-symmetric** if

$$\phi(Ex) = S(E)\phi(x) \qquad \forall\, E \in G, \quad x \in \mathbb{R}^s, \tag{2.29}$$

where the $(\#\Lambda_r) \times (\#\Lambda_r)$ matrix $S(E)$ is defined in (2.5).

Given a symmetry group G with respect to dilation matrix M, a Hermite interpolatory mask $\mathbf{a}$ is said to be G**-symmetric** with respect to M if

$$\mathbf{a}(E\alpha) = S(M^{-1}EM)\,\mathbf{a}(\alpha)S(E^{-1}) \qquad \forall\, \alpha \in \mathbb{Z}^s,\ E \in G. \tag{2.30}$$

Proposition 2.8. *Let $\phi = \phi_{\mathbf{a},M}$ be a refinable Hermite interpolant and G be a symmetry group with respect to M. The following are equivalent.*

1) *The interpolation operator $\mathcal{I}_\phi$ respects change of variables by $E \in G$ in the following sense. For any $F \in C^r$ and $G := F(E\cdot)$, $E \in G$, $\tilde{F} := \mathcal{I}_\phi\left(\partial^{\le r} F\right)$, and $\tilde{G} := \mathcal{I}_\phi\left(\partial^{\le r} G\right)$ are again related by $\tilde{G} := \tilde{F}(E\cdot)$.*

2) $\phi_{\mathbf{a},M}$ *satisfies* (2.29).
3) $\mathbf{a}$ *satisfies* (2.30).

Proof. 1) $\Leftrightarrow$ 2) is basically a consequence of (2.6) and is irrelevant to refinability of ϕ: Assuming 2), $\tilde{G} = \sum_\alpha D^{\leq r} G(\alpha)\phi(\cdot - \alpha) = \sum_\alpha D^{\leq r} F(E\alpha)S(E)\phi(\cdot - \alpha) = \sum_\alpha D^{\leq r} F(E\alpha)\phi(E\cdot - E\alpha) = \sum_\alpha D^{\leq r} F(\alpha)\phi(E\cdot - \alpha) = \tilde{F}(E\cdot)$. Conversely, pick any Hermite interpolant $F = (F_\nu)_{\nu\in\Lambda_r}$, then 2) follows from 1) by applying 1) to each F_ν and $G_\nu := F_\nu(E\cdot)$. (Notice that $\tilde{F_\nu} = \phi_\nu$ and $\tilde{G_\nu} = D^{\leq r} F_\nu(0)S(E)\phi$, yielding (2.29).)

2) $\Leftrightarrow$ 3) When (2.29) holds, we have from the refinement equation

$$\phi(x) = \sum_\beta S(E^{-1})\mathbf{a}(MEM^{-1}\beta)S(MEM^{-1})\phi(Mx - \beta)$$

for all $E \in G$. Since ϕ is a Hermite interpolant, we must have

$$S(E^{-1})\mathbf{a}(MEM^{-1}\beta)S(MEM^{-1}) = \mathbf{a}(\beta)$$

for all β. This is equivalent to (2.30). Conversely, let $\phi_E(x) := S(E^{-1})\phi(Ex)$. By (2.30), we have $\phi_E = Q_\mathbf{a}\phi_{MEM^{-1}} = Q^n_\mathbf{a}\phi_{ME^nM^{-1}}$ for all $E \in G$ and n. Since G is a finite group, there is an integer n such that $E^n = I$. So $\phi_E = Q^n_\mathbf{a}\phi_{ME^nM^{-1}} = Q^n_\mathbf{a}\phi = \phi$ for all $E \in G$. This is (2.29). □

3. A construction recipe and examples

The development in Section 2 suggests the following recipe for constructing Hermite subdivision schemes when given a dilation matrix M and a symmetry group G w.r.t. M.

(i) Pick a finite G-symmetric support of the mask $\mathbf{a}$, i.e., choose $\text{supp}(\mathbf{a}) \subset \mathbb{Z}^s$ such that $\alpha \in \text{supp}(\mathbf{a})$ implies $E\alpha \in \text{supp}(\mathbf{a})$, for all $E \in G$.

(ii) Pick a target polynomial reproduction order l.

(iii) Solve the system of *linear* equations determined by (2.15) and (2.30).

Note: By Lemma (2.1)(a) and (2.30), the unknowns of this linear system can be chosen to be the entries of $\mathbf{a}(k)$ for only those $\alpha \neq \mathbf{0}$ with distinct orbits under G ($\text{orbit}(\alpha) := \{E\alpha : E \in G\}$).

Armed with our construction recipe, the number of new schemes one can create is, of course, infinite. We consider in this paper a few examples which the authors believe are of potential interest in applications.

3.1. $G = \{[1], [-1]\}$, $M = [2]$. In the univariate case, we have the following Hermite mask of order 1 reproducing Π_2, with $\text{supp}(\mathbf{a}) = \{0, \pm 1\}$:

$$\mathbf{a}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix},$$

$$\mathbf{a}(1) = \begin{bmatrix} 1/2 & 2t - 1/2 \\ 1/8 & t \end{bmatrix},$$

$$\mathbf{a}(-1) = N\mathbf{a}(1)N,\ N = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to see that when $t = -1/4$ the mask reproduces Π_2 and the corresponding refinable function vector consists of C^1 piecewise quadratic polynomials at half intervals; when $t = -1/8$ the mask reproduces Π_3 and the refinable function vector consists of C^1 piecewise cubic polynomials at unit intervals.

We shall see a couple of nontensor-product extensions of these spline examples in bivariate settings.

3.2. $G = D_6$, $M = 2I_2$.

Example 3.1. Using our construction recipe with $G = D_6$, $M = 2I_2$, and the D_6-symmetric support,

$$\text{supp}(\mathbf{a}) = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1)\},$$

we find that solutions exist when $l \leq 2$. When $l = 2$, the parametric form of such a mask $\mathbf{a}$ is given by

$$\mathbf{a}(1,1) = \begin{bmatrix} 1/2 & 2t & 2t \\ 1/8 & t + 1/4 & t \\ 1/8 & t & t + 1/4 \end{bmatrix},$$
$$\mathbf{a}(E(1,1)^T) = S(E)\mathbf{a}(1,1)[S(E)]^{-1}, \qquad E \in D_6.$$

The associated subdivision algorithm is a simple "two-point scheme". This algorithmic interpretation will be useful for establishing a

Connection to Powell-Sabin spline. Somewhat surprisingly, when $t = -1/4$ the associated refinable Hermite interpolant ϕ coincides with what is given by a specific application of the classical Powell-Sabin method for spline Hermite interpolation [32], a method originally invented for interpolation on *irregular* triangulations. We show that when $t := -1/4$ the associated subdivision scheme converges to a refinable function vector ϕ consisting of C^1 piecewise quadratic polynomials. While computation based on (2.25) gives $\nu_2(\mathbf{a}, M) = 2.5$ and, by a Sobolev embedding theorem, $\nu_\infty(\phi) \geq \nu_\infty(\mathbf{a}, M) \geq 2.5 - 1 = 1.5$. This spline connection gives $\nu_\infty(\phi) = 2$. Another simple observation is that ϕ is supported at a hexagon around the origin. By symmetry property (2.29), $\phi_{(0,1)}(x,y) = \phi_{(1,0)}(y,x)$ and $\phi_{(1,0)}(-x,-y) = -\phi_{(1,0)}(x,y)$.

To establish such a connection, we first recall

Theorem 3.2 (Powell-Sabin)**.** *Let $T = \triangle V_1V_2V_3$ be a triangle on the plane, $E_1 = \overline{V_1V_2}$, $E_2 = \overline{V_2V_3}$, $E_3 = \overline{V_3V_1}$, $x_0 \in \text{int}(T)$, and $x_i \in \text{int}(E_i)$. By a Powell-Sabin (PS) split of T induced by $(x_i)_{i=0}^3$, we mean the partitioning of T into six smaller triangles based on connecting x_0 to V_i and to x_i.*

For any given set of order 1 Hermite data, $\partial^\nu f(V_i)$, $i = 1, 2, 3$, $|\nu| \leq 1$, and any PS split of T, $T = \bigcup_{i=1}^6 T_i$, there exists a unique C^1 spline function $f : T \to \mathbb{R}$ with $f|_{T_i} \in \Pi_2$ which interpolates the given Hermite data at the vertices of T.

Moreover, if $T = \triangle V_1V_2V_3$ and $T' = \triangle V_1'V_2'V_3'$ share a common edge in such a way that $V_2 = V_1'$ and $V_3 = V_2'$. Assume that $(x_i)_{i=0}^3$ and $(x_i')_{i=0}^3$ induce PS splits on T and T', respectively. Assume also that a set of order 1 Hermite data at the 4 vertices of $T \cup T'$ is given. Let $f : T \to \mathbb{R}$ and $f' : T' \to \mathbb{R}$ be the corresponding Powell-Sabin Hermite interpolants. Then the composite function $F : T \cup T' \to \mathbb{R}$, $F|_T = f$, $F|_{T'} = f'$ is C^1 (across $\overline{V_2V_3}$) for arbitrarily prescribed Hermite data if and only if x_0, $x_2 = x_2'$, x_0' are collinear.

When the PS splits of two adjacent triangles (i.e., the two triangles share a common edge) satisfy the last condition in this theorem, we say that the two PS splits are *compatible*.

The connection between the refinable Hermite interpolant constructed in Example 3.1 and Powell-Sabin Hermite interpolation on triangles is given in the following result.

Proposition 3.3. *Let $h_\alpha (\in \mathbb{R}^3)$ be order 1 Hermite data prescribed on $\mathbb{Z}^2$. Tile $\mathbb{R}^2$ by the uniform triangulation $T_{i,j} = \text{Convex Hull}\{(i,j),(i+1,j),(i+1,j+1)\}$. Introduce a PS split to each $T_{i,j}$ based on barycentric subdivision (i.e., $(x_i)_{i=0}^3$ that induces each PS split is such that x_0 is the barycenter of $T_{i,j}$ and x_i, $i=1,2,3$, are midpoints of the edges of $T_{i,j}$). Then obviously the PS splits of every two adjacent triangles are compatible and hence there is a globally C^1 Hermite interpolant given by Theorem 3.2. Call it P.*

We have $P = \sum_{\alpha\in\mathbb{Z}^2} h_\alpha^T \phi(\cdot - \alpha)$, where ϕ is the refinable Hermite interpolant constructed in Example 3.1, with the choice $t = -1/4$.

Proof. See Appendix. □

Proposition 3.3 can be restated using the language of SIS (shift-invariant subspaces) and refinable SIS. Following the spline literature, denote by

$$\Pi^\rho_{k,\Delta} = \Pi^\rho_{k,\Delta}(\mathbb{R}^s)$$

the space of all C^ρ piecewise degree k polynomials with some partition Δ. Let Δ_{PS} be the triangulation of $\mathbb{R}^2$ based on barycentric subdivision of $T_{i,j}$. Then $\Pi^1_{2,\Delta_{\text{PS}}}$ is exactly the refinable shift-invariant subspace generated by $\phi(\cdot - \alpha)$.

Our next example is a C^2 "Hermite Butterfly Scheme", a vector version of the C^1 Butterfly scheme by Dyn, Levin, and Gregory [17].

Example 3.4. With $G = D_6$, $M = 2I_2$ and

$$\begin{aligned}\text{supp}(\mathbf{a}) = \{&(0,0), \pm(1,0), \pm(1,2), \pm(-1,2), \pm(3,2), \pm(0,1), \pm(2,1),\\ &\pm(-2,1), \pm(2,3), \pm(1,1), \pm(-1,1), \pm(1,3), \pm(3,1)\},\end{aligned}$$

a D_6-symmetric Hermite interpolatory mask of order 1 reproducing Π_4 is given by

$$\mathbf{a}(0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix},$$

$$\mathbf{a}(1,0) = \begin{bmatrix} 1/2 & -25/32 & 25/64 \\ 1/8 & -11/64 & 9/64 \\ 0 & 0 & 7/64 \end{bmatrix},$$

$$\mathbf{a}(1,2) = \begin{bmatrix} 1/32 & 0 & -77/256 \\ 0 & 9/64 & -13/128 \\ 0 & 0 & -1/16 \end{bmatrix},$$

$$\mathbf{a}(-1,2) = \begin{bmatrix} -1/64 & -23/256 & -23/256 \\ 1/128 & 5/128 & -5/128 \\ 0 & 1/128 & -1/128 \end{bmatrix}.$$

Note: There is in fact a 7-parameter family of Hermite interpolatory masks reproducing Π_4. The above mask is obtained by an exhaustive search method (see Section 4). By (2.25), we found by computation that $\nu_2(\mathbf{a}, 2I_2) \approx 3.18383$ and, therefore, $\nu_2(\phi) \geq \nu_2(\mathbf{a}, 2I_2) \approx 3.18383$. So, by a Sobolev embedding theorem, $\phi \in C^2(\mathbb{R}^2)$ since $\nu_\infty(\phi) \geq \nu_2(\mathbf{a}, 2I_2) - 1 = 2.18383$.

3.3. $M = 2I_2$, $G = D_4$.

Example 3.5. Applying our recipe with $M = 2I_2$, $G = D_4$, and the D_4-symmetric support

$$\mathrm{supp}(\mathbf{a}) = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1), \pm(-1,1)\},$$

we find that solutions exist when $l \leq 2$. When $l = 2$, the parametric form of such a mask $\mathbf{a}$ is given by

$$\mathbf{a}(1,0) = \begin{bmatrix} 1/2 & 2t_1 - 1/2 & 0 \\ 1/8 & t_1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}, \quad \mathbf{a}(1,1) = \begin{bmatrix} 1/4 & 2t_3 - 1/4 & 2t_3 - 1/4 \\ 1/16 & t_3 & t_2 \\ 1/16 & t_2 & t_3 \end{bmatrix},$$

and $\mathbf{a}(E(1,1)^T) = S(E)\mathbf{a}(1,1)[S(E)]^{-1}$, $\mathbf{a}(E(1,0)^T) = S(E)\mathbf{a}(1,0)[S(E)]^{-1}$, $E \in D_4$.

We show that when $t_1 = -1/4$, $t_2 = t_3 = -1/8$ the associated refinable function is a C^1 quadratic spline supported on $[-1,1]^2$. Like in the previous section, we relate our subdivision scheme to a known spline scheme. In this case, the scheme is due to Sibson [19].

While Powell-Sabin's scheme is designed for interpolation on irregular triangulations, Sibson's scheme is for interpolation on regular rectangular grids:

Theorem 3.6 (Sibson)**.** *Let $(h_\alpha)_{\alpha\in\mathbb{Z}^s}$ be arbitrary order* 1 *Hermite data prescribed on $\mathbb{Z}^2$. For each unit square $[i,i+1]\times[j,j+1]$, there exists a C^1 piecewise quadratic P with partition lines shown in the left panel of Figure 1 or a C^1 piecewise cubic with partition lines shown in the middle panel which interpolates the Hermite data at the four corners.*

In either case, by imposing the extra assumption that the cross boundary derivatives are linear (equivalently $\partial P(\cdot,j)/\partial y$, $\partial P(\cdot,j+1)/\partial y$, $\partial P(i,\cdot)/\partial x$, $\partial P(i+1,\cdot)/\partial x$ are continuous at $i+1/2$, $i+1/2$, $j+1/2$, $j+1/2$ respectively), the interpolant is unique.

By using such a unique interpolant on each unit square, the overall interpolant is C^1 on $\mathbb{R}^2$.

See [19] for a simple proof based on the B-B form. We now establish a connection analagous to Proposition 3.3:

Proposition 3.7. *Let $h_\alpha(\in \mathbb{R}^3)$ be order* 1 *Hermite data prescribed on $\mathbb{Z}^2$. Tile $\mathbb{R}^2$ by $R_{i,j} = [i,i+1]\times[j,j+1]$. Introduce a Sibson I split (Left Panel, Figure 1) to each $R_{i,j}$. Let P be the unique Sibson's Hermite interpolant. We have $P = \sum_{\alpha\in\mathbb{Z}^2} h_\alpha^T \phi(\cdot - \alpha)$, where ϕ is the refinable Hermite interpolant constructed in Example 3.5, with the choice $(t_1,t_2,t_3) = (-1/4,-1/8,-1/8)$.*

One may be tempted to conjecture that with a different choice of parameter values (t_1,t_2,t_3) one connects the subdivision scheme in Example 3.5 with Sibson's piecewise cubic scheme. This temptation may be further intensified when one notices that by setting $t_1 = -1/8$, the subdivision scheme in Example 3.5, when restricted to one of the horizontal lines $y = j$ or vertical lines $x = i$, produces piecewise cubic curves—the same as what Sibson's cubic scheme produces. Moreover, along a horizontal line (resp. a vertical line), the scheme produces y-derivatives

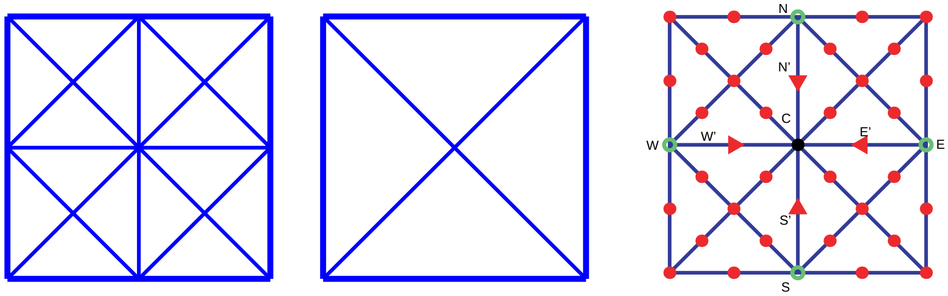

FIGURE 1. **Left**: Partition lines of Sibson's C^1 piecewise quadratic interpolant. **Middle**: Partition lines of Sibson's C^1 piecewise cubic interpolant. **Right**: B-B ordinates of Sibson's quadratic interpolant.

(resp. x-derivatives), which are linear on each unit interval—again the same as what Sibson's scheme produces.

This conjecture is not true.

To see this and also to give a proof for Propositon 3.7, we compare the SIS defined implicitly by the two spline interpolation schemes. Denote by Δ_{I} and Δ_{II} the triangulations of $\mathbb{R}^2$ implicitly defined by Sibson's quadratic and cubic schemes. Associated with Sibson's two spline interpolation schemes are the following vector spaces:

$$\mathcal{S}_{\mathrm{I}} := \left\{ p \in \Pi^1_{2,\Delta_{\mathrm{I}}}(\mathbb{R}^2) : \frac{\partial p(\cdot,j)}{\partial y}, \frac{\partial p(i,\cdot)}{\partial x} \in \Pi^0_{1,\Delta_{\mathbb{Z}}}(\mathbb{R}^1) \right\},$$
$$\mathcal{S}_{\mathrm{II}} := \left\{ p \in \Pi^1_{3,\Delta_{\mathrm{II}}}(\mathbb{R}^2) : \frac{\partial p(\cdot,j)}{\partial y}, \frac{\partial p(i,\cdot)}{\partial x} \in \Pi^0_{1,\Delta_{\mathbb{Z}}}(\mathbb{R}^1) \right\}.$$

(In the above, i,j are integers and $\Delta_{\mathbb{Z}}$ is the partition of $\mathbb{R}$ by unit intervals $[i,i+1]$.)

Both spaces are SIS containing Π_2. We note that $\mathcal{S}_{\mathrm{II}}$ does *not* contain Π_3.

The key observation: $\mathcal{S}_{\mathrm{I}}$ is a *refinable* SIS with respect to the dilation matrix $2I_2$; $\mathcal{S}_{\mathrm{II}}$ is not. As a consequence, there is no hope that the integer shifts of any of our refinable Hermite interpolants can span $\mathcal{S}_{\mathrm{II}}$.

We now calculate the subdivision mask needed to reproduce $\mathcal{S}_{\mathrm{I}}$—which also gives a proof of Proposition 3.7.

As in the proof of Proposition 3.3, we establish the corresponding relation (5.1) between Sibson's interpolant and our subdivision scheme, and we only need to do so for those α and n such that $2^{-n}\alpha \in [0,1)^2$. Write $H_{\epsilon_1} = A(\epsilon_1,0)^T$, $V_{\epsilon_2} = V(0,\epsilon_2)^T$ $D_{\epsilon_1,\epsilon_2} = A(\epsilon_1,\epsilon_2)^T$, for $\epsilon_1,\epsilon_2 \in \{-1,1\}$.

In our case (5.1) is equivalent to that for $P \in \mathcal{S}_{\mathrm{I}}$,

$$H_1 D_n^{\leq 1} P(\alpha) + H_{-1} D_n^{\leq 1} P(\alpha + [1,0]^T) = D_{n+1}^{\leq 1} P(2\alpha + [1,0]^T), \tag{3.1}$$
$$V_1 D_n^{\leq 1} P(\alpha) + V_{-1} D_n^{\leq 1} P(\alpha + [0,1]^T) = D_{n+1}^{\leq 1} P(2\alpha + [0,1]^T), \tag{3.2}$$
$$\sum_{\epsilon_1,\epsilon_2 \in \{-1,1\}} D_{\epsilon_1,\epsilon_2} D_n^{\leq 1} P(\alpha + [1-\epsilon_1, 1-\epsilon_2]^T) = D_{n+1}^{\leq 1} P(2\alpha + [1,1]^T). \tag{3.3}$$

Refinability of $\mathcal{S}_{\rm I}$ assures that if (3.1)–(3.3) holds for $n = 0$, then it holds for all n. Consequently we only need to check it for $n = 0$, $\alpha = 0$—a finite computation that can be carried out by hand calculations. We do so below, again using a B-B form.

We had explained earlier that with the choice $t_1 = -1/8$, (3.1)–(3.2) hold true.

Let $f(\alpha),\ f_x(\alpha),\ f_y(\alpha),\ \alpha = [0,0],[0,1],[1,1],[1,0]$ be order 1 Hermite data defined at the four corners of $[0,1]^2$. Let P be the corresponding Sibson's piecewise quadratic interpolant, with Bézier ordinates of its pieces depicted and labelled in the rightmost panel of Figure 1.

We deduce that $S = \frac{1}{2}\left(f(0,0) + \frac{1}{4}f_x(0,0) + f(1,0) - \frac{1}{4}f_x(1,0)\right)$, with similar expressions for E, W, and N. Sibson's schemes force P to satisfy

$$P_y(1/2,0) = (f_y(0,0) + f_y(1,0))/2, \quad P_y(1/2,1) = (f_y(0,1) + f_y(1,1))/2,$$
$$P_x(0,1/2) = (f_x(0,0) + f_x(0,1))/2, \quad P_x(1,1/2) = (f_x(1,0) + f_x(1,1))/2.$$

Then we get $S' = S + \frac{1}{4}P_y(1/2,0)$ and similarly for E', N', and W'. Finally, C is the average of these four B-ordinates (Farin's C^1 condition.) Using, for example, the B-ordinates C, E', and N' one can calculate P, P_x, and P_y at $[1/2,1/2]$.

This calculation gives the set of parameter values in Proposition 3.7.

We calculate also a subdivision mask which is exact in the *first subdivision step* when applied to arbitrary elements in $\mathcal{S}_{\rm II}$:

$$(t_1,t_2,t_3) = (-1/8,-1/8,-1/16).$$

In other words, the subdivision scheme with this mask satisfies (3.1)–(3.3) for all $P \in \mathcal{S}_{\rm II}$ but only for $n = 0$. The refinable SIS generated by this mask, albeit not the nonrefinable C^1 spline space $\mathcal{S}_{\rm II}$, consists of C^1 elements. In fact, using (2.25), the critical L^2 regularity of this interpolant is $\nu_2 = 2.5$.

In the next two subsections, we consider dilation matrices with determinants equal to 2 or 3, rather than 4 as in the case of $M = 2I_2$. The corresponding subdivision schemes expand data at rates of $2^n/3^n$, rather than 4^n. One of the advantages of this is that in geometric modelling applications more subdivision levels can be computed with a given amount of computer memory; hence, it offers a designer more levels of detail to work with in settings such as interactive editing.

3.4. $M = M_{\mathbf{Quincunx}}$, $G = D_4$.

Example 3.8. As before we first apply our recipe to the smallest D_4-symmetric support which is also nonempty in each coset of $\mathbb{Z}^2/M\mathbb{Z}^2$; in this case,

$$\mathrm{supp}(\mathbf{a}) = \{(0,0), \pm(1,0), \pm(0,1)\}.$$

In this very case, we found that the associated linear system has a *unique* solution when $l = 3$, and the mask is given by

$$\mathbf{a}(0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}, \quad \mathbf{a}(1,0) = \begin{bmatrix} 1/4 & -3/4 & 0 \\ 1/16 & -1/8 & 0 \\ 1/16 & -1/8 & 0 \end{bmatrix},$$

and other entries of the mask are given by (2.30).

Our smoothness computation reveals that the associated refinable Hermite interpolant has critical L^2 smoothness $\nu_2 = 2.7464$, and hence it is at least C^1. The same smoothness information is also obtained in the paper [28].

3.5. $M = M_{\sqrt{3}}$, $G = D_6$.

Example 3.9. Our last example involves a so-called "$\sqrt{3}$-subdivision" scheme with support

$$\text{supp}(\mathbf{a}) = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1)\}.$$

We found that solutions exist as long as $l \le 2$; when $l = 2$, the parametric form of such a mask is given by

$$\mathbf{a}(0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 2/3 \\ 0 & -2/3 & 1/3 \end{bmatrix}, \quad \mathbf{a}(1,0) = \begin{bmatrix} 1/3 & -4t-8/9 & 2t+4/9 \\ -1/18 & 1/9+t & -t \\ -1/9 & 2/9+2t & -1/9-t \end{bmatrix}.$$

Use (2.30) again to get the rest of the mask.

Note: Despite the fact that the support of this scheme is the same as that in Example 3.1, the two subdivision operators have rather different structures. Note that $\mathbb{Z}^2/2\mathbb{Z}^2$ has four cosets, whereas $\mathbb{Z}^2/M_{\sqrt{3}}\mathbb{Z}^2$ has three. In the former case, $\mathbf{a}(0,0)$, $\mathbf{a}(\pm 1,0)$, $\mathbf{a}(0,\pm 1)$, $\mathbf{a}(\pm 1,\pm 1)$ are "responsible" for the subdivision of data at the four cosets, whereas in the $\sqrt{3}$ case, the corresponding partitioning of supp(a) should be $\{\mathbf{a}(0,0)\}$, $\{\mathbf{a}(1,0), \mathbf{a}(0,1), \mathbf{a}(-1,-1)\}$, $\{\mathbf{a}(1,1), \mathbf{a}(-1,0), \mathbf{a}(0,-1)\}$.

Computation shows that when t is in a neighborhood of 0 the associated refinable Hermite interpolant is at least C^1. When $t = 0$, we found that $\nu_2(\mathbf{a}, M_{\sqrt{3}}) = 2.7683$ and therefore, $\nu_2(\phi) = \nu_2(\mathbf{a}, M_{\sqrt{3}}) = 2.7683$.

4. Conclusion

In this paper we define and analyze refinable Hermite interpolants in a general multivariate setting. We also study symmetry properties of such Hermite interpolants. A computational approach is used to construct several new interpolatory Hermite subdivision schemes in dimension 2 and are currently applied to build free-form subdivision surfaces.

In this construction method we deliberately avoid the issue of how one may exploit the free parameters one typically gets from solving the associated linear system. For example, in Example 3.4, the associated linear system has a null space of dimension 7, resulting in a parametric mask with seven parameters. Suppose that a Hermite interpolatory mask $\mathbf{a}$ reproduces Π_k but not Π_{k+1} in the sense of (2.15). Let H_k be given in (2.21). Define $V = \text{span}\{u : \hat{u}(\xi) = \hat{v}(\xi)\overline{\hat{w}(\xi)^T}, v, w \in H_k\} \cap [l(K)]^{m\times m}$, where the compact set K is completely determined by the support of the mask $\mathbf{a}$ and the dilation matrix M. Clearly, the space V is *independent of all parameters* $\mathbf{t}$. The spectral radius $\rho(F_a(\mathbf{t})|_V)$ determines the quantity $\nu_2(\mathbf{a}, M)$ and therefore gives us a lower bound of the critical L^2-Sobolov regularity of the refinable functions (when $\nu_2(a, M) > r + s/2$, its refinable function vector ϕ must be a Hermite interpolant and consequently $\nu_2(\phi) = \nu_2(\mathbf{a}, M)$). As in [25], one can even further take into account symmetry of the mask to significantly reduce the dimension of the finite dimensional space V (see [26]). To optimize the quantity $\nu_2(\mathbf{a}, M)$ within this family, one may consider solving the optimization problem

$$\min_{\mathbf{t}\in\mathbb{R}^7} \rho(F_a(\mathbf{t})|_V). \tag{4.1}$$

Problems of this form are nonsmooth, nonconvex optimization problems and may have many different local minima. A recently developed gradient-bundle method [7], originally designed for solving the spectral abscissa optimization problem in control theory, can be adapted to optimize the spectral radius. This optimization method has been pursued in the companion paper [26].

5. Appendix

Proof of Lemma 2.6. Case 1°: $k \le r$. Choose a large enough R such that $R \ge r'$ and $2R+1 \ge k$. By well-posedness of univariate Hermite interpolation, there exists a unique 1-D Hermite interpolant of order R, denoted by ψ_l, $l = 0, \ldots, R$, such that $\mathrm{supp}(\psi_l) = [-1, 1]$, $\psi_l|_{[-1,0]}, \psi_l|_{[0,1]} \in \Pi_{2R+1}(\mathbb{R}^1)$ and $\psi_l \in C^R(\mathbb{R}^1) \subseteq C^{r'}(\mathbb{R}^1)$. We also have

$$\mathrm{span}\{\psi_l(\cdot - \alpha) : l = 0, \ldots, R, \alpha \in \mathbb{Z}\} \supseteq \Pi_{2R+1}(\mathbb{R}^1) \supseteq \Pi_k(\mathbb{R}^1).$$

Consider $\phi_\mu \in C^{r'}(\mathbb{R}^s)$, $\mu \in \Lambda_R^\sharp := \{\mu \in \mathbb{N}_0^s : \max_i \mu_i \le R\}$, defined by

$$\phi_\mu(x_1, \ldots, x_s) := \psi_{\mu_1}(x_1) \cdots \psi_{\mu_s}(x_s).$$

It is clear (even without the assumption $k \le r$) that $(\phi_\mu)_{\mu \in \Lambda_r}$ is a Hermite interpolant of order r supported at $[-1, 1]^s$. We claim that $\mathrm{span}\{\phi_\mu(\cdot - \alpha) : \alpha \in \mathbb{Z}^s, \mu \in \Lambda_r\} \supseteq \Pi_k(\mathbb{R}^s)$. Let ν be a multi-index with $|\nu| \le k$, then each ν_i satisfies $\nu_i \le k \le 2R + 1$ and $(\cdot)^{\nu_i}$ is spanned by $\psi_l(\cdot - \alpha)$, $l = 0, \ldots, R$, $\alpha \in \mathbb{Z}$. But $(\psi_i)_{i=0,\ldots,R}$ is a 1-D Hermite interpolant, so

$$(\cdot)^{\nu_i} = \sum_\alpha \sum_{l=0}^{R} \partial^l (\cdot)^{\nu_i}|_\alpha \psi_l(\cdot - \alpha) = \nu_i! \sum_\alpha \sum_{l=0}^{\nu_i} \frac{\alpha^{\nu_i - l}}{(\nu_i - l)!} \psi_l(\cdot - \alpha).$$

In other words it is enough to use only $\{\psi_l(\cdot - \alpha)$: $l = 0, \ldots, \nu_i$, $\alpha \in \mathbb{Z}\}$ to span $(\cdot)^{\nu_i}$. As a consequence, $x^\nu = x_1^{\nu_1} \cdots x_s^{\nu_s}$ can be written as a linear combination of shifts of $(\phi_\mu)_{\mu \in \Lambda_k}$. But $\Lambda_k \subseteq \Lambda_r$, so the claim is proved.

Case 2°: $k > r$. It suffices to construct a local shift invariant interpolation operator (see discussions around (2.2)) with accuracy order k and $C^{r'}$ smoothness. We use an old trick called "boolean sum" [6]: we construct two shift-invariant operators I and $A : [l(\mathbb{Z}^s)]^{1\times m} \to [C^{r'}(\mathbb{R}^s)]^m$ with the properties that I interpolates order r Hermite data (i.e., $D_0^{\le r}(Iv) = v$) but may not have the desired accuracy order, A is exact on Π_k (i.e., $Av = p$ if $v = D_0^{\le r} p$ and $p \in \Pi_k$) but may not interpolate. (As before, $m = \#\Lambda_r$.) Consider the operator

$$I \oplus A := I + A - I \cdot A.$$

Here $I \cdot A$ is interpreted as follows. $(I \cdot A)v = Iw$, where $w \in [l(\mathbb{Z}^s)]^m$ is the order r Hermite data sampled from the C^r (recall $r' \ge r$) function Iv.

It is easy to verify that $I \oplus A$ has the desired properties.

The proof of Case 1° takes care of the construction of such an I.

One way to construct such an A is to construct an operator that interpolates *only* the point values $v(\alpha)_{\mathbf{0}}$ in an input Hermite data $v \in [l(\mathbb{Z}^s)]^m$. Consider the tensor product polynomial space

$$\Pi_k(\mathbb{R}^1) \times \cdots \times \Pi_k(\mathbb{R}^1) := \mathrm{span}\{x^\mu : \mu \in \Lambda_k^\sharp\} \supset \Pi_k(\mathbb{R}^s).$$

For each $\alpha \in \mathbb{Z}^s$, there exists a unique $p_\alpha \in \Pi_k(\mathbb{R}^1) \times \cdots \times \Pi_k(\mathbb{R}^1)$ such that $p_\alpha(\alpha + \beta) = v(\alpha + \beta)_{\mathbf{0}}$, for β with $0 \le \beta_i \le k$. Pick a compactly supported $C^{r'}$

function N whose shifts form a smooth partition of unity: $\sum_\alpha N(\cdot-\alpha)=1$. Then define the operator A by

$$(Av)(x) := \sum_{\alpha\in\mathbb{Z}^s} N(x-\alpha)p_\alpha(x).$$

It is clear that A possesses all the desired properties. □

Proof of Proposition 3.3. By Proposition 2.1, it suffices to show

$$(S^n h)(\alpha) = D_n^{\le 1}P(\alpha) \quad \forall\, n,\alpha, \tag{5.1}$$

where $D_n^{\le 1}P(\alpha) := [P(2^{-n}\alpha),\ 2^{-n}P_x(2^{-n}\alpha),\ 2^{-n}P_y(2^{-n}\alpha)]^T$. Also, it suffices to show (5.1) for only those n and α such that $2^{-n}\alpha\in T_{0,0}$.

When $t=-1/4$, write $H_{\pm1}=A(\pm1,0)^T$, $V_{\pm1}=A(0,\pm1)^T$, $D_{\pm1}=A(\pm1,\pm1)^T$. Keep in mind that

$$A(0,0)^T = \mathrm{diag}([1,1/2,1/2]),$$

(5.1) is equivalent to

$$H_1 D_n^{\le 1}P(\alpha) + H_{-1}D_n^{\le 1}P(\alpha+[1,0]^T) = D_{n+1}^{\le 1}P(2\alpha+[1,0]^T) \tag{5.2}$$

together with similar expressions associated with vertical $\overline{(2^{-n}\alpha)(2^{-n}(\alpha+[0,1]^T)}$ and diagonal $\overline{(2^{-n}\alpha)}\ \overline{(2^{-n}(\alpha+[1,1]^T)}$ edges.

The key observation leading to (5.2) is summarized in Figure 2. The interior of every dyadic horizontal edge $\overline{(2^{-n}\alpha)(2^{-n}(\alpha+[1,0]^T)}$ either intersects the PS split line $\overline{CM_1}$ of $T_{0,0}$ at $2^{-n}(\alpha+[1/2,0]^T)$ or lies entirely inside one of the six sub-triangles of $T_{0,0}$. In either case—*the second being a special case of the first*—proving (5.2) boils down to answering the following question: Let $p,q\in\Pi_2$ be such that the composite function F defined to be p on the left of $\overline{CM_1}$ and q on the

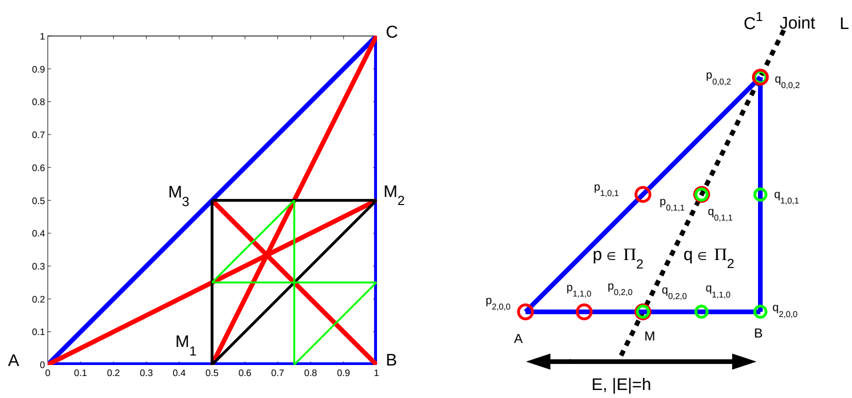

FIGURE 2. **Left**: Interaction of barycentric subdivision and recursive midpoint subdivision: the interior of any horizontal (resp. vertical and diagonal) edge of any triangle derived from recursive midpoint subdivision of $T_{0,0}$, if intersects at all, intersects only the barycentric subdivision line $\overline{CM_1}$ (resp. $\overline{AM_2}$ and $\overline{BM_3}$) and at exactly the midpoint of the edge. **Right**: Two quadratic polynomials meeting at a C^1 joint, and are represented in B-B form with respect to the left and right triangles

right is C^1. Let E be a horizontal line segment intersecting $\overline{CM_1}$ at E's midpoint. Can one uniquely determine the value and the first partial derivatives of F at the midpoint of E, from those at its two endpoints?

The answer is affirmative. Moreover, if the partial derivatives are appropriately scaled, the mapping (which is necessarily linear) is independent of E, yielding the *stationary* subdivision rule (5.2).

To see this, we use Bernstein-Bézier form of multivariate polynomials [19], [13].[1] The two facts we use from this theory are (i) a relation, expressed in terms of the de Casteljau algorithm, between Bézier ordinates and partial derivatives (Lemma 2.6 of [19]) and (ii) a C^k condition of a piecewise polynomial expressed in terms of Bézier ordinates (Corollary 2.10 of [19].)

We return to the situation we are facing, which is now depicted in the right panel of Figure 2. Quadratic polynomials p and q are represented by their Bézier ordinates $p_{i,j,k}$, $q_{i,j,k}$, $i+j+k=2$, respectively, relative to the left and right triangles in the figure. Let p, p_x, p_y, q, q_x, q_y be order 1 Hermite data of p and q at the left and right endpoints of E. An easy application of Lemma 2.6 of [19] gives $p_{2,0,0} = p$, $p_{1,1,0} = p + \frac{h}{4}p_x$, $p_{1,0,1} = p + \frac{h}{2}p_x + \frac{h}{2}p_y$, $q_{2,0,0} = q$, $q_{1,1,0} = q - \frac{h}{4}q_x$, $q_{1,0,1} = q + \frac{h}{2}q_y$.

By Farin's C^1 condition, $p_{0,2,0} = q_{0,2,0} = \frac{1}{2}(p_{1,1,0} + q_{1,1,0})$, $p_{0,1,1} = q_{0,1,1} = \frac{1}{2}(p_{1,0,1} + q_{1,0,1})$.

Again use Lemma 2.6 of [19] to obtain the following one-to-one correspondence between Bézeir ordinates $q_{0,2,0}, q_{1,1,0}, q_{0,1,1}$ and order 1 Hermite data at the midpoint of E: $q(M) = q_{0,2,0}$, $q_{0,2,0} + \frac{h}{4}q_x(M) = q_{1,1,0}$, $q_{0,2,0} + \frac{h}{4}q_x(M) + \frac{h}{2}q_y(M) = q_{0,1,1}$.

The above expressions combine to give

$$\begin{bmatrix} q(M) \\ \frac{h}{2}q_x(M) \\ \frac{h}{2}q_y(M) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/8 & 0 \\ -1 & -1/4 & 0 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} p \\ hp_x \\ hp_y \end{bmatrix} + \begin{bmatrix} 1/2 & -1/8 & 0 \\ 1 & -1/4 & 0 \\ -1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} q \\ hq_x \\ hq_y \end{bmatrix}.$$

This proves (5.2) when setting $h = 2^{-n}$.

The vertical and diagonal versions of (5.2) can be either verified directly or inferred from (5.2) by symmetry. □

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[1]Both [13] and [19] have coverage of the proof of Theorem 3.2 in the language of B-B form.

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