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Smoothing nonlinear subdivision schemes by averaging

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Abstract In the theory of linear subdivision algorithms, it is well-known that the regularity of a linear subdivision scheme can be elevated by one order (say, from C^k to C^{k+1}) by composing it with an averaging step (equivalently, by multiplying to the subdivision mask $a(z)$ a $(1 + z)$ factor. In this paper, we show that the same can be done to nonlinear subdivision schemes: by composing with it any nonlinear, smooth, 2-point averaging step, the lifted nonlinear subdivision scheme has an extra order of regularity than the original scheme. A notable application of this result shows that the classical Lane-Riesenfeld algorithm for uniform B-Spline, when extended to Riemannian manifolds based on geodesic midpoint, produces curves with the same regularity as their linear counterparts. (In particular, curvature does not obstruct the nonlinear Lane-Riesenfeld algorithm to inherit regularity from the linear algorithm.) Our main result uses the recently developed technique of differential proximity conditions.

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1 Introduction

In [7], Lane and Riesenfeld presented a subdivision scheme for constructing B-splines by repeated averaging. Later, Noakes [8] extended the Lane-Riesenfeld algorithm for C^1, B-splines to the manifold setting, by replacing the averaging step by bisection of geodesics, and showed that the limiting curves were C^1.

More generally, nonlinear subdivision schemes based on repeated nonlinear averaging were studied in the functional setting by Dyn and Goldman in [4]. They proved that the limiting curves of these averaging schemes were also C^1, and they conjectured that increasing the number of averaging steps would lead to increased regularity. In [1], we showed that this conjecture is false in the most general case considered in [4] (see below). However, we show here that the conjecture holds for a subclass of these schemes. This result is a corollary to a general averaging result, which we now describe. Recall that a manifold-valued subdivision scheme S with values in a smooth manifold M is defined as follows. Let $U \subset M \times M$ be an open neighborhood of the diagonal $\Delta_M = \{(x, x) : x \in M\}$. A sequence

$$\mathbf{x} : \mathbb{Z} \to M : i \mapsto x_i$$

is said to be *sufficiently dense with respect to U* if $(x_i, x_{i+1}) \in U$ for all $i \in \mathbb{Z}$. Let $\ell_U \subset \ell(\mathbb{Z} \to M)$ denote the set of sufficiently dense sequences with respect to U. We also use the notation ℓ_U^L to refer to the set of sufficiently dense tuples of length L with respect to U, i.e., $\ell_U^L := \{(x_1, \ldots, x_L) \in M \times \cdots \times M : (x_i, x_{i+1}) \in U\}$.

Let us recall the definition of a stationary binary subdivision scheme. Let $L_\sigma, m_\sigma \in \mathbb{Z}$, $L_\sigma > 1$, $\sigma = 0, 1$, be integers and let $U \subset M \times M$ be an open neighborhood of the diagonal, and let

$$q_\sigma : \ell_U^{L_\sigma+1} \to M, \tag{1.1}$$

be continuous maps fixing the hyper-diagonal $M_\Delta \subset M \times \cdots \times M$, i.e.,

$$q_\sigma(x, \ldots, x) = x. \tag{1.2}$$

Definition 1.1 A map $S : \ell_U \to \ell(\mathbb{Z} \to M)$ is called a *stationary binary subdivision scheme on M*, or simply a subdivision scheme on M, if it is given by the formula

$$(S\mathbf{x})_{2i+\sigma} = q_\sigma(x_{i-m_\sigma}, \cdots, x_{i-m_\sigma+L_\sigma}), \quad \sigma = 0, 1, \ i \in \mathbb{Z}. \tag{1.3}$$

The maps q_0, q_1 are called the *even and odd rules* of S, and L_σ and m_σ are called (respectively) the *locality factors* and *phase factors* of S.

Definition 1.2 We say that S is a C^k*-smooth* subdivision scheme, if there exists an open neighborhood V of the diagonal Δ_M, such that for every sequence $\mathbf{x} \in \ell_V$, all the iterates $S^j\mathbf{x}$ are sufficiently dense, and there is a C^k-map $F : \mathbb{R} \to M$ such that

$$\lim_{j\to\infty} (S^{j+n}\mathbf{x})_{2^{j+n}t_0} = F(t_0)\,,$$

for every dyadic integer $t_0 = k/2^n$. The map F is called the *subdivision curve* defined by $\mathbf{x}$, and $\mathbf{x}$ is called the *control data* defining F.

Given a binary subdivision scheme S on M, we can obtain a new subdivision scheme by following S with an "averaging" step. More precisely, let $U_A \subset M \times M$ be an open neighborhood of the diagonal of $M \times M$ as above, which satisfies the additional symmetry property

$$(x, y) \in U_A \implies (y, x) \in U_A \tag{1.4}$$

for all $(x, y) \in U_A$.

Definition 1.3 A *(nonlinear) averaging rule* on M is a C^∞-map[1] $A : U_A \to M$ satisfying the following conditions: (i) $A(x, x) = x$ for all $x \in M$, and (ii) $A(x, y) = A(y, x)$ for all $(x, y) \in U_A$.

Remark 1.4 It follows from (i) and (ii) that

$$dA|_{(x,x)}\,(X, Y) = \frac{1}{2}X + \frac{1}{2}Y. \tag{1.5}$$

Condition (1.5) is weaker than condition (ii) and is sufficiently strong for our proofs. It is more natural and more practical, however, to impose the stronger condition (ii) (and the associated condition (1.4)).

Given a nonlinear, binary subdivision rule S and a nonlinear averaging rule A, one can form a new nonlinear subdivision rule $\overline{S}$ by composition:

$$\begin{aligned}(\overline{S}\mathbf{x})_{2i} &= A\left((S\mathbf{x})_{2i}, (S\mathbf{x})_{2i+1}\right) && \text{(1.6a)}\\ &= A\left(q_0(x_{i-m_0}, \ldots, x_{i-m_0+L_0}), q_1(x_{i-m_1}, \ldots, x_{i-m_1+L_1})\right)\\ (\overline{S}\mathbf{x})_{2i+1} &= A\left((S\mathbf{x})_{2i+1}, (S\mathbf{x})_{2i+2}\right) && \text{(1.6b)}\\ &= A\left(q_1(x_{i-m_1}, \ldots, x_{i-m_1+L_1}), q_0(x_{i+1-m_0}, \ldots, x_{i+1-m_0+L_0})\right).\end{aligned}$$

[1] For simplicity, we assume that A is C^∞, however, a careful examination of our results shows that they hold under weaker regularity assumptions.

Remark 1.5 Notice that $\overline{S}\mathbf{x}$ is only well-defined when $\mathbf{x} \in S^{-1}(\ell_{U_A})$. Fortunately, there is an open neighborhood U' of the diagonal of $M \times M$, such that $\ell_{U'} \subset S^{-1}(\ell_{U_A})$, as required by Definition 1.1. To see this, consider the maps

$$q_{01} : \ell_U^{L_0+1} \times \ell_U^{L_1+1} \to M \times M \; : \; (x_0, \ldots, x_{L_0}, x'_0, \ldots, x'_{L_1}) \mapsto (q_0(x_0, \ldots, x_{L_0}), q_1(x'_0, \ldots, x'_{L_1})) \, ,$$

and

$$q_{10} : \ell_U^{L_1+1} \times \ell_U^{L_0+1} \to M \times M \; : \; (x_0, \ldots, x_{L_1}, x'_0, \ldots, x'_{L_0}) \mapsto (q_1(x_0, \ldots, x_{L_1}), q_0(x'_0, \ldots, x'_{L_0})) \, .$$

By continuity, $q_{01}^{-1}(U_A)$ and $q_{10}^{-1}(U_A)$ are open neighborhoods of the hyper-diagonal (i.e., the constant sequences) of $\ell_U^{L_0+L_1+2}$. As we show in Appendix B, there exists an open neighborhood U' of the diagonal of $M \times M$ such that $\ell_{U'}^{L_0+L_1+2} \subset q_{01}^{-1}(U_A) \cap q_{10}^{-1}(U_A)$. It follows that $\overline{S}\mathbf{x}$ is well-defined for every $\mathbf{x} \in \ell_{U'}$.

The goal of this paper is to prove the following regularity result.

Theorem 1.6 *Let A be a nonlinear averaging rule, and let S be a C^k-smooth, nonlinear, binary subdivision scheme, compatible with a C^k-smooth, stable, linear, binary subdivision scheme S_{lin}. Then the averaged scheme $\overline{S}$ defined in* (1.6a) *is a C^{k+1}-smooth, binary linear subdivision scheme.*

Remark 1.7 Following the approach in [4], one could consider the more general nonlinear subdivision rule given by

$$(\overline{S}\mathbf{x})_{2i} = A\left((S\mathbf{x})_{2i}, (S\mathbf{x})_{2i+1}\right) \tag{1.7a}$$

$$(\overline{S}\mathbf{x})_{2i+1} = B\left((S\mathbf{x})_{2i+1}, (S\mathbf{x})_{2i+2}\right) \tag{1.7b}$$

where A and B are *different* nonlinear averaging rules. However, as we showed in [1] and the end of Section 5, in general the resulting rule will *not* be C^{k+1}.

By the main result of [1], to prove Theorem 1.6, it suffices to check that the averaged rule $\overline{S}$ is compatible with a stable, C^{k+1}, linear subdivision rule and that it satisfies the so-called order $k+1$ ***differential proximity condition***.

Organization of paper In Section 2, we review the differential proximity condition of [1]. In Section 3, we prove some elementary facts about linear subdivision schemes needed in the proof of Theorem 1.6. Section 4 contains the proof of Theorem 1.6. In Section 5, we apply our main result Theorem 1.6 to prove smoothness results for a wide class of nonlinear Lane-Riesenfeld schemes.

2 The differential proximity condition

2.1 The compatibility condition

Recall that the data defining a (binary) linear subdivision scheme consist of locality and phase factors, L_σ, m_σ, together with linear functionals

$$q_{\text{lin},\sigma} : \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R} : (x_0, \ldots, x_{L_\sigma}) \mapsto \sum_{i=0}^{L_\sigma} a_{\sigma,i} x_i, \quad \sigma = 0, 1,$$

satisfying the sum rules $\sum_i a_{\sigma,i} = 1$. Notice that $q_{\text{lin},\sigma}$ extend to linear maps

$$q_{\text{lin},\sigma} : V \times V \times \cdots \times V \to V : (v_0, \ldots, v_{L_\sigma}) \mapsto \sum_{i=0}^{L_\sigma} a_{\sigma,i} v_i,$$

where V denotes any vector space over $\mathbb{R}$. The sum rules imply that $q_{\text{lin},\sigma}$ satisfies the condition (1.2), and formula (1.3) defines a subdivision scheme S_{lin} on $M = V$ for any vector space V.

Definition 2.1 We say that a subdivision scheme S is *smoothly compatible with the linear subdivision scheme* S_{lin} if S_{lin} and S have the same phase and locality factors, and the maps q_σ are at least C^1-smooth with derivative $dq_\sigma|_{(x,\ldots,x)} : T_x M \times \cdots \times T_x M \to T_x M$ satisfying the identity

$$dq_\sigma|_{(x,\ldots,x)}(X_0, \ldots, X_{L_\sigma}) = q_{\text{lin},\sigma}(X_0, \ldots, X_{L_\sigma}), \quad \sigma = 0, 1, \tag{2.1}$$

for all $x \in M$.

2.2 The differential proximity condition

We can encode both of the maps q_0, q_1 into a single map Q as follows. First notice that (1.1)-(1.3) imply that there is a smallest positive integer K such that any $K+1$ consecutive entries in any (sufficiently dense) sequence $\mathbf{x}$ determines *exactly* $K+1$ consecutive entries in $S\mathbf{x}$. We then say that S has a *minimal invariant neighborhood of size* $K+1$, and the map

$$Q : U_K \to \underbrace{M \times \cdots \times M}_{K+1 \text{ copies}}, \tag{2.2}$$

is defined by the following property: For any dense enough sequence $\mathbf{x} \in \ell_U$ and $\mathbf{y} = S\mathbf{x}$,

$$Q(x_i, \ldots, x_{i+K}) = (y_{2i+s}, \ldots, y_{2i+s+K}), \quad \text{for all } i, \tag{2.3}$$

and a uniquely determined "shift factor" s (related to the phase and locality factors of q_0 and q_1.) Here $U_K \subset \underbrace{M \times \cdots \times M}_{K+1 \text{ copies}}$ denotes an open set of sufficiently dense $(K+1)$-tuples of points in M.

It is useful to note that when the input sequence $\mathbf{x}$ is shifted by one entry, the output sequence $\mathbf{y}$ is shifted by two entries. This characteristic of a subdivision operator is also reflected in (2.3) above.

The differential proximity condition is most easily written in local coordinates. Choose local coordinates for M defined on a neighborhood of an arbitrary point $p_0 \in M$ and centered so that p_0 is identified with the origin, and we now let $Q(x_0, x_1, \ldots, x_K)$ denote the local coordinate expression for Q, which is now defined on a neighborhood of the origin. In these coordinates, Q fixes the hyper-diagonal $\{(x, x, \ldots, x) : x \in \mathbb{R}^n\}$ in $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$.

Next make a linear change of coordinates. Let ∇, $\Sigma = \nabla^{-1} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the linear maps defined by the correspondence

$$(x_0, x_1, \ldots, x_K) \underset{\Sigma}{\overset{\nabla}{\rightleftarrows}} (\delta_0 = x_0, \delta_1 = x_1 - x_0, \ldots, \delta_K), \tag{2.4}$$

where δ_k := k-th order difference of $x_0, x_1, \ldots, x_k$, so $\delta_k = \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} x_\ell$, and $x_k = \sum_{\ell=0}^{k} \binom{k}{\ell} \delta_\ell$.

Finally, for $W \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{K+1 \text{ copies}}$ a sufficiently small neighborhood of the origin, define

$$\Psi : W \to \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{K+1 \text{ copies}}$$

by the formula

$$\Psi := \nabla \circ Q \circ \Sigma. \tag{2.5}$$

We write

$$\Psi = (\Psi_0, \Psi_1, \ldots, \Psi_K), \quad \Psi_\ell : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n,$$

when referring to the different components of Ψ. Observe that, in these coordinates, the fixed point set of Ψ is $\{(\delta_0, 0, \ldots, 0) : \delta_0 \in \mathbb{R}^n\} \cap W$; and the compatibility condition now assumes the form

$$d\Psi|_{(x,0,\ldots,0)} = \Psi_{\text{lin}} := \nabla \circ Q_{\text{lin}} \circ \Sigma, \quad \text{for all } x\ . \tag{2.6}$$

Definition 2.2 Let S be a subdivision scheme on M smoothly compatible with S_{lin}. Let $k \geq 1$. We say that S satisfies the *order k differential proximity condition* if for every point $p_0 \in M$ and for local coordinates as above,

$$D^J \Psi_\ell|_{(\delta_0,0,\ldots,0)} = 0, \quad \text{when } |J| \geq 2, \quad \text{weight } (J) := \sum_{i=1}^{K} i j_i \leq \ell, \quad \text{for } 1 \leq \ell \leq k, \tag{2.7}$$

for all $(\delta_0, 0, \ldots, 0) \in W$, where $J = (j_1, \ldots, j_K)$, $|J| = \sum_i j_i$.

Remark 2.3 In the above definition, we have adopted the following notation. For f any smooth, $\mathbb{R}^N$-valued function of $\boldsymbol{\delta}$, the symbol $D^J f$ denotes the mixed partial derivative

$$D^J f = \frac{\partial^{|J|} f}{\partial \delta_1^{j_1} \partial \delta_2^{j_2} \ldots \partial \delta_K^{j_K}} : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{|J|\text{ copies}} \to \mathbb{R}^N ,$$

which we view as a multi-linear map. The Taylor expansion of f in $(\delta_1, \ldots, \delta_K)$ can then be written as

$$f(x_0, 0, \ldots, 0) + \sum_{m=1}^{\infty} \sum_{|J|=m} \frac{1}{J!} \, D^J f\Big|_{(x_0,0,\ldots,0)} (\underbrace{\delta_1, \cdots, \delta_1}_{j_1}, \cdots, \underbrace{\delta_K, \cdots, \delta_K}_{j_K}) ,$$

where $J! = j_1! j_2! \ldots j_K!$. In this paper, the derivatives $D^J f$ will always be computed at $(x_0, 0, \ldots, 0)$, we therefore drop the subscript and let $D^J f = D^J\big|_{(x_0,0,\ldots,0)}$.

The main result of [1] is the following necessary and sufficient condition for C^k-smoothness.

Theorem 2.4 *Let S be a subdivision scheme on a manifold smoothly compatible with the stable, C^k-smooth, linear, binary subdivision scheme S_{lin}. Then S is C^k-smooth* ***if and only if*** *it satisfies the order k differential proximity condition.*

2.3 A trick for checking the proximity condition

To prove Theorem 1.6, we must verify the proximity condition for the averaged subdivision scheme. To set the stage for the final proof in Section 4, we use a trick from our earlier work [1].

Consider the function $Q \circ \Sigma$ in local coordinates and set

$$\mathbf{y} = (y_0, \ldots, y_K) = Q \circ \Sigma(\boldsymbol{\delta}) ,$$

where $\boldsymbol{\delta} = (x_0, \delta_1, \delta_2, \ldots, \delta_K) \in \underbrace{\mathbb{R}^n \times \ldots \mathbb{R}^n}_{(K+1)\text{ copies}}$, and let

$$\Delta^\ell(\mathbf{y}) = (\Delta^\ell(\mathbf{y})_1, \ldots, \Delta^\ell(\mathbf{y})_{K-\ell})$$

the sequence of differences of order ℓ. Thus,

$$\Psi(\boldsymbol{\delta}) = (x_0, \Delta^1(\mathbf{y})_0, \Delta^2(\mathbf{y})_0, \ldots, \Delta^K(\mathbf{y})_0)$$

and

$$D^J \Psi = (0, D^J \Delta^1(\mathbf{y})_0, D^J \Delta^2(\mathbf{y})_0, \ldots, D^J \Delta^K(\mathbf{y})_0) = (0, \Delta^1(D^J \mathbf{y})_0, \Delta^2(D^J \mathbf{y})_0, \ldots, \Delta^K(D^J \mathbf{y})_0). \tag{2.8}$$

The order k differential proximity condition can then be rephrased as follows:

The equality $(\Delta^\ell D^J \mathbf{y})_0 = 0$ *holds for all* ℓ *and* J *such that* $1 < \ell \leq k$, $|J| > 1$, *and* $w(J) \leq \ell$.

A straightforward induction argument shows that this is equivalent to

The equality $(\Delta^\ell D^J \mathbf{y})_i = 0$ *holds for all* i, ℓ *and* J *such that* $1 < \ell \leq k$, $|J| > 1$, $w(J) \leq \ell$, *and* $0 \leq i \leq k - \ell$.

We need the following fact, which can be easily shown by induction. (See the Appendix.)

Lemma 2.5 *Let* $\mathbf{v} = (v_0, v_1, \ldots, v_k)$, $k > 1$, *be a finite sequence of vectors in a vector space* V, *and let* ℓ, $1 \leq \ell \leq k$ *be an integer. Consider the sequence of differences of order* ℓ:

$$\Delta^\ell \mathbf{v} = ((\Delta^\ell \mathbf{v})_0, \ldots, (\Delta^\ell \mathbf{v})_{k-\ell})$$

Then $(\Delta^\ell \mathbf{v})_i = 0$ *for all* i *if and only if there is a* V*-valued polynomial* $P : \mathbb{R} \to V$ *of degree less than* ℓ *such that* $v_i = P(i)$ *for* $0 \leq i \leq k$.

Applying Lemma 2.5 yields the following version of the differential proximity condition:

Lemma 2.6 *Let* S *be a nonlinear subdivision scheme compatible with a stable,* C^k, *linear scheme* S_{lin}. *Let* $\mathbf{y} = Q \circ \Sigma(\boldsymbol{\delta})$ *be as above. Then* S *is* C^k *if and only if the following condition is satisfied:*

For every multi-index J *such that* $|J| > 1$ *and* $w(J) \leq k$, *there is a polynomial* $P_J(t)$ *of degree less than* $w(J)$ *such that* $D^J y_i = P_J(i)$.

3 Averaging linear subdivision schemes

The two lemmas in this section are known, we include them for the sake of completeness.

Let S_{lin} be a linear subdivision rule and let $\overline{S}_{\text{lin}}$ be the linear subdivision rule defined by

$$(\overline{S}_{\text{lin}}\mathbf{x})_i := \frac{1}{2}((S_{\text{lin}}\mathbf{x})_i + (S_{\text{lin}}\mathbf{x})_{i+1}) \,.$$

Lemma 3.1 *If* S_{lin} *is a* C^k *linear subdivision scheme, then* $\overline{S}_{lin}$ *is a* C^{k+1} *linear subdivision scheme.*

Proof Let ϕ be the C^k basis function for S. The basis function ϕ_{avg} for $\overline{S}_{\text{lin}}$ is obtained by convolution of ϕ with $\chi_{[-1,0]}$,the characteristic function for the closed interval $[-1, 0]$, i.e.,

$$\phi_{\text{avg}}(t) = \int_t^{t+1} \phi(s) ds \,,$$

which is clearly C^{k+1}. □

Lemma 3.2 *If S_{lin} is a stable linear subdivision rule, then so is $\overline{S}_{lin}$.*

Proof Let ϕ be the basis function of the linear subdivision rule S_{lin}, and let ϕ_{avg} be the basis function of the linear subdivision rule $\overline{S}_{\text{lin}}$ obtained by composing S_{lin} with one averaging step. Then, the Fourier transforms of ϕ and ϕ_{avg} satisfy

$$\widehat{\phi_{\text{avg}}}(\xi) = e^{i\xi/2}\text{sinc}(\frac{\xi}{2})\hat{\phi}(\xi)$$

According to the stability criterion of Jia-Micchelli [6, Theorem 3.5], $\overline{S}_{\text{lin}}$ is stable if and only if

$$\sup_{\alpha\in\mathbb{Z}} |\widehat{\phi_{\text{avg}}}(\xi + 2\pi\alpha)| > 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Since S_{lin} is stable, it follows that

$$\sup_{\alpha\in\mathbb{Z}} |\hat{\phi}(\xi + 2\pi\alpha)| > 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Note that

$$|\widehat{\phi_{\text{avg}}}(\xi + 2\pi\alpha)| = |\text{sinc}(\frac{\xi + 2\pi\alpha}{2})||\hat{\phi}(\xi + 2\pi\alpha)|\,.$$

If $\xi \neq 2k\pi$ where $k \in \mathbb{Z}$, then

$$|\text{sinc}(\frac{\xi + 2\pi\alpha}{2})| > 0 \quad \text{for any } \alpha \in \mathbb{Z}.$$

Hence,

$$\sup_{\alpha\in\mathbb{Z}} |\widehat{\phi_{\text{avg}}}(\xi + 2\pi\alpha)| > 0 \quad \text{when } \xi \neq 2k\pi.$$

If $\xi = 2k\pi$ for some $k \in \mathbb{Z}$, then choose $\alpha = -k$. In this case,

$$|\widehat{\phi_{\text{avg}}}(\xi + 2\pi\alpha)| = |\widehat{\phi_{\text{avg}}}(2k\pi + 2\pi\alpha)| = |\widehat{\phi_{\text{avg}}}(0)| = |\text{sinc}(0)||\hat{\phi}(0)| = |\hat{\phi}(0)| > 0.$$

□

4 Proof of theorem 1.6

Recall that S is assumed to be compatible with the stable, C^k subdivision rule S_{lin}. By Lemmas 3.1 and 3.2 $\overline{S}_{\text{lin}}$ is a stable, C^{k+1} linear subdivision rule. To conclude the proof of Theorem 1.6, we need to show that $\overline{S}$ is compatible with $\overline{S}_{\text{lin}}$ and that $\overline{S}$ satisfies the order $k+1$ differential proximity condition. Lemma 4.1 and Theorem 4.2 below complete the proof.

Lemma 4.1 *If a nonlinear subdivision operator S is smoothly compatible with a linear subdivision scheme S_{lin}, and A is a nonlinear averaging rule, then $\overline{S}$ is smoothly compatible with $\overline{S}_{lin}$.*

Proof It is obvious that $\overline{S}$ and $\overline{S}_{\text{lin}}$ have the same phase and locality factors. Suppose they are $\bar{m}_\sigma$ and $\bar{L}_\sigma$, respectively. Then

$$(\overline{S}\mathbf{x})_{2i} = \bar{q}_0(x_{i-\bar{m}_0}, \cdots, x_{i-\bar{m}_0+\bar{L}_0}),$$

$$(\overline{S}\mathbf{x})_{2i+1} = \bar{q}_1(x_{i-\bar{m}_1}, \cdots, x_{i-\bar{m}_1+\bar{L}_1}).$$

It follows from (1.6a) and (1.3) that

$$\bar{q}_0(x_0, \cdots, x_{\bar{L}_0}) = A(q_0(x_{\bar{m}_0-m_0}, \cdots, x_{\bar{m}_0-m_0+L_0}), q_1(x_{\bar{m}_0-m_1}, \cdots, x_{\bar{m}_0-m_1+L_1}))$$

and

$$\bar{q}_1(x_0, \cdots, x_{\bar{L}_1}) = A(q_1(x_{\bar{m}_1-m_1}, \cdots, x_{\bar{m}_1-m_1+L_1}), q_0(x_{\bar{m}_1-m_0+1}, \cdots, x_{\bar{m}_1-m_0+1+L_0}))\,.$$

Combining with the fact that $q_\sigma(x, \cdots, x) = x$, we have

$$\begin{aligned} & d\bar{q}_0|_{(x,\cdots,x)}(X_0, \cdots, X_{\bar{L}_0}) \\ = \; & dA|_{(x,x)}(dq_0|_{(x,\cdots,x)}(X_{\bar{m}_0-m_0}, \cdots, X_{\bar{m}_0-m_0+L_0}), dq_1|_{(x,\cdots,x)}(X_{\bar{m}_0-m_1}, \cdots, X_{\bar{m}_0-m_1+L_1})) \end{aligned}$$

and

$$\begin{aligned} & d\bar{q}_1|_{(x,\cdots,x)}\,(X_0, \cdots, X_{\bar{L}_1}) \\ = \; & dA|_{(x,x)}\,(dq_1|_{(x,\cdots,x)}\,(X_{\bar{m}_1-m_1}, \cdots, X_{\bar{m}_1-m_1+L_1}),\, dq_0|_{(x,\cdots,x)}\,(X_{\bar{m}_1-m_0+1}, \cdots, X_{\bar{m}_1-m_0+1+L_0})). \end{aligned}$$

Hence, by (1.5),

$$\begin{aligned} & d\bar{q}_0|_{(x,\cdots,x)}\,(X_0, \cdots, X_{\bar{L}_0}) \\ = \; & \frac{1}{2}\, dq_0|_{(x,\cdots,x)}\,(X_{\bar{m}_0-m_0}, \cdots, X_{\bar{m}_0-m_0+L_0}) + \frac{1}{2}\, dq_1|_{(x,\cdots,x)}\,(X_{\bar{m}_0-m_1}, \cdots, X_{\bar{m}_0-m_1+L_1}) \\ = \; & \frac{1}{2} q_{\text{lin},0}(X_{\bar{m}_0-m_0}, \cdots, X_{\bar{m}_0-m_0+L_0}) + \frac{1}{2} q_{\text{lin},1}(X_{\bar{m}_0-m_1}, \cdots, X_{\bar{m}_0-m_1+L_1}) \\ = \; & \bar{q}_{\text{lin},0}(X_0, \cdots, X_{\bar{L}_0}). \end{aligned}$$

Similarly,

$$d\bar{q}_1|_{(x,\cdots,x)}\,(X_0, \cdots, X_{\bar{L}_1}) = \bar{q}_{\text{lin},1}(X_0, \cdots, X_{\bar{L}_1}).$$

□

To prepare for the main result of this section, namely Theorem 4.2, recall from the comments around (2.2) that the subdivision scheme S, for which $\overline{S}$ is based on, has a minimal invariant neighborhood of which its size is denoted by $K+1$. We claim that $\overline{S}$, then, has a minimal invariant neighborhood of size $K+2$. To see this, notice that when the input sequence $\mathbf{x}$ is shifted by one entry, the subdivided sequence $\mathbf{y} = S\mathbf{x}$ is shifted by two entries. Consequently, a sequence

$$(x_0, \ldots, x_K, x_{K+1})$$

of $K+2$ sufficiently dense points in M, determines a sequence

$$(y_0, \ldots, y_K, y_{K+1}, y_{K+2})$$

of $K+3$ points. This allows us to define the map $\overline{Q}$ via the formula

$$\overline{Q}(x_0, \ldots, x_{K+1}) = (A(y_0, y_1), \ldots, A(y_{K+1}, y_{K+2}))\,. \tag{4.1}$$

And it is easy to see that $K+2$ is the minimum number of data point for which the same number of data point can be generated by $\overline{S}$; i.e., $K+2$ is the size of the minimum invariant neighborhood of $\overline{S}$. Moreover, $\overline{Q}$ represents $\overline{S}$ according to the property set by (2.3).

Theorem 4.2 *Let A be a smooth averaging rule, let S_{lin} be a stable, C^k-smooth, linear subdivision operator, and let S be a nonlinear subdivision rule, smoothly compatible with S_{lin} that satisfies the order k differential proximity condition. Then $\overline{S}$ satisfies the order $k+1$ differential proximity condition.*

Proof Let $\boldsymbol{\delta} = (x_0, \delta_1, \cdots, \delta_K, \delta_{K+1})$, $\mathbf{z} = \overline{Q} \circ \Sigma(\boldsymbol{\delta})$ is the length $K+2$ sequence obtained from applying the subdivision scheme $\overline{S}$ to the data $\Sigma(\boldsymbol{\delta})$. Similarly, define $\mathbf{y}$ to be the length $K+3$ sequence obtained from applying the subdivision scheme S to the same data $\Sigma(\boldsymbol{\delta})$. Then

$$z_i = A(y_i, y_{i+1}) = A((y(\boldsymbol{\delta})_i, (y(\boldsymbol{\delta}))_{i+1})\,. \tag{4.2}$$

By Lemma 2.6, we need to prove that

$$\left.\frac{\partial^{|J|} z_i}{\partial \delta^J}\right|_{(x_0,0,\ldots,0)} \text{ is a polynomial in} i \text{ with degree} \le w(J)-1 \tag{4.3}$$

whenever $|J| > 1$ and $w(J) \le k+1$. We accomplish this by substituting the Taylor expansions of $y_i(\boldsymbol{\delta})$ and $y_{i+1}(\boldsymbol{\delta})$ into the Taylor expansion of $A(x, y)$.

Consider the Taylor expansion

$$y_i(x_0, \delta_1, \cdots, \delta_K) \sim y_i(x_0, 0, \cdots, 0) + \sum_{|\alpha|=1}^{+\infty} \frac{1}{\alpha!} D^\alpha y_i(\delta^\alpha).$$

Since $y_i(x_0, 0, \cdots, 0) = x_0$ and $A(x_0, x_0) = x_0$, we have the Taylor expansion

$$z_i = A(y_i, y_{i+1}) \sim x_0 + \sum_{m=1}^{+\infty} \frac{1}{m!} \sum_{t_j \in \{x,y\}} \left.\frac{\partial^m A}{\partial t_m \cdots \partial t_1}\right|_{(x_0,x_0)} \left(y_{\lambda(t_n)} - x_0, \cdots, y_{\lambda(t_1)} - x_0\right), \tag{4.4}$$

where $\lambda(x) = i$ and $\lambda(y) = i+1$ and

$$y_{\lambda(t_j)} - x_0 \sim \sum_{|\alpha_j|=1}^{+\infty} \frac{1}{\alpha_j!} D^{\alpha_j} y_{\lambda(t_j)}(\delta^{\alpha_j}), \tag{4.5}$$

where $\alpha_j \in \mathbb{N}^K$ are multi-indices.

Now assume $|J| > 1$ and $w(J) \leq k+1$. Hence $J = (j_1, \cdots, j_k, 0, \cdots, 0)$. Combining (4.4) and (4.5) yields the formula

$$\frac{1}{J!} D^J z_i(\delta^J) = \sum_{m=1}^{|J|} \frac{1}{m!} Z_m(\delta) \tag{4.6}$$

where

$$Z_m(\delta) = \sum_{\substack{\sum_{j=1}^m \alpha_j = J \\ |\alpha_j| \geq 1}} \frac{1}{\prod_{j=1}^n \alpha_j!} \sum_{t_j \in \{x,y\}} \left.\frac{\partial^m A}{\partial t_m \cdots \partial t_1}\right|_{(x_0,x_0)} \left(D^{\alpha_n} y_{\lambda(t_n)}(\delta^{\alpha_n}), \ldots, D^{\alpha_1} y_{\lambda(t_1)}(\delta^{\alpha_1})\right). \tag{4.7}$$

The right hand side of (4.6) can be divided into three parts:

$$Z_1(\delta) + Z_{|J|}(\delta) + \sum_{1<n<|J|} Z_n(\delta). \tag{4.8}$$

which we consider individually.

Consider the first term:

$$Z_1(\delta) = \frac{1}{J!} \left(\left.\frac{\partial A}{\partial x}\right|_{(x_0,x_0)} \cdot D^J y_i(\delta^J) + \left.\frac{\partial A}{\partial y}\right|_{(x_0,x_0)} \cdot D^J y_{i+1}(\delta^J) \right)$$

Differentiating the identity $A(x,x) = x$ yields the identity

$$\left.\frac{\partial A}{\partial x}\right|_{(x,x)} + \left.\frac{\partial A}{\partial y}\right|_{(x,x)} = I\,;$$

and differentiating the identity $A(x,y) = A(y,x)$ with respect to x yields the identity

$$\left.\frac{\partial A}{\partial x}\right|_{(x,y)} = \left.\frac{\partial A}{\partial y}\right|_{(y,x)}.$$

Combining these shows that

$$\left.\frac{\partial A}{\partial x}\right|_{(x,x)} = \left.\frac{\partial A}{\partial y}\right|_{(x,x)} = \frac{1}{2} I\,, \text{ for all } x.$$

Substituting this into the expression for $Z_1(\delta)$ above gives

$$Z_1(\delta) = \frac{1}{2J!} \left(D^J y_i(\delta^J) + D^J y_{i+1}(\delta^J) \right)$$

Since S satisfies the order k differential proximity condition, for $|J| > 1$ and $w(J) \leq \ell$ we have

$$D^J \Delta^\ell \mathbf{y} = c_{i,J}^\ell(x_0) = 0, \quad \text{for } \ell = 2, \cdots, k\,, \tag{4.9}$$

Equivalently (see Lemma 2.6), when $|J| > 1$ and $w(J) \leq k$,

$$D^J y_i \text{ is a polynomial in} i \text{ with degree} \leq w(J) - 1. \tag{4.10}$$

It follows from (4.10) that when $|J| > 1$ and $w(J) \leq k$, the multi-linear maps $D^J y_i$ and $D^J y_{i+1}$ are polynomials in i of degree at most $\leq w(J) - 1$.

Suppose

$$(\Delta^{k+1}\mathbf{y})_i = \text{linear terms} + \sum_{|J|>1} c_{i,J}^{k+1}(x_0)\delta^J.$$

By (4.9) $c_{i,J}^k(x_0) = 0$ when $|J| > 1$ and $w(J) \leq k$. Part(2) of the Alternating Sign Lemma in [1] then shows that

$$c_{i,J}^{k+1}(x_0)\delta^J + c_{i+1,J}^{k+1}(x_0)\delta^J = 0,$$

for all $|J| > 1$ with $w(J) = k+1$. Hence, when $|J| > 1$ and $w(J) = k+1$,

$$D^J(\Delta^{k+1}\mathbf{y})_i(\delta^J) + D^J(\Delta^{k+1}\mathbf{y})_{i+1}(\delta^J) = 0\,.$$

Therefore, when $|J| > 1$ and $w(J) = k+1$,

$$D^J y_i(\delta^J) + D^J y_{i+1}(\delta^J)$$

is a polynomial in i of degree at most k.

We have shown that when $|J| > 1$ and $w(J) \leq k+1$, $Z_1(\delta)$ is a polynomial in i of degree at most $w(J) - 1$.

We next consider the term $Z_{|J|}(\delta)$. Notice that when $n = |J|$, it follows from the identities $\sum_{j=1}^{|J|} \alpha_j = J$ and $|\alpha_j| \geq 1$ that $|\alpha_1| = \cdots = |\alpha_{|J|}| = 1$. Therefore,

$$Z_{|J|}(\delta) = \frac{1}{|J|!} \sum_{\substack{\sum_{j=1}^{|J|} \alpha_j = J \\ |\alpha_j|=1}} \sum_{t_j \in \{x,y\}} \left.\frac{\partial^{|J|} A}{\partial t_{|J|} \cdots \partial t_1}\right|_{(x_0,x_0)} \left(\left.\frac{\partial y_{\lambda(t_{|J|})}}{\partial \delta^{\alpha_{|J|}}}\right|_{(x_0,0,\cdots,0)} \delta^{\alpha_{|J|}}, \cdots, \left.\frac{\partial y_{\lambda(t_1)}}{\partial \delta^{\alpha_1}}\right|_{(x_0,0,\cdots,0)} \delta^{\alpha_1} \right).$$

Since $J = (j_1, \cdots, j_k, 0, \cdots, 0)$, $\sum_{j=1}^{|J|} \alpha_j = J$ and $|\alpha_j| = 1$, we have $w(\alpha_j) \leq k$.

Since S is smoothly compatible with S_{lin} and S_{lin} is stable and C^k, it follows that $(\Delta^\ell \mathbf{y})_i$ has a Taylor expansion of the form

$$(\Delta^\ell \mathbf{y})_i = 2^{-\ell}\delta_\ell + \sum_{j=\ell+1}^{K} \lambda_{ij}^\ell(x_0)\delta_j + \sum_{|J|>1} c_{i,J}^\ell(x_0)\delta^J,\ \ \ell = 1, \cdots, k,$$

where $c_{i,J}^\ell(x_0) := D^J \Delta^\ell \mathbf{y}$. Hence,

$$\left(\Delta^\ell \left.\frac{\partial \mathbf{y}}{\partial \delta_\ell}\right|_{(x_0,0,\cdots,0)}\right)_i = \left.\frac{\partial(\Delta^\ell \mathbf{y})_i}{\partial \delta_\ell}\right|_{(x_0,0,\cdots,0)} = 2^{-\ell} I,\ \ \ell = 1, \cdots, k\,.$$

It follows that $\left(\Delta^{\ell+1}(\frac{\partial \mathbf{y}}{\partial \delta_\ell})\right)_i = 0$. Therefore, for $\ell = 1, \cdots, k$ and any $a = (a_1, \cdots, a_N)^{\mathrm{T}} \in \mathbb{R}^N$,

$$\left.\frac{\partial y_i}{\partial \delta_\ell}\right|_{(x_0,0,\cdots,0)} \cdot a \text{ is a degree } \ell \text{ polynomial in } i \text{ with leading coefficient } \frac{2^{-\ell}}{\ell!} a_j. \tag{4.11}$$

It follows from (4.11) that for $j = 1, \cdots, |J|$,

$$\left.\frac{\partial y_{\lambda(t_j)}}{\partial \delta^{\alpha_j}}\right|_{(x_0,0,\cdots,0)} \delta^{\alpha_j} = \frac{2^{-w(\alpha_j)}}{w(\alpha_j)!} i^{w(\alpha_j)} \delta^{\alpha_j} + r_j,$$

where r_j is a polynomial in i of degree $< w(\alpha_j)$. Since $w(\alpha_1) + \cdots + w(\alpha_{|J|}) = w(J)$,

$$Z_{|J|}(\delta) = \frac{2^{-w(J)}}{|J|!} i^{w(J)} \sum_{\substack{\sum_{j=1}^{|J|} \alpha_j = J \\ |\alpha_j|=1}} \frac{1}{\prod_{j=1}^{|J|} w(\alpha_j)!} \sum_{t_j \in \{x,y\}} \left. \frac{\partial^{|J|} A}{\partial t_{|J|} \cdots \partial t_1} \right|_{(x_0,x_0)} \left(\delta^{\alpha_{|J|}}, \cdots, \delta^{\alpha_1}\right) + R,$$

where R is a polynomial in i with degree $\leq w(J) - 1$. Repeatedly differentiating the identity $A(x, x) = x$ yields the identity

$$\sum_{t_i \in \{x,y\}} \left. \frac{\partial^k A}{\partial t_k \cdots \partial t_1} \right|_{(x_0,x_0)} = 0, \quad k = 2, 3, \cdots \tag{4.12}$$

where the summation is over all sequences $(t_1, \ldots, t_k)$, $t_i \in \{x, y\}$. Consequently,

$$\sum_{t_j \in \{x,y\}} \left. \frac{\partial^{|J|} A}{\partial t_{|J|} \cdots \partial t_1} \right|_{(x_0,x_0)} \left(\delta^{\alpha_{|J|}}, \cdots, \delta^{\alpha_1}\right) = 0.$$

Hence, $Z_{|J|}(\delta) = R$. Therefore, when $|J| > 1$ and $w(J) \leq k + 1$, $Z_{|J|}(\delta)$ is a polynomial in i of degree $\leq w(J) - 1$.

It remains to consider the terms $Z_m(\delta)$ for $1 < m < |J|$. The conditions $m > 1$, $\sum_{j=1}^{m} \alpha_j = J$ and $|\alpha_j| \geq 1$ together imply that

$$w(\alpha_j) < w(J) \leq k + 1 \text{ for all } j.$$

Condition (4.11) implies that when $|\alpha_j| = 1$,

$$\left. \frac{\partial^{|\alpha_j|} y_{\lambda(t_j)}}{\partial \delta^{\alpha_j}} \right|_{(x_0,0,\cdots,0)} \delta^{\alpha_j} \text{ is a polynomial in} i \text{ of degree } w(\alpha_j).$$

While (4.10) implies that when $|\alpha_j| > 1$,

$$\left. \frac{\partial^{|\alpha_j|} y_{\lambda(t_j)}}{\partial \delta^{\alpha_j}} \right|_{(x_0,0,\cdots,0)} \delta^{\alpha_j} \text{ is a polynomial in} i \text{ of degree } \leq w(\alpha_j) - 1\,.$$

Since $n < |J|$ and $\sum_{j=1}^{n} \alpha_j = J$, there exists at least one multi-index α_j with $|\alpha_j| > 1$. Hence,

$$\left. \frac{\partial^n A}{\partial t_n \cdots \partial t_1} \right|_{(x_0,x_0)} \left(D^{\alpha_n} y_{\lambda(t_n)}(\delta^{\alpha_n}), \cdots, D^{\alpha_1} y_{\lambda(t_1)}(\delta^{\alpha_1})\right)$$

is a polynomial in i with degree $\leq \sum_{j=1}^{n} w(\alpha_j) - 1 = w(J) - 1$. Therefore, when $|J| > 1$ and $w(J) \leq k + 1$,

$$\sum_{1<n<|J|} Z_n(\delta) \text{ is a polynomial in} i \text{ of degree } \leq w(J) - 1. \tag{4.13}$$

Combining the cases $n = 1$, $n = |J|$ and $1 < n < |J|$ implies the condition (4.3), which in turn implies the order $k + 1$ proximity condition. □

Theorem 1.6 now follows from [1, Theorem 1.19] applied to $\overline{S}$ and $\overline{S}_{\text{lin}}$.

5 Application: smoothness of nonlinear Lane-Riesenfeld schemes

As an application of Theorem 1.6, we study here a class of nonlinear Lane-Riesenfeld subdivision schemes on Riemannian manifolds, which we call *geodesic Lane-Riesenfeld subdivision schemes*.

Let M be a Riemannian manifold and let g denote its Riemannian metric. Then, there exists an open neighborhood $U \subset M \times M$ such that for all $(x, y) \in U$ there is a unique geodesic segment from x to y. Let $A_g(x, y)$ denote the midpoint of this geodesic segment. Then, an elementary argument in differential geometry shows that the map

$$A_g : U \to M \,:\, (x, y) \mapsto A_g(x, y)$$

is a C^∞ averaging rule.

We can now inductively define a sequence of nonlinear subdivision schemes on M as follows. Let $\mathbf{x} : \mathbb{Z} \to M$ be a sufficiently dense control sequence of points in M. Let S^0 be the nonlinear subdivision rule defined as follows:

$$(S^0\mathbf{x})_{2i} = x_i \quad (S^0\mathbf{x})_{2i+1} = A_g(x_i, x_{i+1})\,, \text{ for } i \in \mathbb{Z}.$$

Then define S^{k+1} for $k \geq 0$ by

$$(S^{k+1}\mathbf{x})_{2i} = A_g\left((S^k\mathbf{x})_{2i}, (S^k\mathbf{x})_{2i+1}\right) \quad (S^{k+1}\mathbf{x})_{2i+1} = A_g\left((S^k\mathbf{x})_{2i+1}, (S^k\mathbf{x})_{2i+2}\right).$$

We call S^k the *order k geodesic Lane-Riesenfeld subdivision scheme*. Inductively applying Lemma 4.1 below shows that S^k is compatible with the (linear) C^k B-spline. The order 1 scheme was introduced by Noakes in [8], who showed that the limiting curves are C^1.

More generally, we have the following theorem.

Theorem 5.1 *For all k, the limiting curves of the order k geodesic Lane-Riesenfeld subdivision scheme are C^k.*

Proof As we already mentioned, the case $k = 1$ was proved by Noakes. Since S^1 is compatible with the C^1 B-spline, which is known to be a stable linear scheme, inductively applying Theorem 1.6 shows that the limiting curves of S^k are C^k for all k. □

Remark 5.2 Although midpoint averaging along geodesics is perhaps the most natural nonlinear averaging scheme to use, Theorem 1.6 applies to *any* averaging rule. Consequently, one could consider a more general class of subdivision rules inductively defined by

$$(S^{k+1}\mathbf{x})_{2i} = A_{k+1}\left((S^k\mathbf{x})_{2i}, (S^k\mathbf{x})_{2i+1}\right), \quad (S^{k+1}\mathbf{x})_{2i+1} = A_{k+1}\left((S^k\mathbf{x})_{2i+1}, (S^k\mathbf{x})_{2i+2}\right),$$

where A_k, $k = 2, \ldots$ is any sequence of nonlinear averaging rules on M. Theorem 1.6 applies in this case as well to show that the limiting curves are C^k in this more general case.

Remark 5.3 We have proved that the aforementioned nonlinear variants of the Lane-Riesenfeld subdivision scheme inherits the regularity from the corresponding linear Lane-Riesenfeld scheme. This is a property also enjoyed by the nonlinear schemes proposed in [10] and [5]. In contrast, it is known from [2, 3] that *curvature* can obstruct such a smoothness equivalence in the single basepoint scheme first proposed in [9].

In [4], Dyn-Goldman consider an even more general class of schemes, in the special case where $M = \mathbb{R}$. In their framework, possibly different averaging rules can be used at each averaging step. However, as we showed in [1] the limiting curves may fail to be C^k. Let $\mathbf{x}$ be a bi-infinite sequence of points in $\mathbb{R}$, and define the nonlinear subdivision S^2 as follows:

$$\begin{array}{llll} \left(S^0\mathbf{x}\right)_{2i} &= x_i & \left(S^0\mathbf{x}\right)_{2i+1} &= \frac{x_i+x_{i+1}}{2} \\ \left(S^1\mathbf{x}\right)_{2i} &= \frac{(S^0\mathbf{x})_{2i}+(S^0\mathbf{x})_{2i+1}}{2} & \left(S^1\mathbf{x}\right)_{2i+1} &= \frac{(S^0\mathbf{x})_{2i+1}+(S^0\mathbf{x})_{2i+2}}{2} \\ \left(S^2\mathbf{x}\right)_{2i} &= \frac{(S^1\mathbf{x})_{2i}+(S^1\mathbf{x})_{2i+1}}{2} & \left(S^2\mathbf{x}\right)_{2i+1} &= B(\left(S^1\mathbf{x}\right)_{2i+1}, \left(S^1\mathbf{x}\right)_{2i+2}) \end{array},$$

where B is a (nonlinear) smooth averaging rule. Dyn-Goldman [4] show that this rule is C^1. The even and odd rules of the algorithm are easily seen to be

$$q_0(x_1, x_2, x_3) = B\left(\frac{x_1 + 3x_2}{4}, \frac{3x_2 + x_3}{4}\right) \text{ and } q_1(x_1, x_2) = \frac{x_1 + x_2}{2}.$$

In [1] we computed the Taylor expansion of Ψ with respect to δ_1 and δ_2 to be

$$\Psi(x_0, \delta_1, \delta_2) = \begin{pmatrix} \Psi^0(x_0, \delta_1, \delta_2) \\ \Psi^1(x_0, \delta_1, \delta_2) \\ \Psi^2(x_0, \delta_1, \delta_2) \end{pmatrix} = \begin{pmatrix} x_0 \\ \frac{\delta_1}{2} \\ \frac{\delta_2}{4} + \frac{B^{(1,1)}(x_0, x_0)}{4}\delta_1^2 \end{pmatrix} + \begin{pmatrix} \text{terms of weight} > 0 \\ \text{terms of weight} > 1 \\ \text{terms of weight} > 2 \end{pmatrix}. \quad (5.1)$$

By Theorem 2.4, the limit functions of this scheme are C^2 if and only if $B^{(1,1)}(x_0, x_0) = 0$ for all x_0. Therefore, in general, this subdivision is not C^2.

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Appendix

A Proof of Lemma 2.5

Proof (By finite induction on ℓ.) When $\ell = 1$, $\Delta\mathbf{v} = (0, 0, \dots, 0)$ if and only if $v_i = v_0$ for all i, which is a degree $\ell - 1 = 0$ polynomial $P(t) = v_0$.

Assume that lemma holds for $\ell < k$ and consider a sequence $\mathbf{v}$. Since $\Delta^{\ell+1}\mathbf{v} = \Delta^{\ell}\Delta^{1}\mathbf{v} = (0, 0, \dots, 0)$ there is a polynomial $P(t)$ of degree $\ell - 1$ such that

$$\Delta\mathbf{v}_i = P(i) \quad \text{for } 0 \le i \le k-1.$$

But then $v_i = v_0 + \sum_{j=1}^{i} \Delta\mathbf{v}_{i-j} = v_0 + \sum_{j=1}^{i} P(i-j)$, which is a polynomial in i of degree ℓ in i.

Conversely, assume that $v_i = P(i)$ for P a polynomial of degree ℓ. Then $(\Delta\mathbf{v})_i = P(i+1) - P(i)$, which is a polynomial in i of degree $\ell - 1$. Therefore, by the induction hypothesis, $\Delta^{\ell}\mathbf{v} = (0, 0, \dots, 0)$. Consequently,

$$\Delta^{\ell+1}\mathbf{v} = \Delta(\Delta^{\ell}\mathbf{v}) = \Delta(0, 0, \dots, 0) = (0, 0, \dots, 0)\,.$$

□

B A point-set topology result

Proposition B.1 *Let (M, d) be a metric space, and $\times^{k+1}M = \overbrace{M \times M \times \cdots \times M}^{(k+1)-times}$ be equipped with the product topology. Let $U \subset \times^{k+1}M$ be a neighborhood of the hyper-diagonal $\{(x, x, \dots, x) : x \in M\}$. When $k \ge 2$, there is a smaller open neighborhood $V' \subset U$ of the hyper-diagonal of the form*

$$V' = \{(x_0, x_1, \dots, x_k) : (x_i, x_{i+1}) \in V\,, 0 \le i < k\}$$

for $V \subset M \times M$ an open neighborhood of the diagonal of $M \times M$.

Proof For the product space $\times^{k+1}M$, we use the metric $\overline{d}((x_0, \dots, x_k), (x_0', \dots, x_k')) = \sum_i d(x_i, x_i')$ which is consistent with the usual product topology on $\times^{k+1}M$.

For any $x \in M$, there is an open ball with radius $r_x > 0$ in $\times^{k+1}M$ such that $B((x, \dots, x), r_x) \subset U$. Let

$$V = \bigcup_{x \in M} B((x, x), r_x/L), \text{ where } L = k(k+1)/2.$$

Then V is an open neighborhood of the diagonal of $M \times M$. Let

$$V' = \{(x_0, x_1, \dots, x_k) : (x_i, x_{i+1}) \in V\,, 0 \le i < k\}.$$

For $i = 1, \cdots, k$, the projection map $p_i : (x_0, x_1, \cdots, x_k) \mapsto (x_{i-1}, x_i)$ is continuous. Hence, $p_i^{-1}(V)$ is open in $\times^{k+1}M$. Since $V' = \bigcap_{i=1}^{k} p_i^{-1}(V)$, it follows that V' is an open neighborhood of the hyper-diagonal of $\times^{k+1}M$.

Next, we are going to show that $V' \subset U$.

For any $(x_0, x_1, \cdots, x_k) \in V'$, there exist $y_1, \cdots, y_k \in M$ such that $(x_{i-1}, x_i) \in B((y_i, y_i), r_{y_i}/L)$ for $i = 1, \cdots, k$. Let $r := \max_{1\le i\le k} r_{y_i}$. Then for $i = 1, \cdots, k$, we have

$$d(x_{i-1}, y_i) + d(x_i, y_i) < r/L. \tag{5.2}$$

Assume $j = \arg\max_{1\le i\le k} r_{y_i}$, i.e., $r_{y_j} = r$. Note that

$$\sum_{i=0}^{k} d(x_i, y_j) = \sum_{i=0}^{j-2} d(x_i, y_j) + \underbrace{d(x_{j-1}, y_j) + d(x_j, y_j)}_{<r/L} + \sum_{i=j+1}^{k} d(x_i, y_j).$$

It follows from the triangle inequality that, for $i > j$,

$$\begin{aligned} d(x_i, y_j) &\le \underbrace{d(x_i, y_i) + d(y_i, x_{i-1})}_{<r/L} + \underbrace{d(x_{i-1}, y_{i-1}) + d(y_{i-1}, x_{i-2})}_{<r/L} \\ &\quad + \cdots + \underbrace{d(x_{j+1}, y_{j+1}) + d(y_{j+1}, x_j)}_{<r/L} + \underbrace{d(x_j, y_j)}_{<r/L} \\ &< (i-j)r/L. \end{aligned}$$

Similarly, if $i \le j-2$, $d(x_i, y_j) < (j-i)r/L$. Hence,

$$\begin{aligned} \sum_{i=0}^{k} d(x_i, y_j) &= \sum_{i=0}^{j-2} \underbrace{d(x_i, y_j)}_{<(j-i)r/L} + \underbrace{d(x_{j-1}, y_j) + d(x_j, y_j)}_{<r/L} + \sum_{i=j+1}^{k} \underbrace{d(x_i, y_j)}_{<(i-j)r/L} \\ &< \frac{r}{L}\Big[\sum_{i=0}^{j-1}(j-i) + \sum_{i=j+1}^{k}(i-j)\Big]. \end{aligned} \tag{5.3}$$

It is not hard to see that the sum inside the bracket of (5.3) is the largest when $j = k$, for which the sum is $1+2+\cdots+k$, which is the value we chose for L. In other words, $(x_0, x_1, \cdots, x_k) \in B((y_j, \cdots, y_j), r_{y_j}) \subset U$. Since the choice of $(x_0, x_1, \cdots, x_k)$ is arbitrary, it follows that $V' \subset U$. □

As an example of the above proposition (and its proof), consider $M = \mathbb{R}$, $k = 2$. Consider the set

$$U = \bigcup_{t\in\mathbb{R}} \Big\{(x_1, x_2, x_3) + (t, t, t) : |x_1| + |x_2| + |x_3| < \frac{1}{1+t^2}\Big\},$$

which forms an open neighborhood of the line $\{(t, t, t) : t \in \mathbb{R}\}$ in $\mathbb{R}^3$. In this case, the V guaranteed by Proposition B.1 can be chosen to be

$$V = \bigcup_{t\in\mathbb{R}} \Big\{(x, y) + (t, t) : |x| + |y| < \frac{1}{3}\frac{1}{1+t^2}\Big\}.$$

To see why this works (and why the shrinking factor 1/3 is introduced), notice that if (x_1, x_2, x_3) is such that $(x_1, x_2), (x_2, x_3) \in V$, then for some $t_1, t_2 \in \mathbb{R}$, $|x_1 - t_1|+$

$|x_2 - t_1| < 1/(1 + t_1^2)$ and $|x_2 - t_2| + |x_3 - t_2| < 1/(1 + t_2^2)$. Assume $t_1 < t_2$ (the opposite case is complete analogous), we have

$$
\begin{aligned}
|x_1 - t_1| + |x_2 - t_1| + |x_3 - t_1| &< \frac{1}{3}\frac{1}{1+t_1^2} + |x_3 - t_3| < \frac{1}{3}\frac{1}{1+t_1^2} + |x_3 - t_2| + |t_2 - x_2| + |x_2 - t_1| \\
&< \frac{1}{3}\frac{1}{1+t_1^2} + \frac{1}{3}\frac{1}{1+t_2^2} + \frac{1}{3}\frac{1}{1+t_1^2} < \frac{1}{1+t_1^2}.
\end{aligned}
$$

This means (x_1, x_2, x_3) is in U. And we have established that V has the desirable property.

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