SINGLE BASEPOINT SUBDIVISION SCHEMES FOR MANIFOLD-VALUED DATA: TIME-SYMMETRY WITHOUT SPACE-SYMMETRY

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Abstract. This paper establishes smoothness results for a class of nonlinear subdivision schemes, known as the ***single basepoint*** manifold-valued subdivision schemes, which shows up in the construction of wavelet-like transform for manifold-valued data. This class includes the (single basepoint) Log-Exp subdivision scheme as a special case. In these schemes, the exponential map is replaced by a so-called retraction map f from the tangent bundle of a manifold to the manifold. It is known that any choice of retraction map yields a C^2 scheme, provided the underlying linear scheme is C^2 (this is called "C^2 equivalence"). But when the underlying linear scheme is C^3, Navayazdani and Yu have shown that to guarantee C^3 equivalence, a certain tensor P_f associated to f must vanish. They also show that P_f vanishes when the underlying manifold is a symmetric space and f is the exponential map. Their analysis is based on certain "C^k proximity conditions" which are known to be sufficient for C^k equivalence.

In the present paper, a geometric interpretation of the tensor P_f is given. Associated to the retraction map f is a torsion-free affine connection, which in turn defines an exponential map. The condition $P_f = 0$ is shown to be equivalent to the condition that f agrees with the exponential map of the connection up to the 3rd order. In particular, when f is the exponential map of a connection, one recovers the original connection and P_f vanishes. It then follows that the condition $P_f = 0$ is satisfied by a wider class of manifolds than was previously known. Under the additional assumption that the subdivision rule satisfies a ***time-symmetry***, it is shown that the vanishing of P_f implies that the C^4 proximitiy conditions hold, thus guaranteeing C^4 equivalence. Finally, the analysis in the paper shows that for $k \geq 5$, the C^k proximity condtions imply vanishing curvature. This suggests that vanishing curvature of the connection associated to f is likely to be a necessary condition for C^k equivalence for $k \geq 5$.

1. Introduction

Motivated by the vast development in wavelet analysis and the proliferation of manifold-valued data in several areas of science and engineering, such as diffusion tensor imaging and motion capturing, Donoho *et al* [11, 20] introduced a framework for a nonlinear wavelet transform for multiscale representations of data living on a manifold, which he assumed was either a Lie group or a symmetric space. Underlying this framework is a nonlinear subdivision scheme on a manifold M of the form

$$(S\mathbf{x})_{2h+\sigma} = \exp_{x_h}\Big(\sum_{\ell} a_{2\ell+\sigma}\ \log_{x_h}(x_{h-\ell})\Big), \quad \sigma = 0, 1, \ \ h \in \mathbb{Z}, \tag{1.1}$$

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where (a_ℓ) is the mask for a linear subdivision scheme $S_{\rm lin}$, $\exp : TM \to M$ is the exponential map of M, $\log_x$ is the local inverse of $\exp_x : TM_x \to M$, the restriction of exp to TM_x, and $\mathbf{x} = \{x_h\}$ is a sequence of points[1] in M. (Recall that $\exp_x$ is a diffeomorphism between a neighborhood of $0_x \in TM_x$ and a neighborhood of $x \in M$.)

Notice that the nonlinear scheme S depends on three data: the underlying manifold M, the map exp, and the linear subdivision rule $S_{\rm lin}$. A basic problem in analysis is to determine conditions under which S shares the same smoothness as the underlying linear scheme $S_{\rm lin}$. This is the so-called ***smoothness equivalence problem***.

In previous work [31, 29, 25, 28], it was found that Donoho's original conjecture that S is always as smooth as $S_{\rm lin}$ is most likely not true in general. The conjecture does hold in the following two cases:

(i) If $S_{\rm lin}$ (and hence also S) is interpolatory, then S and $S_{\rm lin}$ are C^k equivalent for any k [28, 25, 14].
(ii) If we use two different (and carefully constructed) basepoints x_i and $x_{i+1/2}$ for the even and odd rules, then a modified version of (1.2) satisfies the C^k equivalence property for arbitrary k [29, 15].

Neither the interpolatory nor the two basepoint scheme is desirable for the wavelet-like transform in [20]: The former leads to L^2-instability already in the linear setting (see the unpublished article [9]), while the latter forces us to give up non-redundancy (a.k.a. 'critical sampling' in the wavelet literature.) For these reasons, we consider here the single basepoint plane scheme (1.1)) with $S_{\rm lin}$ non-interpolatory. In this paper we prove the following:

(iii) The non-interpolatory single basepoint scheme S can satisfy C^k equivalence up to C^4, but our analysis indicates that, for many manifolds of interest, the equivalence is doomed to breakdown at degree 5.

One would expect the interpolatory and the non-interpolatory schemes to have similar smoothness equivalence properties. Also, it is surprising that the smoothness equivalence properties of the single basepoint strategy are so different from those of the two basepoint strategy—in the latter strategy, the choice of retraction (see below), time-symmetry, and curvature play no role in the analysis, but as we shall see, all three play a role in smoothness properties of the single basepoint subdivision scheme.

Our analysis is based on a generalization of (1.1). Let M denote any smooth, n-dimensional manifold without boundary. Let 0_x denote the zero tangent vector based at $x \in M$. Recall that $\exp(0_x) = x$ for all $x \in M$ and that its restriction $\exp_x : TM_x \to M$ is a local diffeomorphism between a neighborhood of 0_x in TM_x and a neighborhood of x in M. We replace the exponential map by a smooth map $f : TM \to M$, satisfying these two conditions. Thus, the restriction $f_x : TM_x \to M$ has a local inverse $g(x, \cdot) : V_x \to TM_x$ with $g(x, x) = 0_x$, where $V_x \subset M$ is an open neighborhood of x; and we can now define a nonlinear subdivision rule as follows:

$$(S\mathbf{x})_{2h+\sigma} = f_{x_h}\Big(\sum_\ell a_{2\ell+\sigma}\, g(x_h, x_{h-\ell})\Big), \quad \sigma = 0, 1, \quad h \in \mathbb{Z}. \tag{1.2}$$

We shall view M as a subset of TM by identifying each point $x \in M$ with $0_x \in TM$, the zero tangent vector based at x. With this identification, the restriction of f to M is the identity map, and so f is a smooth retraction in the standard topological sense. Because in this paper we only consider retraction maps whose restrictions to TM_x are local diffeomorophisms, we abuse notation and refer to this special class of maps maps as ***retraction maps***. (This is consistent with the terminology in the applied mathematics literature [3, 2].) In Section 2 we give a more detailed discussion of such maps.

[1] For $S\mathbf{x}$ to be well defined, adjacent points must be sufficiently close. We implicitly assume this condition throughout.

Recall that a subdivision scheme is called C^k if it yields C^k curves for any initial data. We say that the nonlinear scheme S has the ***C^k equivalence*** property if it is C^k whenever the underlying linear scheme S_{lin} is C^k.

1.1. **Main results.** In [31] it was shown that S has the C^3 equivalence property if the retraction map f satisfies a condition of the form $P_f = 0$, where P_f is a certain tensor constructed from f and independent of S_{lin}; and numerical evidence was presented suggesting that C^3 equivalence fails for $P_f \neq 0$, and it was conjectured that the condition $P_f = 0$ is necessary and sufficient for C^3 equivalence.

In this paper, we show that the condition $P_f = 0$ has a simple geometric interpretation. As we shall see in Section 2, a retraction f defines a torsion-free affine connection, which in turn defines an exponential map, which we denote by $\exp_f$. In the case where the retraction is the exponential map of a torsion free, affine connection, we recover the original connection; but in general f and $\exp_f$ only agree to second order along $M \subset TM$, as illustrated in the following diagram:

$$f \longrightarrow \text{(affine connection)} \longrightarrow \exp_f \qquad \text{(1.3)}$$
$$\text{agree up to 2nd order along } M \subset TM \text{ (between } f \text{ and } \exp_f\text{)}$$

In Section 3, we present a simplified proof that the condition $P_f = 0$ is sufficient for S to have the C^3 equivalence property. We then prove that $P_f = 0$ if and only if f agrees with $\exp_f$ to the 3rd order along $M \subset TM$. This enables us to show that the condition is satisfied not only by the (standard) exponential maps defined on Lie groups and symmetric spaces, but also by the exponential map of any torsion-free affine connection on any manifold. This significantly generalizes the results of [31, Theorem 8] and shows that, the only role played by the symmetric space structure is through its exponential map, the symmetric space structure, itself, has little to do with the C^3 equivalence condition. In particular, the condition $P_f = 0$ holds true if f is the standard exponential map on any Riemannian manifold and even more generally if f is the exponential map of any torsion free, affine connection on any manifold.

We next consider C^4 equivalence. Using the "proximity conditions" of [29], we prove in Section 4 that the condition $P_f = 0$ implies C^4 equivalence provided that the underlying linear subdivision scheme has a natural ***time-symmetry***. In the absence of time-symmetry, we show that the proximity conditions force the curvature of the affine connection associated with f to vanish and also force the retraction f to agree with $\exp_f$ up to 4-th order. Although in numerical analysis imposing a natural symmetry in a numerical scheme often implies an additional order of accuracy, it is however surprising that this can happen without *any* requirement on the 4th order behavior of f.

Finally, in Section 5, we prove that the C^5 proximity conditions imply vanishing curvature, even for linear schemes with a time-symmetry. It is well-known that vanishing curvature imposes stringent conditions on the topology of the underlying manifold. By a classical result of Auslander and Markus [4], if a manifold has a complete, torsion free, flat affine connection then its universal cover is $\mathbb{R}^n$. This means that many manifolds of interest do not have a retraction that satisfies the C^5 proximity conditions. In particular, the C^5 proximity condition is automatically violated on all spheres, all non-abelian Lie groups, and all Grassmannians. Moreover, even in cases, such as $GL(n)$, where the manifold admits a torsion free, flat affine connection, the "natural" retraction map may be ruled out. One can show, for example, that the exponential maps on $GL(n)$ and the symmetric space POS_n of positive definite symmetric matrices both define affine connections with non-vanishing curvature.

We remark that the above results apply, in particular, to the case where f is the exponential map of a torsion-free affine connection. Consequently, our results apply to the important special cases where f is the exponential map of a Lie group or a symmetric spaces. They also apply to certain homogeneous spaces that are not symmetric spaces (see [19] for details), and to all Riemannian manifolds, and to all affine manifolds.

The reader should note, however, that the proximity conditions we study here are only known to be sufficient conditions for C^k equivalence. We conjecture that they are also necessary conditions, but this remains an open problem. In Section 6, we discuss necessity in a special case.

1.2. **Time- and Space-Symmetry.** In this and our previous paper [31], we use the term 'time-symmetry' to refer to an invariance property of a subdivision scheme under a 'time'-reversal $t \mapsto -t$ in the *domain*. This form of symmetry comes in two kinds: primal and dual. For any (linear or nonlinear) subdivision scheme S, we say that S has a **primal time-symmetry** if $S \circ R_0 = R_0 \circ S$ where R_0 is the reflection operation about 0, i.e. $(R_0\mathbf{x})_k = \mathbf{x}_{-k}$. Similarly, we say that S has a **dual time-symmetry** if $S \circ R_{1/2} = R_{1/2} \circ S$ where $R_{1/2}$ is the reflection operation about 1/2, i.e. $(R_{1/2}\mathbf{x})_k = \mathbf{x}_{1-k}$.

We summarize linear subdivision schemes with these two kinds of time-symmetry in Table 1.

	Primal	Dual
Examples:	Odd degree B-Spline, Dubuc's scheme	Even degree B-Spline, Donoho's AI scheme
Data:	Associated with dyadic points	Assocated with dyadic intervals
Property of S_{lin}:	$S_{\text{lin}} \circ R_0 = R_0 \circ S_{\text{lin}}$	$S_{\text{lin}} \circ R_{1/2} = R_{1/2} \circ S_{\text{lin}}$
	$\Updownarrow$	$\Updownarrow$
Property of mask:	$a_{-k} = a_k$	$a_{1-k} = a_k$
	$\Updownarrow$	$\Updownarrow$
Property of refinable function:	$\phi(-x) = \phi(x)$	$\phi(1-x) = \phi(x)$

TABLE 1. Primal and dual time-symmetry for linear subdivision schemes

The term 'space-symmetry', on the other hand, refers to invariance of the subdivision scheme under a transitive group action on the *range* space. In the linear case, where the range space is $\mathbb{R}^n$, we of course have $S_{\text{lin}}(A\mathbf{x} + b) = AS_{\text{lin}}\mathbf{x} + b$ for any affine transformation $x \mapsto Ax + b$ in $\mathbb{R}^n$ – even when S_{lin} does not possess any time-symmetry. More generally, when the range space is a homogeneous space M acted upon by a transitive group action G, then the space-symmetry refers to the property $S(g \cdot \mathbf{x}) = g \cdot S(\mathbf{x})$ for all $g \in G$.

Although the proof in [31] of the main result makes essential use of space-symmetry and while space-symmetry is a desirable property in practice, we shall see in Section 3 and 4 of this paper that the main results in [31] are valid without assuming any space-symmetry. On the other hand, we shall also see in Section 4 that the notion of dual time-symmetry in the linear subdivision scheme underlying the nonlinear S (1.2) plays an interesting role in the smoothness properties of S – hence the title of this article.

In Donoho's original use of the Log-Exp scheme, the underlying linear subdivision schemes is either an interpolatory Deslauriers-Dubuc scheme, which has a primal time-symmetry, or an average-interpolating (AI) subdivision scheme [10], which has a dual time-symmetry. In the former case, we know from previous results [29, 25, 28] that, due to the interpolatory property, S is always as smooth as S_{lin}. So Donoho's conjecture is true in this special case. For the latter case, the results in this paper tell us that, although Donoho's original smoothness equivalence conjecture is most likely incorrect, C^k equivalence holds in the 'practical range' $k \leq 4$.

2. Retraction maps

Recall from the introduction that the nonlinear subdivision schemes we consider here are defined in terms of a ***retraction map*** f. In this section, we show how to construct a torsion free affine connection ∇_f associated to each retraction map.

Let $B \subset TM$ be an open neighborhood of the set of zero vectors and let $f : B \to M$ be a smooth map satisfying the condition $f(0_x) = x$ for all $x \in M$. Let $f_x : B_x := TM_x \cap B \to M$ denote the restriction of f to the tangent space to M at $x \in M$. We assume that f satisfies the additional requirement that $f_x : B_x \to M$ is a diffeomorphism onto its image.

In local coordinates, we can express f in the form

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \, : \, (x, X) \mapsto f(x, X) = (f^1(x, X), \dots, f^n(x, X)) \,. \tag{2.1}$$

with $f(x, 0) = x$ for all x. Because $f(x, 0) = x$, the Taylor expansion of f with respect to X at $X = 0$ has the form

$$f^\ell(x, X) = x^\ell + f^\ell_i(x)\, X^i + \frac{1}{2!} f^\ell_{ij}(x)\, X^i X^j + \frac{1}{3!} f^\ell_{ijk}(x)\, X^i X^j X^k + \dots\,, \tag{2.2}$$

where $f^\ell_i(x) = \frac{\partial f^\ell(x,0)}{\partial X^i}$, $f^\ell_{ij}(x) = \frac{\partial^2 f^\ell(x,0)}{\partial X^i \partial X^j}$, etc.

Remark 1. The Taylor expansion of f in (2.2) is for a fixed x but varying X. If we consider the more general Taylor expansion of f in local coordinates with both x and X varying, then we encounter the multilinear maps (see [31, Section 2])

$$F_{\alpha,\beta} \, : \, \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\alpha-\text{times}} \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\beta-\text{times}} \longrightarrow \mathbb{R}^n$$

defined in a component-free, basis-independent way by the formula

$$F_{\alpha,\beta}(u_1, \dots, u_\alpha; v_1, \dots, v_\beta) = \frac{d}{ds_1} \cdots \frac{d}{ds_\alpha} \frac{d}{dt_1} \cdots \frac{d}{dt_\beta}\Big|_{s_i = t_j = 0} f\Big(x + \sum_{i=1}^{\alpha} s_i u_i, \sum_{j=1}^{\beta} t_j v_j\Big)\,. \tag{2.3}$$

We shall use the notation $F_{\alpha,\beta}$ extensively in the next section.

Let e_i, $i = 1, \dots, n$ denote the standard basis for $\mathbb{R}^n$. Then in local coordinates, using the component-wise notation with Einstein convention,

$$F_{0,1}(X) = f^\ell_i(x) X^i \, e_\ell \qquad F_{0,2}(X, Y) = f^\ell_{ij}(x)\, X^i Y^j \, e_\ell\,,$$

$$F_{0,3}(X, Y, Z) = f^\ell_{ijk}(x)\, X^i Y^j Z^k \, e_\ell \qquad F_{1,2}(X; Y, Z) = \frac{\partial f^\ell_{jk}(x)}{\partial x^i} X^i Y^j Z^k e_\ell$$

for tangent vectors $X = X^i e_i$, $Y = Y^i e_i$, and $Z = Z^i e_i$. In this notation, the Taylor expansion (2.2) assumes the form

$$f(x, X) = x + F_{0,1}(X) + \frac{1}{2!} F_{0,2}(X, X) + \frac{1}{3!} F_{0,3}(X, X, X) + \dots\,. \tag{2.4}$$

A standard comptuation shows that the map

$$A_f : TM \to TM \, : \, X \mapsto F_{0,1}(X)$$

is coordinate independent. Recall that $f_x : B_x \to M$ is a diffeomorphism from a neighborhood of 0_x to a neighborhood of x. By the Inverse Function Theorem, this implies that the $n \times n$ matrices $f^j_i(x)$ are all invertible, so $A_f : TM \to TM$ is an automorphism of TM.

We may, therefore, use A_f to normalize f as follows. Let $B_0 = A_f(B)$ and let $f' = f \circ A_f^{-1}$. Then f' is also a retraction map, and by construction, $A_{f'}$ is the identity map.

Moreover, f' defines the same nonlinear subdivision rule as f does. To see this note that the local inverse of f'_x is $g'(x,\cdot) = A_f \circ g(x,\cdot)$, and so the subdivision rule S' defined by f' is given by

$$\begin{aligned}(S'\mathbf{x})_{2h+\sigma} = f'_{x_h}\left(\sum_\ell a_{2\ell+\sigma}\, g'(x_h, x_{h-\ell})\right) &= f\left(A_f^{-1}(\sum_\ell a_{2\ell+\sigma}\, A_f(g(x_h, x_{h-\ell})))\right)\\ = f\left(\sum_\ell a_{2\ell+\sigma}\, g(x_h, x_{h-\ell})\right) &= S(\mathbf{x})_{2h+\sigma}\,,\end{aligned}$$

where we have used linearity of the map A_f. Therefore, without loss of generality, we may assume that A_f is the identity. This leads to the following formal definition:

Definition 2. A ***retraction*** is a smooth map $f : B \to M$, defined on a neighborhood of the zero vectors such that $f(0_x) = x$ for all $x \in M$ and such that $A_f = \mathrm{id}_{TM} : TM \to TM$.

For the remainder of this paper we assume that f satisfies Definition 2. Consequently, the Taylor expansion (2.2) reduces to the form

$$f^\ell(x, X) = x^\ell + X^\ell + \frac{1}{2!} f^\ell_{ij}(x)\, X^i X^j + \frac{1}{3!} f^\ell_{ijk}(x)\, X^i X^j X^k + \dots\,. \tag{2.5}$$

Equivalently, using component-free notation

$$f(x, X) = x + X + \frac{1}{2!} F_{0,2}(X, X) + \frac{1}{3!} F_{0,3}(X, X, X) + \dots\,.$$

Remark 3. From a computational point of view, a retraction is usually regarded as an approximation to the standard exponential map on a matrix Lie group or a symmetric space; from this point of view, the exponential map comes first, and the approximating retraction comes afterward. For example, the following diagonal Padé approximations $R_{m,m}$ of e^z

$$e^z = \underbrace{\frac{1 + \frac{1}{2}z + \frac{1}{12}z^2}{1 - \frac{1}{2}z + \frac{1}{12}z^2}}_{R_{2,2}(z)} + O(z^5) = \underbrace{\frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}}_{R_{1,1}(z)} + O(z^3)$$

have the remarkable property that they map $so(n)$ to $SO(n)$, and, when combined with the group operation on $SO(n)$, then can be used to define retractions on $SO(n)$ that are cheaper to compute than the exponential map, see [31, Section 4.4].

2.1. **The affine connection of a retraction.** We next show that the quantities $\Gamma^k_{ij} := -f^k_{ij}$ in Equation 2.5 define a torsion-free, affine connection on M. It suffices to check that the quantities $-f^\ell_{ij}$ satisfy the following transformation identity for connection coefficients:

$$\Gamma^k_{ij} = \frac{\partial x^k}{\partial \overline{x}^c} \left\{ \frac{\partial \overline{x}^a}{\partial x^i} \frac{\partial \overline{x}^b}{\partial x^j} \overline{\Gamma}^c_{ab} + \frac{\partial^2 \overline{x}^c}{\partial x^i \partial x^j} \right\}, \tag{2.6}$$

where $\overline{\Gamma}^c_{ab}$ are the connection coefficients in $\overline{x}$-coordinates.

To see this, let $\overline{x}$ be another set of local coordinates, let $\overline{x} = \phi(x)$ denote the change of coordinates map, and let $\overline{f}(\overline{x}, \overline{X})$ denote the expression for f in $\overline{x}$-coordinates. Taking into account the change of coordinates formula for tangent vectors,

$$\overline{X}^j \frac{\partial}{\partial \overline{x}^j} = \left(\frac{\partial \overline{x}^j}{\partial x^i} X^i \right) \frac{\partial}{\partial \overline{x}^j} = X^i \left(\frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial}{\partial \overline{x}^j} \right) = X^i \frac{\partial}{\partial x^i}\,,$$

gives the following identity relating f and $\overline{f}$:

$$\overline{f}^{\ell}(\overline{x},\overline{X}) = \overline{f}^{\ell}\left(\phi(x), \frac{\partial \overline{x}^1}{\partial x^j}X^j, \ldots, \frac{\partial \overline{x}^n}{\partial x^j}X^j\right) = \phi^{\ell}\left(f^1(x,X),\ldots,f^n(x,X)\right) . \tag{2.7}$$

Differentiate Equation 2.7 twice with respect to X^i and X^j at $X = 0$ and use the chain rule to obtain the formula

$$\begin{aligned}\frac{\partial^2 \overline{f}^{\ell}(\overline{x},0)}{\partial \overline{X}^a \partial \overline{X}^b}\frac{\partial \overline{x}^a}{\partial x^i}\frac{\partial \overline{x}^b}{\partial x^j} &= \frac{\partial^2 \overline{x}^{\ell}}{\partial x^a \partial x^b}\frac{\partial f^a(x,0)}{\partial X^i}\frac{\partial f^b(x,0)}{\partial X^j} + \frac{\partial \overline{x}^{\ell}}{\partial x^k}\frac{\partial^2 f^k(x,0)}{\partial X^i \partial X^j} \\ &= \frac{\partial^2 \overline{x}^{\ell}}{\partial x^i \partial x^j} + \frac{\partial \overline{x}^{\ell}}{\partial x^k}\frac{\partial^2 f^k(x,0)}{\partial X^i \partial X^j} .\end{aligned}$$

(Note that we have used the identity $\frac{\partial f^i(x,0)}{\partial X^j} = \delta_i^j$.) Rewriting this in terms of f^k_{ik} and $\overline{f}^k_{ij}$ gives

$$\overline{f}^{\ell}_{ab}\frac{\partial \overline{x}^a}{\partial x^i}\frac{\partial \overline{x}^b}{\partial x^j} = \frac{\partial^2 \overline{x}^{\ell}}{\partial x^i \partial x^j} + \frac{\partial \overline{x}^{\ell}}{\partial x^k} f^k_{ij} .$$

Finally, solving for f^k_{ij} using the fact that the Jacobian matrix $(\frac{\partial \overline{x}^{\ell}}{\partial x^k})$ is invertible, gives the final transformation identity,

$$-f^k_{ij} = \frac{\partial x^k}{\partial \overline{x}^c}\left\{\frac{\partial \overline{x}^a}{\partial x^i}\frac{\partial \overline{x}^b}{\partial x^j}(-\overline{f}^c_{ab}) + \frac{\partial^2 \overline{x}^c}{\partial x^i \partial x^j}\right\} .$$

Setting $\Gamma^k_{ij} = -f^k_{ij}$ yields precisely the identity (2.6).

Because Γ^k_{ij} are mixed partial derivatives, the connection is torsion-free, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all i, j, k. We summarize the above discussion in the next lemma.

Lemma 4. *Every retraction $f : B \to M$ induces a torsion-free affine connection on M, with connection coefficients given in local coordinates $x = (x^1, \ldots, x^n)$ by the formula $\Gamma^k_{ij}(x) = -\dfrac{\partial^2 f^k(x,0)}{\partial X^i \partial X^j}$.*

2.2. **The exponential map of an affine connection.** In this section, we recall some standard facts about the exponential map of an affine connection. See [16] for a more complete exposition. Assume that Γ^k_{ij} are the connection coefficients of any affine connection (not necessarily defined by f). Consider the initial value problem:

$$\ddot{x}^{\ell} + \Gamma^{\ell}_{i,j}\dot{x}^i\dot{x}^j = 0 , \quad x(0) = x , \quad \dot{x}(0) = X \tag{2.8}$$

for $X = (X^1, \ldots, X^n) \in \mathbb{R}^n$. The solution

$$\gamma_X(t) = (x^1(t), \ldots, x^n(t))$$

of (2.8) is called an ***autoparallel curve***.[2] Because γ_X is the solution of a differential equation with smooth coefficients Γ^k_{ij}, it depends smoothly on the initial condition. Also, for X sufficiently small, $\gamma(t)$ is defined for $0 \le t \le 1$. So for sufficiently small X, the equation

$$\exp(X) = \gamma_X(1)$$

makes sense. This defines a map $\exp : \mathbb{R}^n \supset B_0 \to \mathbb{R}^n$ on an open neighborhood of $0 \in \mathbb{R}^n$ and it is not difficult to show that it satisfies the following properties:

(i) $\exp(0) = x$
(ii) $d_0 \exp = \mathrm{id}_x : \mathbb{R}^n \to \mathbb{R}^n$,

[2] We do not assume here that the connection is the Levi-Civita connection of an underlying Riemannian metric. Consequently, these curves are not necessarily geodesics in the sense of Riemannian geometry.

where $d_0 \exp$ denoted the derivative at $0 \in \mathbb{R}^n$. We can reinterpret exp as the map

$$\exp : TM_x \supset B_x \to M$$

defined on a neighborhood of the zero vector $0_x \in TM_x$.

A standard argument using the transformation rules for tangent vectors and connection coefficients shows that exp is coordinate independent, and letting x vary over all points on M gives a map

$$\exp : TM \supset B \to M \tag{2.9}$$

defined on an open neighborhood B of the set of zero vectors, called the ***exponential map*** of the connection.

Remark 5. A connection is said to be ***complete*** when every autoparallel curve can be extended indefinitely. It is well-known that when M is compact, all affine connections are complete.

Remark 6. In coordinate-free form, properties (i) and (ii) above assume the form

(i) $\exp(0_x) = x$ for all $x \in M$
(ii) $d_0 \exp = \mathrm{id} : TM_x \to TM_x$ for all x, where the tangent space to TM_x at 0_x is identified with TM_x, itself.

In future sections we will need to use a Taylor expansion for autoparallel curves. Suppose that $x(t)$ is an autoparallel curve, satisfying the initial value problem (2.8). Because $x(0) = x$, and $\dot{x}(0) = X$, the Taylor expansion of $x(t)$ is

$$x^\ell(t) = x^\ell + t\,X^\ell - \frac{t^2}{2!}\,\Gamma^\ell_{ij} X^i X^j + \dots\,,$$

where we have used the differential equation for $x(t)$ to express $\ddot{x}(0)$ in terms of $\dot{x}(0)$. Differentiating Equation (2.8) with respect to t yields a formula for the third derivative:

$$\begin{aligned} \dddot{x}^\ell &= -\frac{\partial \Gamma^\ell_{ij}}{\partial x^k} \dot{x}^\ell \dot{x}^i \dot{x}^j - \Gamma^\ell_{ij} \ddot{x}^i \dot{x}^j - \Gamma^\ell_{ij} \dot{x}^i \ddot{x}^j \\ &= -\frac{\partial \Gamma^\ell_{ij}}{\partial x^k} \dot{x}^k \dot{x}^i \dot{x}^j + \Gamma^\ell_{ij} \Gamma^i_{st} \dot{x}^s \dot{x}^t \dot{x}^j + \Gamma^\ell_{ij} \dot{x}^i \Gamma^j_{st} \dot{x}^s \dot{x}^t \\ &= -\left\{ \frac{\partial \Gamma^\ell_{ij}}{\partial x^k} - 2\Gamma^\ell_{is}\Gamma^s_{jk} \right\} \dot{x}^i \dot{x}^j \dot{x}^k\,, \end{aligned} \tag{2.10}$$

from which we obtain the Taylor expansion

$$x^\ell(t) = x^\ell + tX^\ell - \frac{t^2}{2}\Gamma^\ell_{ij} X^i X^j - \frac{t^3}{3!} \left\{ \frac{\partial \Gamma^\ell_{ij}}{\partial x^k} - 2\Gamma^\ell_{is}\Gamma^s_{jk} \right\} X^i X^j X^k + O(t^4)\,. \tag{2.11}$$

Definition 7. We denote by $\exp_f$ the exponential map of the connection defined by the retraction map f.

As we noted above, the exponential map of any affine connection is, itself, a retraction map. Consequently, it in turn defines a torsion-free, affine connection. The next proposition shows that this process stops:

Proposition 8. *If* exp *is the exponential map of a torsion free affine connection, then* $\exp_{\exp} = \exp$.

Proof. We need only show that $\Gamma^k_{ij}(x) = -\dfrac{\partial^2 \exp^k(x,0)}{\partial X^i \partial X^j}$. But Equation (2.8), which defines autoparallel curves, shows that exp has the Taylor expansion

$$\exp^k(x, X) = x^k + X^k - \frac{1}{2!}\Gamma^k_{ij}(x) X^i X^j + \dots\,.$$

□

3. The Geometric Interpretation of the condition $P_f = 0$

In [31] the following invariant of the retraction f was found:

$$P_f(u) := F_{0,2}(u, F_{0,2}(u,u)) + \frac{1}{2}F_{1,2}(u;u,u) - \frac{1}{2}F_{0,3}(u,u,u)\,, \tag{3.1}$$

and the condition $P_f = 0$ was shown to be sufficient for S to have the C^3 equivalence property.

Here we give a geometric interpretation of this condition. As a corollary, we show that the exponential map of every torsion-free affine connection has the C^3 equivalence property.

Definition 9. We say that a retraction $f : TM \supset B \to M$ satisfies the ***order k condition*** if it agrees up to k-th order with the exponential map $\exp_f$ along $M \subset TM$. Equivalently, f satisfies the order k condition if the Taylor expansions of the two families of curves on M:

$$(i) \quad \mu_X \,:\, t \mapsto f(tX) \;\text{ and }\; (ii) \quad \gamma_X \,:\, t \mapsto \exp_f(tX)$$

agree up to order k for all $X \in B$.

In light of the Taylor expansions (2.5) and (2.11), $\mu_X(t)$ and $\gamma_X(t)$ have Taylor expansions of the forms

$$\mu^\ell_X(t) = x^\ell + tX^\ell + \frac{t^2}{2!}f^\ell_{ij}X^iX^j + \frac{t^3}{3!}\dddot{\mu}^\ell(0) + \dots$$

and

$$\gamma^\ell_X(t) = x^\ell + tX^\ell + \frac{t^2}{2!}f^\ell_{ij}X^iX^j + \frac{t^3}{3!}\dddot{\gamma}^\ell(0) + \dots\,,$$

respectively, and so $\mu_X(t)$ and $\gamma_X(t)$ always agree up to second order for all X and so f and $\exp_f$ always satisfy the order 2 condition.

We now show that $P_f = 0$ is equivalent to the condition that f satisfy the order 3 condition.

Theorem 10. *Let f be a retraction. Then $P_f = 0$ if and only if $\dddot{\mu}_X(0) = \dddot{\gamma}_X(0)$ for all $X \in B$.*

Proof. Choose an arbitrary vector X. From Equation (2.11), we have

$$\dddot{\gamma}^\ell_X(0) = -\left\{\frac{\partial \Gamma^\ell_{ij}}{\partial x^k} - 2\Gamma^\ell_{is}\Gamma^s_{jk}\right\}X^iX^jX^k\,,$$

where $\Gamma^k_{ij} = -f^k_{ij}$. On the other hand, from (2.5), we have

$$\dddot{\mu}^\ell_X(0) = f^\ell_{ijk}X^iX^jX^k\,.$$

Thus, $\dddot{\gamma}_X(0) = \dddot{\mu}_X(0)$ if and only if

$$f^\ell_{ijk}X^iX^jX^k = \left\{\frac{\partial f^\ell_{ij}}{\partial x^k} + 2f^\ell_{is}f^s_{jk}\right\}X^iX^jX^k\,. \tag{3.2}$$

On the other hand, in component-free notation, the invariant P_f is given by the formula in (3.1), which in component-wise notation assumes the form

$$\begin{aligned} P^\ell_f(x,X) &= f^\ell_{is}X^i\left(f^s_{jk}X^jX^k\right) + \frac{1}{2}\frac{\partial f^\ell_{ij}(x)}{\partial x^k}X^iX^jX^k - \frac{1}{2}f^\ell_{ijk}X^iX^jX^k \\ &= \left\{f^\ell_{is}f^s_{jk} + \frac{1}{2}\frac{\partial f^\ell_{ij}}{\partial x^k} - \frac{1}{2}f^\ell_{ijk}\right\}X^iX^jX^k\,. \end{aligned} \tag{3.3}$$

Comparing (3.2) and (3.3) shows immediately that $P_f(x,X) = 0$ if and only if $\dddot{\gamma}_X(0) = \dddot{\mu}_X(0)$. □

Remarks 11. (1) Note that the terms $f^\ell_{is} f^s_{jk}$ and $\frac{\partial f^\ell_{ij}}{\partial x^k}$ are *not* symmetric in ijk; the summation over all (i, j, k) in Equation (3.3) uses only the symmetric parts of these quantities.

(2) Since $P_f(u)$ is a homogeneous polynomial of degree 3, by the polarization theorem (see [8, page 8] or [1]), there is a unique symmetric trilinear map, which by abuse of notation we again denote by P_f, such that $P_f(u, u, u) = P_f(u)$:

$$P_f(u, v, w) = \frac{1}{3!} \frac{\partial}{\partial \lambda_1} \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial \lambda_3} P_f(\lambda_1 u + \lambda_2 v + \lambda_3 w)\Big|_{\lambda=0}.$$

It is given by the formula[3] :

$$\begin{aligned} P_f(u, v, w) = {} & \frac{1}{3} \left[F_{0,2}(u, F_{0,2}(v, w)) + F_{0,2}(v, F_{0,2}(u, w)) + F_{0,2}(w, F_{0,2}(u, v))\right] + \\ & \frac{1}{6} \left[F_{1,2}(u; v, w) + F_{1,2}(v; u, w) + F_{1,2}(w; u, v)\right] - \frac{1}{2} F_{0,3}(u, v, w). \end{aligned} \tag{3.4}$$

We call $P_f(u, v, w)$ the ***depolarized form*** of $P_f(u)$.

Theorem 10, when combined with Proposition 8, has an immediate corollary:

Corollary 12. *Let* $\exp : TM \to M$ *be the exponential map of a symmetric connection on* M*. Then the invariant* $P_{\exp}$ *vanishes identically.*

In particular, if M *is a Lie group, a symmetric space, or a Riemannian manifold and* $f : TM \supset B \to M$ *is its exponential map, then* S *and* S_{lin} *satisfy the* C^3 *equivalence property.*

The key point of this corollary is that each of these 'standard' exponential maps is actually the exponential map of a symmetric affine connection of the underlying manifold. The Riemannian case is of course well-known from the Levi-Civita connection. For the cases of Lie group and symmetric space, see Loos [17].

4. C^4 ANALYSIS WITH AND WITHOUT TIME-SYMMETRY

Given Theorem 10, it is natural to ask if the order 4 proximity condition is guaranteed by the order 4-th condition on f. This speculation turns out to be false. In this section, we prove the following result:

Theorem 13. *Assume that the retraction* $f : TM \to M$ *satisfies the condition* $P_f = 0$ *for* C^3 *equivalence. Then* S *and* S_{lin} *satisfy* C^4 *equivalence if either of the following two conditions is satisfied:*

(a) The linear scheme S_{lin} *has a* **dual time-symmetry***, i.e.* $a_k = a_{1-k}$*.*

(b) The curvature R_f *of the affine connection defined by* f *vanishes and in addition* f *satisfies the order 4 condition given by Definition 9.*

As we mentioned in the introduction, the vanishing curvature condition rules out many manifolds of interest. As such, Theorem 13 has a ***dichotomous*** flavor: with time-symmetry in the linear scheme, C^4 equivalence is guaranteed without any constraint on the 4-th order behavior of f or any symmetry property whatsoever on the manifold M. Without time-symmetry, the theorem suggests that we can only get C^4 equivalence on a flat affinely connected manifold.

[3]In [31, Appendix] P_f is shown to be independent of the choice of coordinates, and therefore a trilinear map of the tangent bundle of M. In differential geometry jargon, this is also called a tensor field of type $(1, 3)$ on M.

4.1. **Review of previous results.** The proof of Theorem 13 is based on the now standard proximity approach, introduced in [24, 23].

We recall the following result from [29].

Theorem 14 ([29, Theorem 2.4])**.** *Assume that the linear scheme $S_{\rm lin}$ is stable and C^k, $k \geq 1$. If S and $S_{\rm lin}$ satisfy the* **order k proximity condition**[4] *(in some local coordinates), i.e. there exists a constant $C > 0$ such that for any dense enough bounded sequence $\mathbf{x}$, we have*

$$\|\Delta^{j-1} S\mathbf{x} - \Delta^{j-1} S_{\rm lin}\mathbf{x}\|_\infty \leq C\,\Omega_j(\mathbf{x}), \quad j = 1, \dots, k, \tag{4.1}$$

where

$$\Omega_j(\mathbf{x}) := \sum_{\gamma \in \Gamma_j} \prod_{i=1}^{j} \|\Delta^i \mathbf{x}\|_\infty^{\gamma_i}, \quad \Gamma_j := \left\{ \gamma = (\gamma_1, \cdots, \gamma_j) \;\Big|\; \gamma_i \in \mathbb{Z}^+,\ \sum_{i=1}^{j} i\,\gamma_i = j+1 \right\}, \tag{4.2}$$

then S is also C^k.

Remark 15. In [30], we prove that the proximity condition is invariant under change of coordinates, i.e. the proximity condition is satisfied in one coordinate system if and only if it is satisfied in any other coordinate system. This result will be exploited in the proof of Theorem 13 and again in Section 5.

We use Theorem 14 here in the same manner as it was used in [31] to study C^3 equivalence. Our result is local, so we may work in local coordinates. We write f in the form $f(x, X)$, with local inverse $g(x, y)$, satisfying $g(x, x) = 0$.

By the locality and shift-invariant properties of both S and $S_{\rm lin}$, it suffices to assume that the sequence $\mathbf{x} \subset \mathbb{R}^n$ in (4.1) above is a finite sequence indexed by $\{0, 1, \dots, L\}$ for a $L \geq k$ large enough[5] so that $x_0, \dots, x_L$ determine $(S\mathbf{x})_{2h+\sigma}$ and $(S_{\rm lin}\mathbf{x})_{2h+\sigma}$, for k consecutive indices of $2h+\sigma$. Then at least one entry of the sequence $\Delta^{k-1}S\mathbf{x} - \Delta^{k-1}S_{\rm lin}\mathbf{x}$ can be determined from $x_0, \dots, x_L$, which is all we need to determine.

We view $(S\mathbf{x} - S_{\rm lin}\mathbf{x})_{2h+\sigma}$ as an $\mathbb{R}^n$-valued function of $\mathbf{x} = (x_0, \dots, x_L)$. In [31] it is shown that $(S\mathbf{x} - S_{\rm lin}\mathbf{x})_{2h+\sigma}$ can be written in the form

$$(S\mathbf{x} - S_{\rm lin}\mathbf{x})_{2h+\sigma} = \Phi_{k,2h+\sigma}(D_1, \dots, D_{k-1}) + O(\Omega_k(\mathbf{x}))\,,$$

where $\Phi_{k,2h+\sigma}$ is a certain $\mathbb{R}^n$-valued polynomial in the variables

$$D_1 = (\Delta\mathbf{x})_0 = \mathbf{x}_1 - \mathbf{x}_0,\ \ D_2 = (\Delta^2\mathbf{x})_0 = \mathbf{x}_2 - 2\mathbf{x}_1 + \mathbf{x}_0\,, \dots,\ D_{k-1} = (\Delta^{k-1}\mathbf{x})_0\,. \tag{4.3}$$

Observe that $D_j = O(\|\Delta^j\mathbf{x}\|_\infty)$ for $j < k$ and $D_j = O(\|\Delta^k\mathbf{x}\|_\infty)$ for $j \geq k$. Thus, we can write

$$x_h - x_0 = \sum_{j=1}^{k} \binom{h}{j} D_j + O(\|\Delta^k\mathbf{x}\|_\infty)\,.$$

The polynomial $\Phi_{k,2h+\sigma}$ is obtained by computing the Taylor polynomial of degree k of $(S\mathbf{x} - S_{\rm lin}\mathbf{x})_{2h+\sigma}$ at the constant sequence $x_h = x_0$, changing to the variables $D_1, \dots, D_L$, and absorbing as many terms as possible into $O(\Omega_k(\mathbf{x}))$.

To obtain a more precise expression for $\Phi_{k,2h+\sigma}$, we need the following multi-index notation. Let

$$D_J := (D_{j_1}, \dots, D_{j_q}), \quad A_J^h := \prod_{k=1}^{q} \binom{h}{j_k}, \quad |J| := j_1 + \cdots + j_q\,.$$

[4]This is not to be confused with the order k condition in Definition 9.

[5]For the smallest support C^k subdivision scheme, namely the dyadic subdivision scheme coming from the degree $k+1$ B-spline, L is exactly k.

for $J = (j_1, \ldots, j_q)$ any ordered list of integers between 1 and $k-1$. For integers α, m with $0 \le \alpha \le m$, $2 \le m \le k$, consider the set of multi-indices of the form $I = (J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^{\alpha})$, where $q = m - \alpha$ and

$$J = (j_1, j_2, \ldots, j_{m-\alpha}), \quad J_1^i = (j_{1,1}^i, \ldots, j_{1,n^i-\beta^i}^i), \quad J_2^i = (j_{2,1}^i, \ldots, j_{2,\beta^i}^i),$$

where $1 \le j_i, j_{1,a}^i, j_{2,a}^i \le k$, $1 \le n_i \le k - m + 1$, and $0 \le \beta_i \le n_i$. We need only consider multi-indices I satisfying the additional condition

$$|I| := |J| + \sum_{i=1}^{\alpha}(|J_1^i| + |J_2^i|) \le k\,. \tag{4.4}$$

Finally, let $N_I = N_{(J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha})}$ denote the multilinear map

$$N_I(D) := N_{(J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha})}(D) = F_{\alpha}^{(m)}\Big(D_J; G_{\beta_1}^{(n_1)}(D_{J_1^1}; D_{J_2^1}), \ldots, G_{\beta_\alpha}^{(n_\alpha)}(D_{J_1^\alpha}; D_{J_2^\alpha})\Big)\,, \tag{4.5}$$

where $D = (D_1, \ldots, D_{k-1})$ are as in (4.3); and let $c_I^{h,\sigma} := c_{(J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha})}^{h,\sigma}$ be the real number

$$c_{(J,(n_i,\beta_i,J_1^i,J_2^i)_{i=1}^{\alpha})}^{h,\sigma} = \left[A_J^h \prod_{i=1}^{\alpha} A_{J_1^i}^h\right]\left[\prod_{i=1}^{\alpha}\sum_{\ell} a_{2\ell+\sigma}A_{J_2^i}^{h-\ell} - \sum_{\ell} a_{2\ell+\sigma}\prod_{i=1}^{\alpha} A_{J_2^i}^{h-\ell}\right]\,, \tag{4.6}$$

where

$$F_{\alpha}^{(m)} := \frac{1}{(m-\alpha)!\alpha!}F_{m-\alpha,\alpha}|_{(x_0,0)}, \quad G_{\beta}^{(n)} := \frac{1}{(n-\beta)!\beta!}G_{n-\beta,\beta}|_{(x_0,x_0)}\,,$$

and $F_{\alpha,\beta}$ are the derivatives of f defined in (1) and $G_{\alpha,\beta}$ are the analogously defined derivatives of g.

With this notation in place, we are ready to recall the following lemma.

Lemma 16 ([31, Lemma 3]). *For any $k \ge 2$,*

$$\begin{aligned}(S\mathbf{x} - S_{\text{lin}}\mathbf{x})_{2h+\sigma} &= \Phi_{k,2h+\sigma}(D_1, \ldots, D_{k-1}) + O(\Omega_k(\mathbf{x})) \\ &= \sum_{m=2}^{k}\sum_{|I|=m} c_I^{h,\sigma} N_I(D_1, \ldots, D_{k-1}) + O(\Omega_k(\mathbf{x}))\,.\end{aligned} \tag{4.7}$$

Two remarks are in order:

Remark 17. First note that to use Lemma 16 together with Theorem 14, we need to analyze not only $S\mathbf{x} - S_{\text{lin}}\mathbf{x}$, but also $\Delta^j(S\mathbf{x} - S_{\text{lin}}\mathbf{x})$ for all differencing orders $j \le k-1$. Assume that we have already established the order $k-1$ proximity condition,[6] To establish the next higher order, we need only prove (4.1) for $j = k$. Now, since the spatial indices (h, σ) only show up in (4.6), the operator Δ^{k-1} only acts on the sequences of coefficients defined by (4.6). Therefore, it is sufficient to prove that the polynomial

$$\Delta^{k-1}\Phi_{k,2h+\sigma}(D_1, \ldots, D_{k-1}) := \sum_{m=2}^{k}\sum_{|I|=m} \Delta^{k-1}c_I^{h,\sigma}\, N_I(D_1, \ldots, D_{k-1})$$

vanishes. In light of the $k-1$ proximity conditions, the sum of terms of weight less than k already vanishes. It, therefore, suffices to determine only the cases of $I = (J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^{\alpha})$ with $|I| = k$ and for which the sequence (in h, σ)

$$\Delta^{k-1}c_I^{h,\sigma} \tag{4.8}$$

is non-zero. We refer to such an index $I = (J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^{\alpha})$ as a ***non-vanishing case***.

[6]Recall that S and S_{lin} always satisfy order 2 proximity condition.

Remark 18. This lemma sheds some light on why the nonlinear scheme S suffers from a breakdown of smoothness equivalence. By construction, the coefficients (4.6) are polynomials in h for each fixed $\sigma = 0$ or 1. If all the sequences in (4.6) *were* polynomial sequences with degree not exceeding $k-2$, then the $k-1$-order differences in Equation (4.8) would always be zero. Therefore by Theorem 14, the nonlinear scheme (1.2) would satisfy the C^k equivalence property for any k. Unfortunately, this is too good to be true. And we may view this as a strong indication of why these schemes appear to suffer from a breakdown of smoothness equivalence (see [29, Section 1.1]).

We may attribute the breakdown to the *first bracket* in (4.6). The problem occurs only when at least one of the lists J and J_1^i, $i = 1, \ldots, \alpha$, is non-empty. In this case, since the index σ (= 0 or 1) does not show up in the first bracket of (4.6), the sequence (4.6) is not even a polynomial sequence, but consists of *two interlacing polynomial sequences*. In the case when all of J and J_1^i, $i = 1, \ldots, \alpha$ are empty, the first bracket becomes the constant unit sequence, and we are left with the sequence in the second bracket. In this case, the sequence is a polynomial sequence (in h) and, moreover, is one which makes (4.8) vanish. This is the content of our next lemma below.

Lemma 19 (Essentially borrowed from [29])**.** *Assume that the linear scheme S_{lin} reproduces Π_k. The second square bracket on the right-hand side of* (4.6) *is always a polynomial of degree not exceeding $\sum_{i=1}^{\alpha} |J_2^i| - 2$. (Note also that $\sum_{i=1}^{\alpha} |J_2^i| - 2 \le k-2$.)*

Consequently, (4.8) *vanishes when J, J_1^i, $i = 1, \ldots, \alpha$ are all empty; therefore we need not consider these cases.*[7] *We may also ignore the cases when $\sum_{i=1}^{\alpha} |J_2^i| \le 1$, as this implies* (4.6) *vanishes.*

Proof. See Appendix A. ☐

4.2. **Non-vanishing cases.** In this section, we enumerate the non-vanishing cases in Lemma 16 when $k = 4$. Under the $P_f = 0$ condition, we already have the order 3 proximity condition. Therefore, we only need to prove (4.1) for $j = 4$, and we only need to consider those $(J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^{\alpha})$ such that $|J| + \sum_{i=1}^{\alpha} |J_1^i| + |J_2^i| = 4$.

Our enumeration will be divided into two parts. Part I follows from an observation valid for any $k \ge 4$, while Part II simply consists of those non-trivial cases for $k = 4$ not included in Part I.

4.2.1. *Part I.* Fix a $k \ge 4$. According to the constraints, in any non-vanishing case there are at least three D_j's, each being at least 1, and (consequently) is at most $k-2$. In fact, the only way we can see a term involving D_{k-2} is when the map $F_\alpha^{(m)}\big(\cdot; \big(G_{\beta_i}^{(n_i)}(\cdot;\cdot)\big)_{i=1}^{\alpha}\big)$ arising from (4.5) is 3-linear and it acts on the arguments D_1, D_1, D_{k-2} (in any order.)

By the symmetries of the multilinear maps $F_\alpha^{(m)}$ and $G_\beta^{(n)}$, different combinations of $(J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^{\alpha})$ lead to the same term in (4.5). For example, since $F_\alpha^{(m)}$ is invariant under permutation of the last α arguments, for any permutation σ of $\{1, \ldots, \alpha\}$,

$$F_\alpha^{(m)}\big(D_J; G_{\beta_{\sigma(1)}}^{(n_{\sigma(1)})}(D_{J_1^{\sigma(1)}}; D_{J_2^{\sigma(1)}}), \ldots, G_{\beta_{\sigma(\alpha)}}^{(n_{\sigma(\alpha)})}(D_{J_1^{\sigma(\alpha)}}; D_{J_2^{\sigma(\alpha)}})\big)$$

is the same.

We now enumerate such terms in Table 2. The observation is that, after exploiting symmetry, there are always 7 such terms regardless of the value of k. In Table 2, each of the 7 cases is assigned a case label

[7] This lemma was overlooked in the order 3 proximity analysis in [31]. Notice that the zeros in the last column of [31, Table 1] correspond exactly to the cases where all J, J_1^i, $i = 1, \ldots, \alpha$ are all empty.

without any prime ($'$); any case with a label followed by prime(s) is one leading to a term identical to that coming from the corresponding unprimed case. As such, each of these 7 cases has a multiplicity associated to it, which we record in the last column of the table.

Case	(m,α)	J	$(n_i)_{i=1}^{\alpha}$	$(\beta_i)_{i=1}^{\alpha}$	$(J_1^i, J_2^i)_{i=1}^{\alpha}$	$c^{h,\sigma}$	$F_\alpha^{(m)}\left(D_J; (G_{\beta_i}^{(n_i)}(D_{J_1^i}; D_{J_2^i}))_{i=1}^{\alpha}\right)$	Mult.
(A)	$(2,2)$	J	(n_1,n_2)	(β_1,β_2)	$J_1^1, J_2^1, J_1^2, J_2^2$			
(AI)			$(1,2)$					
(AI.1)		$()$		$(1,1)$	$(), (k-2), (1), (1)$	Υ_1	$F_2^{(2)}(G_1^{(1)}(D_{k-2}), G_1^{(2)}(D_1; D_1))$	2
(AI.2)		$()$		$(1,1)$	$(), (1), (1), (k-2)$	Υ_1	$F_2^{(2)}(G_1^{(1)}(D_1), G_1^{(2)}(D_1; D_{k-2}))$	2
(AI.3)		$()$		$(1,1)$	$(), (1), (k-2), (1)$	Υ_2	$F_2^{(2)}(G_1^{(1)}(D_1), G_1^{(2)}(D_{k-2}; D_1))$	2
(AI')			$(2,1)$					
(B)	$(3,2)$	J	(n_1,n_2)	(β_1,β_2)	$J_1^1, J_2^1, J_1^2, J_2^2$			
(BI)			$(1,1)$					
(BI.1)		(1)		$(1,1)$	$(), (1), (), (k-2)$	Υ_1	$F_2^{(3)}(D_1; G_1^{(1)}(D_1), G_1^{(1)}(D_{k-2}))$	2
(BI.1$'$)		(1)		$(1,1)$	$(), (k-2), (), (1)$			
(BI.2)		$(k-2)$		$(1,1)$	$(), (1), (), (1)$	Υ_2	$F_2^{(3)}(D_{k-2}; G_1^{(1)}(D_1), G_1^{(1)}(D_1))$	1
(C)	$(3,3)$	J	(n_1,n_2,n_3)	$(\beta_1,\beta_2,\beta_3)$	$J_1^1, J_2^1, J_1^2, J_2^2, J_1^3, J_2^3$			
(CI)			$(1,1,1)$					
(CI.1)		$()$		$(1,1,0)$	$(), (k-2), (), (1), (1), ()$	Υ_1	$F_3^{(3)}(G_1^{(1)}(D_{k-2}), G_1^{(1)}(D_1), G_0^{(1)}(D_1))$	6
(CI.1$'$)		$()$		$(1,1,0)$	$(), (1), (), (k-2), (1), ()$			
(CI.1$''$)		$()$		$(1,0,1)$	$(), (k-2), (1), (), (), (1)$			
(CI.1$'''$)		$()$		$(1,0,1)$	$(), (1), (1), (), (), (k-2)$			
(CI.1$''''$)		$()$		$(0,1,1)$	$(1), (), (), (k-2), (), (1)$			
(CI.1$'''''$)		$()$		$(0,1,1)$	$(1), (), (), (1), (), (k-2)$			
(CI.2)		$()$		$(1,1,0)$	$(), (1), (), (1), (k-2), ()$	Υ_2	$F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_{k-2}))$	3
(CI.2$'$)		$()$		$(1,0,1)$	$(), (1), (k-2), (), (), (1)$			
(CI.2$''$)		$()$		$(0,1,1)$	$(k-2), (), (), (1), (), (1)$			

TABLE 2. The seven non-vanishing cases involving only D_{k-2} when $k \geq 4$. If $\Delta^{k-1}\Upsilon_1 = \Delta^{k-1}\Upsilon_2$, these cases impose a condition equivalent to the $P_f = 0$ condition, otherwise they force us to impose the zero curvature condition. Note: When $k = 3$, there are only 3 cases (see [31, Table 1]) which lead to the $P_f = 0$ condition.

4.2.2. *Part II.* When $k = 4$, the possible combinations for (m, α) in (4.7) are:

$$(2,2), (3,2), (3,3), (4,2), (4,3), (4,4)\,.$$

In Table 3, we enumerate for each of these six (m, α) all the non-vanishing cases $(J, (n_i, \beta_i, J_1^i, J_2^i)_{i=1}^{\alpha})$ with $|J| + \sum_{i=1}^{\alpha} |J_1^i| + \sum_{i=1}^{\alpha} |J_2^i| = 4$ and those not already covered in Part I.

These cases are listed in columns 3-6 in Table 3. We follow a similar convention as in Table 2. For example, case (AII.2$'$) gives the same term as case (AII.2). Cases (CII$'$) and (CII$''$) both lead to the same ***group*** of four terms determined by cases (CII.1)-(CII.4).

4.2.3. *Analysis of $c^{h,\sigma}$.* It should be clear that (4.6) has a stronger invariance w.r.t. the subscript indices than (4.5). While we have a total of $7 + 15$ different cases for (4.5) listed in Tables 2-3, by inspection, there are only two distinct sequences in (4.6) coming from Table 2:

$$\begin{aligned} \Upsilon_1^{h,\sigma} &:= A_1^h (S_{\text{lin}} A_1^h S_{\text{lin}} A_{k-2}^h - S_{\text{lin}} A_1^h A_{k-2}^h)_{2h+\sigma} \\ \Upsilon_2^{h,\sigma} &:= A_{k-2}^h (S_{\text{lin}} A_1^h S_{\text{lin}} A_1^h - S_{\text{lin}} A_1^h A_1^h)_{2h+\sigma} \end{aligned} \tag{4.9}$$

Case	(m,α)	J	$(n_i)_{i=1}^{\alpha}$	$(\beta_i)_{i=1}^{\alpha}$	$(J_1^i, J_2^i)_{i=1}^{\alpha}$	$c^{h,\sigma}$	$F_{\alpha}^{(m)}\left(D_J; (G_{\beta_i}^{(n_i)}(D_{J_1^i}; D_{J_2^i}))_{i=1}^{\alpha}\right)$	Mult.
(A)	$(2,2)$	J	(n_1,n_2)	(β_1,β_2)	$J_1^1, J_2^1, J_1^2, J_2^2$			
(AII)			$(2,2)$					
(AII.1)		$()$		$(1,1)$	$(1),(1),(1),(1)$	Ξ_2	$F_2^{(2)}(G_1^{(2)}(D_1,D_1), G_1^{(2)}(D_1,D_1))$	1
(AII.2)		$()$		$(1,2)$	$(1),(1),(),(1,1)$	Ξ_1	$F_2^{(2)}(G_1^{(2)}(D_1;D_1), G_2^{(2)}(D_1;D_1))$	2
(AII.2′)		$()$		$(2,1)$	$(),(1,1),(1),(1)$			
(AIII)			$(1,3)$					
(AIII.1)		$()$		$(1,1)$	$(),(1),(1,1),(1)$	Ξ_2	$F_2^{(2)}(G_1^{(1)}D_1, G_1^{(3)}(D_1,D_1,D_1))$	2
(AIII.2)		$()$		$(1,2)$	$(),(1),(1),(1,1)$	Ξ_1	$F_2^{(2)}(G_1^{(1)}(D_1), G_2^{(3)}(D_1;D_1,D_1))$	2
(AIII′)			$(3,1)$					
(B)	$(3,2)$	J	(n_1,n_2)	(β_1,β_2)	$J_1^1, J_2^1, J_1^2, J_2^2$			
(BII)			$(1,2)$					
(BII.1)		(1)		$(1,1)$	$(),(1),(1),(1)$	Ξ_2	$F_2^{(3)}(D_1; G_1^{(1)}(D_1), G_1^{(2)}(D_1;D_1))$	2
(BII.2)		(1)		$(1,2)$	$(),(1),(),(1,1)$	Ξ_1	$F_2^{(3)}(D_1; G_1^{(1)}(D_1), G_2^{(2)}(D_1,D_1))$	2
(BII′)			$(2,1)$					
(C)	$(3,3)$	J	(n_1,n_2,n_3)	$(\beta_1,\beta_2,\beta_3)$	$J_1^1, J_2^1, J_1^2, J_2^2, J_1^3, J_2^3$			
(CII)			$(1,1,2)$					
(CII.1)		$()$		$(0,1,1)$	$(1),(),(),(1),(1),(1)$	Ξ_2	$F_3^{(3)}(G_0^{(1)}D_1, G_1^{(1)}D_1, G_1^{(2)}(D_1,D_1))$	2×3
(CII.1′)		$()$		$(1,0,1)$	$(),(1),(1),(),(1),(1)$			
(CII.2)		$()$		$(1,1,1)$	$(),(1),(),(1),(1),(1)$	Ξ_3	$F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(2)}(D_1,D_1))$	3
(CII.3)		$()$		$(0,1,2)$	$(1),(),(),(1),(),(1,1)$	Ξ_1	$F_3^{(3)}(G_0^{(1)}(D_1), G_1^{(1)}(D_1), G_2^{(2)}(D_1,D_1))$	2×3
(CII.3′)		$()$		$(1,0,2)$	$(),(1),(1),(),(),(1,1)$			
(CII.4)		$()$		$(1,1,0)$	$(),(1),(),(1),(1,1),()$	Ξ_2	$F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(2)}(D_1,D_1))$	3
(CII′)			$(1,2,1)$					
(CII″)			$(2,1,1)$					
(D)	$(4,2)$	J	(n_1,n_2)	(β_1,β_2)	$J_1^1, J_2^1, J_1^2, J_2^2$			
		$(1,1)$	$(1,1)$	$(1,1)$	$(),(1),(),(1)$	Ξ_2	$F_2^{(4)}(D_1,D_1; G_0^{(1)}(D_1), G_0^{(1)}(D_1))$	1
(E)	$(4,3)$	J	(n_1,n_2,n_3)	$(\beta_1,\beta_2,\beta_3)$	$J_1^1, J_2^1, J_1^2, J_2^2, J_1^3, J_2^3$			
(E.1)		(1)	$(1,1,1)$	$(1,1,0)$	$(),(1),(),(1),(1),()$	Ξ_2	$F_3^{(4)}(D_1; G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1))$	3
(E.1′)-(E.1″)								
(E.2)		(1)		$(1,1,1)$	$(),(1),(),(1),(),(1)$	Ξ_3	$F_3^{(4)}(D_1; G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(1)}(D_1))$	1
(F)	$(4,4)$	J	(n_1,n_2,n_3,n_4)	$(\beta_1,\beta_2,\beta_3,\beta_4)$	$J_1^1, J_2^1, J_1^2, J_2^2, J_1^3, J_2^3, J_1^4, J_2^4$			
(F.1)		$()$	$(1,1,1,1)$	$(1,1,0,0)$	$(),(1),(),(1),(1),(),(1),()$	Ξ_2	$F_4^{(4)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1), G_0^{(1)}(D_1))$	6
(F.1′)-(F.1′′′′′)								
(F.2)		$()$		$(1,1,1,0)$	$(),(1),(),(1),(),(1),(1),()$	Ξ_3	$F_4^{(4)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1))$	4
(F.2′)-(F.2‴)								

TABLE 3. The non-vanishing cases for $k=4$ for not covered by Table 2. Note: Cases (E.1′)-(E.1″), (F.1′)-(F.1′′′′′) and (F.2′)-(F.2‴) refer to the obvious shuffling of $(J_1^i, J_2^i)_{i=1}^{\alpha}$ of their corresponding unprimed cases.

and three coming from Table 3:

$$\begin{aligned}
\Xi_1^{h,\sigma} &:= A_1^h (S_{\text{lin}} A_1^h S_{\text{lin}} A_1^h A_1^h - S_{\text{lin}} A_1^h A_1^h A_1^h)_{2h+\sigma} \\
\Xi_2^{h,\sigma} &:= A_1^h A_1^h (S_{\text{lin}} A_1^h S_{\text{lin}} A_1^h - S_{\text{lin}} A_1^h A_1^h)_{2h+\sigma} \qquad (4.10)\\
\Xi_3^{h,\sigma} &:= A_1^h (S_{\text{lin}} A_1^h S_{\text{lin}} A_1^h S_{\text{lin}} A_1^h - S_{\text{lin}} A_1^h A_1^h A_1^h)_{2h+\sigma}\,.
\end{aligned}$$

Column 7 in both tables indicates which of the five sequences is obtained in each case. Note that these five sequences are dependent on S_{lin} and independent of the retraction f. After applying the 3rd order differencing operator to them, there are only 4 independent sequences; when S_{lin} has a dual time-symmetry, we are left with only two.

Lemma 20. *Let S_{lin} be a linear subdivision scheme that reproduces Π_3 and has a dual time-symmetry, i.e. $a_i = a_{1-i}$, we have: (i) $\Delta^3\,\Xi_1 = \Delta^3\,\Xi_2 = \frac{2}{3}\Delta^3\,\Xi_3$, and (ii) $\Delta^3\,\Upsilon_1 = \Delta^3\,\Upsilon_2$ when $k=4$ in (4.9).*

Proof. See Appendix B. □

The proof of Theorem 13 uses the following curvature condition.

Lemma 21. *The condition that the curvature of the affine connection induced by f vanishes is equivalent to the condition*

$$F_{1,2}(v;u,v)-F_{1,2}(u;v,v)+F_{0,2}(u,F_{0,2}(v,v))-F_{0,2}(v,F_{0,2}(u,v))=0,\quad \forall u,v. \tag{4.11}$$

Proof. In local coordinates using index notation, and recalling that $\Gamma^k_{ij}=-f^k_{ij}$, condition (4.11) assumes the form

$$\left\{-\frac{\partial\Gamma^\ell_{jk}}{\partial x^i}+\frac{\partial\Gamma^\ell_{ik}}{\partial x^j}+\Gamma^\ell_{ia}\Gamma^a_{j,k}-\Gamma^\ell_{ja}\Gamma^a_{ik}\right\}X^iY^jX^k\frac{\partial}{\partial x^\ell}=0$$

for all X, Y. We now use the following well-known formula for the curvature tensor of a connection

$$R_f(X,Y)Z=R^\ell_{ijk}X^iY^jZ^k\frac{\partial}{\partial x^\ell}\,, \tag{4.12}$$

where

$$R^\ell_{ijk}=\frac{\partial\Gamma^\ell_{jk}}{\partial x^i}-\frac{\partial\Gamma^\ell_{ik}}{\partial x^j}+\Gamma^\ell_{ip}\Gamma^p_{jk}-\Gamma^\ell_{pj}\Gamma^p_{ik}\,.$$

which shows that (4.11) is equivalent to the condition $R_f(X,Y)X=0$ for all X, Y.

Clearly then, vanishing curvature implies (4.11). The converse is the content of the next lemma. □

Lemma 22. *Suppose that $R_f(X,Y)Y=0$ for all X, Y. Then $R_f(X,Y)Z=0$ for all X, Y, Z.*

Proof. [8] Any bilinear mapping $b(u,v)$ with $b(u,u)=0$ is skew-symmetric, so our assumption implies that the curvature $R_f(X,Y)Z$ is skew-symmetric in the variables Y and Z. Since the curvature tensor is also skew-symmetric in X,Y, the first Bianchi identity

$$R_f(X,Y)Z+R_f(Y,Z)X+R_f(Z,X)Y=0$$

transforms to

$$R_f(X,Y)Z+R_f(X,Y)Z+R_f(X,Y)Z=0$$

if we apply two swaps to the second and third terms. This shows $3R_f=0$. □

4.3. **Proof of Theorem 13.** Armed with Tables 2 and 3 and Lemmas 20 and 21, we are now ready to prove Theorem 13.

Proof of Theorem 13. By Remark 17, we only need to show that $\Delta^3\Phi_{4,2h+\sigma}$ vanishes identically. First notice that the variable D_3 does not appear in Table 2 (with $k=4$) nor in Table 3. Further inspection of the tables shows that

$$\Delta^3\Phi_{4,2h+\sigma}(D_1,D_2)=\Delta^3\Upsilon_1H_1+\Delta^3\Upsilon_2H_2+\Delta^3\Xi_1Q_1+\Delta^3\Xi_2Q_2+\Delta^3\Xi_3Q_3\,, \tag{4.13}$$

[8]We wish to thank one of the referees for suggesting this proof.

where

$$H_1(D_1,D_2) := F_2^{(2)}(G_1^{(1)}(D_2), G_1^{(2)}(D_1;D_1)) \times 2 + F_2^{(2)}(G_1^{(1)}(D_1), G_1^{(2)}(D_1;D_2)) \times 2 + F_2^{(3)}(D_1; G_1^{(1)}D_1, G_1^{(1)}D_2) \times 2 + F_3^{(3)}(G_1^{(1)}(D_2), G_1^{(1)}(D_1), G_0^{(1)}(D_1)) \times 6, \tag{4.14a}$$

$$H_2(D_1,D_2) := F_2^{(2)}(G_1^{(1)}(D_1), G_1^{(2)}(D_2;D_1)) \times 2 + F_2^{(3)}(D_2; G_1^{(1)}(D_1), G_1^{(1)}(D_1)) \times 1 + F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_2)) \times 3, \tag{4.14b}$$

$$\begin{aligned} Q_1(D_1) := {} & F_2^{(2)}(G_1^{(2)}(D_1;D_1), G_2^{(2)}(D_1,D_1)) \times 2 \\ & + F_2^{(2)}(G_1^{(1)}(D_1), G_2^{(3)}(D_1;D_1,D_1)) \times 2 \\ & + F_2^{(3)}(D_1; G_1^{(1)}D_1, G_2^{(2)}(D_1,D_1)) \times 2 \\ & + F_3^{(3)}(G_0^{(1)}(D_1), G_1^{(1)}(D_1), G_2^{(2)}(D_1,D_1)) \times 2 \times 3\,, \end{aligned} \tag{4.15a}$$

$$\begin{aligned} Q_2(D_1) := {} & F_2^{(2)}(G_1^{(2)}(D_1;D_1), G_1^{(2)}(D_1;D_1)) \times 1 \\ & + F_2^{(2)}(G_1^{(1)}(D_1), G_1^{(3)}(D_1;D_1,D_1)) \times 2 \\ & + F_2^{(3)}(D_1; G_1^{(1)}(D_1), G_1^{(2)}(D_1,D_1)) \times 2 \\ & + F_3^{(3)}(G_0^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(2)}(D_1,D_1)) \times 2 \times 3 \\ & + F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(2)}(D_1,D_1)) \times 3 \\ & + F_2^{(4)}(D_1; D_1, G_0^{(1)}(D_1), G_0^{(1)}(D_1)) \times 1 \\ & + F_3^{(4)}(D_1; G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1)) \times 3 \\ & + F_4^{(4)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1), G_0^{(1)}(D_1)) \times 6\,, \end{aligned} \tag{4.15b}$$

$$\begin{aligned} Q_3(D_1) := {} & F_3^{(3)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(2)}(D_1;D_1)) \times 3 \\ & + F_3^{(4)}(D_1; G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(1)}(D_1)) \times 1 \\ & + F_4^{(4)}(G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_1^{(1)}(D_1), G_0^{(1)}(D_1)) \times 4. \end{aligned} \tag{4.15c}$$

These expressions simplify considerably if we express the derivatives of g in terms of those of f, and replace D_1 and D_2 by u and v, respectively. By applying the chain rule to the relation $f(x, g(x,y)) = y$, together with our assumptions that $F_0^{(1)} = F_1^{(1)} = \mathrm{id}$, we have

$$G_0^{(1)} = -\mathrm{id}, \;\; G_1^{(1)} = \mathrm{id}, \;\; G_0^{(2)} = -\frac{1}{2}F_{0,2}, \;\; G_2^{(2)} = -\frac{1}{2}F_{0,2}, \;\; G_1^{(2)}(u,v) = F_{0,2}(u,v)\,, \tag{4.16a}$$

and, using (3.1),

$$\begin{aligned} G_1^{(3)}(u;u,u) &= \frac{3}{2}F_{0,2}(u, F_{0,2}(u,u)) + F_{1,2}(u;u,u) - \frac{1}{2}F_{0,3}(u,u,u) \\ &= \frac{1}{2}F_{0,2}(u, F_{0,2}(u,u)) + \frac{1}{2}F_{1,2}(u;u,u), \end{aligned} \tag{4.16b}$$

$$\begin{aligned} G_2^{(3)}(u,u;u) &= -\frac{3}{2}F_{0,2}(u, F_{0,2}(u,u)) - \frac{1}{2}F_{1,2}(u;u,u) + \frac{1}{2}F_{0,3}(u,u,u) \\ &= -\frac{1}{2}F_{0,2}(u, F_{0,2}(u,u)). \end{aligned} \tag{4.16c}$$

(The bilinear map $G_1^{(2)}$ happens to be symmetric in its two arguments. The trilinear maps $G_1^{(3)}$ and $G_2^{(3)}$, on the other hand, are not symmetric. The asymmetries do not concern us for now as we only need the expressions for $G_1^{(3)}(u^3)$ and $G_2^{(3)}(u^3)$.)

Now apply (4.16) to simplify the expressions for H_i and Q_j:

$$H_1(u,v) = F_{0,2}(v, F_{0,2}(u,u)) + F_{0,2}(u, F_{0,2}(u,v)) + F_{1,2}(u;u,v) - F_{0,3}(v,u,u) \tag{4.17a}$$

$$H_2(u,v) = F_{0,2}(u, F_{0,2}(u,v)) + \frac{1}{2}F_{1,2}(v;u,u) - \frac{1}{2}F_{0,3}(u,u,v) \tag{4.17b}$$

$$\begin{aligned} Q_1(u) :=& -\frac{1}{2}F_{0,2}(F_{0,2}(u,u), F_{0,2}(u,u)) - \frac{1}{2}F_{0,2}(u, F_{0,2}(u, F_{0,2}(u,u))) \\ & - \frac{1}{2}F_{1,2}(u;u,F_{0,2}(u,u)) + \frac{1}{2}F_{0,3}(u,u,F_{0,2}(u,u)) \end{aligned} \tag{4.18a}$$

$$\begin{aligned} Q_2(u) :=& \frac{1}{2}F_{0,2}(F_{0,2}(u,u), F_{0,2}(u,u)) + \frac{1}{2}F_{0,2}(u, F_{0,2}(u, F_{0,2}(u,u))) - \frac{1}{2}F_{0,2}(u, F_{1,2}(u;u,u)) \\ & + F_{1,2}(u;u,F_{0,2}(u,u)) - \frac{5}{4}F_{0,3}(u,u,F_{0,2}(u,u)) + \frac{1}{4}F_{2,2}(u,u;u,u) \\ & - \frac{1}{2}F_{1,3}(u;u,u,u) + \frac{1}{4}F_{0,4}(u,u,u,u)\,, \end{aligned} \tag{4.18b}$$

$$Q_3(u) := \frac{3}{4}F_{0,3}(u,u,F_{0,2}(u,u)) + \frac{1}{4}F_{1,3}(u;u,u,u) - \frac{1}{4}F_{0,4}(u,u,u,u)\,. \tag{4.18c}$$

Next use the condition $P_f(u) = 0$ to further simplify Equation (4.13) as follows. Let

$$\begin{aligned} P_1(u,v) :=& H_1(u,v) + H_2(u,v) \\ =& F_{0,2}(v, F_{0,2}(u,u)) + 2F_{0,2}(u, F_{0,2}(u,v)) \\ & + F_{1,2}(u;u,v) + \frac{1}{2}F_{1,2}(v;u,u) - \frac{3}{2}F_{0,3}(u,u,v), \end{aligned} \tag{4.19a}$$

and

$$\begin{aligned} P_2(u) :=& Q_1(u) + Q_2(u) + \frac{3}{2}Q_3(u) \\ =& \frac{1}{2}F_{1,2}(u;u,F_{0,2}(u,u)) + \frac{1}{2}F_{0,2}(u, F_{1,2}(u;u,u) \\ & - \frac{1}{4}F_{1,3}(u;u,u,u) + \frac{1}{4}F_{2,2}(u,u;u,u). \end{aligned} \tag{4.19b}$$

We claim that conditions $P_1(u,v) = 0$ and $P_2(u) = 0$ are satisfied and are simply the de-polarization and the spatial differentiation of the condition $P_f(u) = 0$. First compare (4.19a) and (3.4) to conclude

$$P_1(u,v) = 3\,P_f(u,u,v) = 0\,.$$

Next observe that differentiating (3.1) with respect to the spatial variable x, gives exactly $2P_2(u)$, from which we conclude $P_2(u) = 0$. The equations $P_1(u,v) = 0$ and $P_2(u) = 0$, together imply

$$H_1 + H_2 = 0, \text{ and } Q_1 + Q_2 + \frac{3}{2}Q_3 = 0\,. \tag{4.20}$$

Consequently, Equation (4.13) reduces to

$$\Delta^3\Phi_{4,2h+\sigma}(u,v) = (\Delta^3\Upsilon_1 - \Delta^3\Upsilon_2)\,H_1(u,v) + (\Delta^3\Xi_1 - \frac{2}{3}\Delta^3\Xi_3)\,Q_1(u) + (\Delta^3\Xi_2 - \frac{2}{3}\Delta^3\Xi_3)\,Q_2(u)\,. \tag{4.21}$$

To prove part (a), assume that S_{lin} has a dual time-symmetry and observe that Lemma 20 immediately shows that $\Delta^3\Phi_{4,2h+\sigma}(u,v)$ vanishes identically.

To prove part (b), assume that S_{lin} does not have a dual time-symmetry and that $\Delta^3\Phi_{4,h2+\sigma} = 0$. Setting $v = 0$ gives

$$\Delta^3\Phi_{4,2h+\sigma}(u,0) = (\Delta^3\Xi_1 - \frac{2}{3}\Delta^3\Xi_3)\,Q_1(u) + (\Delta^3\Xi_2 - \frac{2}{3}\Delta^3\Xi_3)\,Q_2(u)\,.$$

It follows that $(\Delta^3\Upsilon_1 - \Delta^3\Upsilon_2)H_1(u,v) = 0$ for all u, v. One can check from examples (e.g. the C^4 degree 5 B-spline scheme) that $\Delta^3\Upsilon_1 \neq \Delta^3\Upsilon_2$. Consequently, the term $H_1(u,v) = 0$ for all u, v.

Using the condition $P_1(u,v) = 0$ to replace the term involving $F_{0,3}$ in $H_1(u,v)$ with lower order expression shows that

$$H_1(u,v) = -\frac{1}{3}[F_{1,2}(v,u,v) - F_{1,2}(u,v,v) + F_{0,2}(u,F_{0,2}(v,v)) - F_{0,2}(v,F_{0,2}(u,v))].$$

Consequently, Equation (4.11) and Lemma 22 imply that the connection induced by f has vanishing curvature.

We have shown that

$$\Delta^3\Phi_{4,2h+\sigma}(u,v) = (\Delta^3\Xi_1 - \frac{2}{3}\Delta^3\Xi_3)\,Q_1(u) + (\Delta^3\Xi_2 - \frac{2}{3}\Delta^3\Xi_3)\,Q_2(u)\,.$$

We now know that the connection induced by f is torsion-free with vanishing curvature. It is a well-known fact in differential geometry (see for instance [4]) that we can choose special coordinates, centered at x_0, in which all connection coefficients $\Gamma^k_{ij} = -\frac{\partial^2 f^\ell(x,0)}{\partial X^i \partial X^j}$ vanish for x in a neighborhood of x_0, i.e.

$$F_{0,2} \equiv 0.$$

We call such coordinates ***flat coordinates***.

By Remark 15, we may compute the proximity condition in flat coordinates. The initial value problem (2.8) defining the exponential map then reduces to the form

$$\ddot{x}^\ell = 0\,, \quad x(0) = x \quad \dot{x}(0) = X\,.$$

Consequently, the exponential map is given by

$$\exp_f(x,X) = x + X\,.$$

and the order k-condition reduces to $F_{0,k} = 0$.

Notice that in flat coordinates, $F_{k,2} = 0$ for all k, and therefore the condition $P_f = 0$ reduces to the equivalent condition

$$F_{0,3} \equiv 0.$$

This in turn implies that $F_{k,3} = 0$ for all k. It follows that (in flat coordinates) $Q_1(u) = 0$, and $\Delta^3\Phi_{4,2h+\sigma}(u,v)$ reduces to the expression

$$\Delta^3\Phi_{4,2h+\sigma}(u,v) = \frac{1}{4}(\Delta^3\Xi_2 - \frac{2}{3}\Delta^3\Xi_3)\,F_{0,4}(u,u,u,u)\,.$$

One can check from examples (e.g. the C^4 degree 5 B-spline scheme) that $(\Delta^3\Xi_2 - \frac{2}{3}\Delta^3\Xi_3) \neq 0$. Hence, the C^4-proximity condition implies $F_{0,4} \equiv 0$. But the condition that f satisfy the order 4 condition[9] also reduces to

$$F_{0,4} \equiv 0.$$

Consequently, (for schemes without dual time-symmetry) the vanishing curvature condition together with the order 4 condition, are equivalent to the 4-th order proximity condition $\Delta^3\Phi_{4,2h+\sigma} \equiv 0$. □

[9]For the reference, the order 4 condition on a general manifold can be expressed in a general coordinate system as $F_{0,4}u^4 - F_{2,2}(u^2;u^2) - 4F_{1,2}(u;u,F_{0,2}u^2) - F_{1,2}(F_{0,2}u^2;u^2) - 2F_{0,2}(F_{1,2}(u;u^2),u) - 2F_{0,2}(F_{0,2}(u^2)^2) - 4F_{0,2}(u,F_{0,2}(u,F_{0,2}(u^2))) = 0$, assuming that the order 3 condition $P_f(u) = F_{0,2}(u,F_{0,2}(u,u)) + \frac{1}{2}F_{1,2}(u;u,u) - \frac{1}{2}F_{0,3}(u,u,u) = 0$ already holds.

5. Order k proximity analysis, $k \geq 5$

When $k = 5$, there are *many* more nontrivial cases than the seven cases listed in Table 2, but since the argument D_3 shows up only in those seven cases, and since D_3 is arbitrary, the order 5 proximity condition implies the condition

$$\Delta^4 \Upsilon_1 H_1(D_1, D_3) + \Delta^4 \Upsilon_2 H_2(D_1, D_3) = 0, \quad \forall\, D_1, D_3\,,$$

where H_1 and H_2 are defined in (4.14a) and (4.14b), and the sequences Υ_i, $i = 1, 2$, are defined in (4.9) with $k = 5$. The condition $P_f(u) = 0$ again implies $H_2(u, v) = -H_1(u, v)$. Consequently, the order 5 proximity condition implies

$$\left(\Delta^4 \Upsilon_1 - \Delta^4 \Upsilon_2\right) H_1(u, v) = 0 \tag{5.1}$$

for all u, v, where we have set $u = D_1$ and $v = D_3$, Moreover, one can check from examples (e.g. the dual-symmetric C^5 degree 6 B-spline scheme would do) that when $k = 5$,

$$\Delta^4 \Upsilon_1 \neq \Delta^4 \Upsilon_2\,.$$

Therefore, H_1 vanishes identically. But we learned in the previous section that this implies that the affine connection induced by f has vanishing curvature. Consequently, the C^5 proximity conditions imply vanishing curvature, *even for subdivision schemes satisfying dual time-symmetry*!

Paradoxically, although the vanishing curvature condition has the unfortunate consequence of ruling out many manifolds, it greatly simplifies the analysis of the remaining proximity conditions. In addition to the curvature terms, are a number of other terms, many more than in Table 3. However, as in the proof of part (b) of Theorem 13, we may choose flat coordinates in which the derivative $F_{0,2}$ vanishes identically. Inductively applying the proximity conditions for $k \leq 5$ yields the conditions

$$F_{0,k} = 0, \text{ for } 2 \leq k \leq 5\,.$$

Notice also that this is exactly the order 5 condition on the retraction f. This induction step can be continued indefinitely to yield the next proposition.

Proposition 23. *If f is a retraction whose connection is flat, then for $k \geq 5$ the order k conditions reduce in flat coordinates to the conditions*

$$F_{0,k'} = 0 \textit{ for all } k' \leq k\,.$$

Moreover, these conditions imply the S and $S_{\rm lin}$ satisfy the order k proximity condition.

Note that $F_{0,k'} = 0$ for all $k' \leq k$ means $f(x, X) = x + X + O(|X|^{k+1})$. If this condition holds for all k and if f is analytic, then $f(x, X) = x + X$. In this very case, our nonlinear subdivision scheme S is really a linear subdivision scheme in disguise: in normal coordinates (i.e. coordinates defined by the exponential map of the connection), S is exactly the linear scheme $S_{\rm lin}$, and we may write: $S = \phi \circ S_{\rm lin} \circ \phi^{-1}$,[10] where ϕ is the change of coordinate map from normal coordinates to whatever coordinates we begin our analysis with. In this special case, C^k equivalence between S and $S_{\rm lin}$ obviously holds.

6. Ongoing and Future Work

As we remarked earlier, our analysis based on the proximity conditions only gives sufficient conditions for C^k equivalence. This raises the following questions, which we partially address here and in our ongoing work [12]:

[10] For $M = \mathbb{R}^n$ or $\mathbb{R}^+$, this kind of "linear subdivision schemes in disguise" are explored in [22].

(Q1) Is the $P_f = 0$ condition derived in [31] truly necessary for C^3 equivalence?

(Q2) Is the vanishing curvature condition and the order 4 condition truly necessary for C^4 equivalence when the linear scheme does not have the right symmetry?

(Q3) Is the vanishing curvature condition and the order 5 condition truly necessary for C^5 equivalence even if we have the right symmetry?

(Q1'-Q3') In each case above, can we at least prove by example that the sufficient conditions cannot be dispensed with?

There is ample numerical evidence supporting the conjecture that these conditions are necessary; but it has defied proof. The fundamental difficulty in our study of nonlinear subdivision thus far [27, 32, 26, 31, 29, 25, 28, 15, 14, 13] is that we do not have any effective way to guarantee the following implication:

$$\|\Delta^k S^j \mathbf{x}\|_\infty \lesssim 2^{-j\nu} \overset{?}{\Leftarrow} S \text{ is } C^\nu \text{ smooth.} \tag{6.1}$$

We recall that, while the converse implication

$$\|\Delta^k S^j \mathbf{x}\|_\infty \lesssim 2^{-j\nu} \Rightarrow S \text{ is } C^\nu \text{ smooth} \tag{6.2}$$

always holds, the implication (6.1) is not guaranteed even when S is linear. In the linear theory (e.g. [5, 21, 7]), the so-called ***stability condition*** in various forms guarantees (6.1). Notice that if a (possibly nonlinear) subdivision scheme S is interpolatory (which is exactly the case we are *not* interested in here), then the subdivision data $S^j\mathbf{x}$ is simply the limit function evaluated at the dyadic grid $2^{-j}\mathbb{Z}$, in this case it is well-known to approximation theorists that (6.1) holds. For nonlinear non-interpolatory schemes, one can replace stability condition by a rate of convergence condition; for example, if $\nu < 1 \le k$, then one can show that

$$S \text{ is } C^\nu \text{ smooth } + \sup_k |\phi(2^{-j}k) - (S^j\mathbf{x})_k| \lesssim 2^{-j\nu} \Rightarrow \|\Delta^k S^j \mathbf{x}\|_\infty \lesssim 2^{-j\nu}.$$

See [12] for more details.

We venture ourselves to answer (Q1') by constructing a concrete example with $M = \mathbb{R}$ where the condition $P_f = 0$ does not hold and C^3 equivalence fails. In this case, $TM = \mathbb{R} \times \mathbb{R}$ and we define f by the formula

$$f(x, X) = x + X + X^3, \tag{6.3}$$

which satisfies $F_{0,2} = 0$ but where $P_f = F_{0,3}$ is clearly non-vanishing. Let $S_{\rm lin}$ be the C^3 degree 4 B-Spline scheme, which happens also to have dual time-symmetry. Our goal is to prove that the associated nonlinear scheme S is *not* C^3.

In [12], we prove the following estimate, using a dynamical system analysis: For generic initial data $\mathbf{x}$,

$$\|\Delta^3 S^j \mathbf{x}\|_\infty \asymp j 2^{-3j}. \tag{6.4}$$

This alone is more than enough to guarantee that the limit function ϕ is C^2 (which we already know from previous results), but is insufficient for arguing that ϕ is *not* C^3 – due to the absence of a "nonlinear stability condition."

Write $x_k^j := (S^j x)_k$. We prove in [12] the following rate of convergence result:

$$\sup_k \left| \frac{x_{k+1}^j - 2x_k^j + x_{k-1}^j}{2^{-2j}} - \phi''\left(2^{-j}(k+1/2)\right) \right| \lesssim 2^{-j}. \tag{6.5}$$

Armed with (6.5), we argue as follows. Suppose ϕ'' is Lipschitz, then

$$\sup_k \left|\phi''\left(2^{-j}(k+1/2)\right) - \phi''\left(2^{-j}(k-1/2)\right)\right| \lesssim 2^{-j}. \tag{6.6}$$

Combining (6.5) and (6.6) and employing the triangle inequality yields the estimate

$$\sup_k \left|2^{2j}(x^j_{k+1} - 2x^j_k + x^j_{k-1}) - 2^{2j}(x^j_k - 2x^j_{k-1} + x^j_{k-2})\right| \lesssim 2^{-j}\,.$$

But this contradicts (6.4). Therefore ϕ'' cannot be Lipschitz, let alone being C^1. In other words, we get a breakdown in C^3 equivalence, as desired.

To appreciate the technical nature of the estimate (6.5), notice that even if S were a *stable linear* subdivision scheme satisfying (6.4), the linear theory tells us that we should only expect the following rate of convergence:

$$\sup_k \left|2^{2j}(x^j_{k+1} - 2x^j_k + x^j_{k-1}) - \phi''\left(2^{-j}k\right)\right| \lesssim j2^{-j}. \tag{6.7}$$

For the nonlinear scheme S at hand, we can indeed prove that (6.7) holds true. This 'standard' estimate, however, is of no use for our purpose here. With this background in mind, the rate of convergence in (6.5) looks unreasonably fast. The underlying reason appears to be that ϕ'' is in the Zygmund class Λ_*, and we effectively exploit the subtle fact that Lip1 is slightly smaller than Λ_* [18, 6].

By the way, this specific result also settles another question: Can the dual time-symmetry condition on $S_{\rm lin}$ guarantee C^3 equivalence without the $P_f = 0$ condition? The answer is negative according to the result above.

Appendix A. Proof of Lemma 19

By the definition of $S_{\rm lin}$, the sequence is the second square bracket of (4.6) is simply $\prod_i S_{\rm lin} A_{J_2^i} - S_{\rm lin}\prod_i A_{J_2^i}$, where A_J is thought of as a sequence on $\mathbb{Z}$ whose h-th entry is A^h_J. Note that A_J is a polynomial sequence of degree $|J|$. Since $\sum_{i=1}^{\alpha} |J_2^i| \le k$, and $S_{\rm lin}$ leaves the polynomial spaces Π_ℓ, $\ell \le k$, invariant, both $\prod_i S_{\rm lin} A_{J_2^i}$ and $S_{\rm lin}\prod_i A_{J_2^i}$, and hence also their difference, must have degrees no bigger than $\sum_{i=1}^{\alpha} |J_2^i|$.

We now prove that, in fact, the difference is two degree less than what we expect. The proof of this part merely requires the simple fact $\sum_\ell a_{2\ell+\sigma} = 1$. Note that

$$A^{h-\ell}_j = \frac{1}{j!}\left(h^j + B_j(\ell)h^{j-1} + \sum_{d<j-1} B^d_j(\ell)h^d\right).$$

If $J = (j_1, \ldots, j_\gamma)$,

$$A^{h-\ell}_J = \prod A^{h-\ell}_{j_i} = \frac{1}{j_1!\cdots j_\gamma!}\left(h^{|J|} + B_J(\ell)h^{|J|-1} + \cdots\right),$$

where $B_J(\ell) = B_{j_1}(\ell) + \cdots + B_{j_\gamma}(\ell)$. Write $J! := j_1!\cdots j_\gamma!$ and $D := \sum_i |J_2^i|$. By aggregating terms one more time we have

$$\prod_{i=1}^{\alpha} A^{h-\ell}_{J_2^i} = \frac{1}{J_2^1!\cdots J_2^\alpha!}\left(h^D + \sum_{i=1}^{\alpha} B_{J_2^i}(\ell)\cdot h^{D-1} + \text{lower deg. terms}\right).$$

It should now be evident that both the h^D and h^{D-1} terms in $\prod_{i=1}^{\alpha}\sum_\ell a_{2\ell+\sigma}A^{h-\ell}_{J_2^i}$ and $\sum_\ell a_{2\ell+\sigma}\prod_{i=1}^{\alpha} A^{h-\ell}_{J_2^i}$ are the same, and hence drop off under the difference. The lemma follows. □

Appendix B. Proof of Lemma 20

Using only the property that S_{lin} reproduces Π_3, by Lemma 19, we already know that for each fixed $\sigma = 0$ or 1, $\Xi_i^{h,\sigma}$ or $\Upsilon_i^{h,\sigma}$ is a *quadratic* polynomial in h, therefore, so are

$$\Xi_1^{h,\sigma} - \Xi_2^{h,\sigma},\ \ \Xi_3^{h,\sigma} - \frac{3}{2}\Xi_1^{h,\sigma},\ \ \Upsilon_1^{h,\sigma} - \Upsilon_2^{h,\sigma}\,.$$

The lemma is proved if we can prove that in each of these three cases, the sequence is actually a *single* quadratic polynomial sampled at $2h+\sigma$. (Recall Remark 18.) But this is equivalent to showing that

$$\Xi_1^{h+\frac{1}{2},\sigma} - \Xi_2^{h+\frac{1}{2},\sigma} = \Xi_1^{h,\sigma} - \Xi_2^{h,\sigma}, \tag{B.1}$$

$$\Xi_3^{h+\frac{1}{2},\sigma} - \frac{3}{2}\Xi_1^{h+\frac{1}{2},\sigma} = \Xi_3^{h,\sigma} - \frac{3}{2}\Xi_1^{h,\sigma}, \tag{B.2}$$

$$\Upsilon_1^{h+\frac{1}{2},\sigma} - \Upsilon_2^{h+\frac{1}{2},\sigma} = \Upsilon_1^{h,\sigma} - \Upsilon_2^{h,\sigma}. \tag{B.3}$$

Our goal is to prove that (B.1)-(B.3) hold under the additional assumption that S_{lin} has a dual time-symmetry.

Preparation. Since S_{lin} reproduces Π_3, we have the sum rules

$$\sum_\ell a_{2\ell} = \sum_\ell a_{2\ell+1} = 1, \tag{B.4}$$

$$\sum_\ell a_{2\ell}\pi(\ell) = \sum_\ell a_{2\ell+1}\pi(\ell+\frac{1}{2}), \quad \forall \pi \in \Pi_3\,. \tag{B.5}$$

Combining (B.4) with the dual time-symmetry of S_{lin}, expressed as $a_{2\ell+1} = a_{-2\ell}$, we have

$$\sum_\ell a_{2\ell+1}\ell = \sum_\ell a_{-2\ell}\ell = -\sum_\ell a_{2\ell}\ell = -\sum_\ell a_{2\ell+1}\big(\ell+\frac{1}{2}\big) = -\frac{1}{2} - \sum_\ell a_{2\ell+1}\ell\,,$$

from which we obtain the identity

$$\sum_\ell a_{2\ell+1}\ell = -\frac{1}{4}. \tag{B.6}$$

We compute as follows using the sum rules:

$$\begin{aligned}\sum_\ell a_{2\ell+1}\ell^3 &= \sum_\ell a_{-2\ell}\ell^3 = -\sum_\ell a_{2\ell}\ell^3 = -\sum_\ell a_{2\ell+1}\big(\ell+\frac{1}{2}\big)^3\\ &= -\sum_\ell a_{2\ell+1}\ell^3 - \frac{3}{2}\sum_\ell a_{2\ell+1}\ell^2 - \frac{3}{4}\sum_\ell a_{2\ell+1}\ell - \frac{1}{8}\\ &= -\sum_\ell a_{2\ell+1}\ell^3 - \frac{3}{2}\sum_\ell a_{2\ell+1}\ell^2 + \frac{1}{16}.\end{aligned}$$

This gives the identity

$$\sum_\ell a_{2\ell+1}\ell^3 = -\frac{3}{4}\sum_\ell a_{2\ell+1}\ell^2 + \frac{1}{32}. \tag{B.7}$$

Finally use Equations (B.4)-(B.7) to obtain the next three identities:

$$\sum_\ell a_{2\ell+1}(h-\ell) = h + \frac{1}{4}, \tag{B.8}$$

$$\sum_\ell a_{2\ell+1}(h-\ell)^2 = h^2 - 2h\sum_\ell a_{2\ell+1}\ell + \sum_\ell a_{2\ell+1}\ell^2 = h^2 + \frac{1}{2}h + \sum_\ell a_{2\ell+1}\ell^2, \tag{B.9}$$

and

$$\begin{aligned}\sum_\ell a_{2\ell+1}(h-\ell)^3 &= h^3 - 3h^2\sum_\ell a_{2\ell+1}\ell + 3h\sum_\ell a_{2\ell+1}\ell^2 - \sum_\ell a_{2\ell+1}\ell^3 \\ &= h^3 + \frac{3}{4}h^2 + 3h\sum_\ell a_{2\ell+1}\ell^2 + \frac{3}{4}\sum_\ell a_{2\ell+1}\ell^2 - \frac{1}{32}, \\ &= h^3 + \frac{3}{4}h^2 - \frac{1}{32} + 3\left(h+\frac{1}{4}\right)\sum_\ell a_{2\ell+1}\ell^2. \qquad \text{(B.10)}\end{aligned}$$

We are now ready to prove (B.1)-(B.3).

Proof of (B.1). By definition,

$$\begin{aligned}\Xi_1^{h,0} &= h\Big[\sum_\ell a_{2\ell}(h-\ell)\sum_\ell a_{2\ell}(h-\ell)^2 - \sum_\ell a_{2\ell}(h-\ell)^3\Big] \\ \Xi_1^{h,1} &= h\Big[\sum_\ell a_{2\ell+1}(h-\ell)\sum_\ell a_{2\ell+1}(h-\ell)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^3\Big] \\ \Xi_2^{h,0} &= h^2\Big[\Big(\sum_\ell a_{2\ell}(h-\ell)\Big)^2 - \sum_\ell a_{2\ell}(h-\ell)^2\Big] \\ \Xi_2^{h,1} &= h^2\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^2\Big].\end{aligned}$$

Hence,

$$\begin{aligned}\Xi_1^{h+\frac{1}{2},0} &= \Big(h+\frac{1}{2}\Big)\Big[\sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)\sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)^2 - \sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)^3\Big] \\ &= \Big(h+\frac{1}{2}\Big)\Big[\sum_\ell a_{2\ell+1}(h-\ell)\sum_\ell a_{2\ell+1}(h-\ell)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^3\Big].\end{aligned}$$

Therefore,

$$\Xi_1^{h+\frac{1}{2},0} - \Xi_1^{h,1} = \frac{1}{2}\Big[\sum_\ell a_{2\ell+1}(h-\ell)\sum_\ell a_{2\ell+1}(h-\ell)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^3\Big]. \qquad \text{(B.11)}$$

Similarly, we have

$$\Xi_2^{h+\frac{1}{2},0} - \Xi_2^{h,1} = \Big(h+\frac{1}{4}\Big)\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^2\Big]. \qquad \text{(B.12)}$$

Substituting (B.8)-(B.10) into (B.11) and (B.12), yields the two identities

$$\Xi_1^{h+\frac{1}{2},0} - \Xi_1^{h,1} = \Big(h+\frac{1}{4}\Big)\Big(\frac{1}{16} - \sum_\ell a_{2\ell+1}\ell^2\Big), \qquad \text{(B.13)}$$

$$\Xi_2^{h+\frac{1}{2},0} - \Xi_2^{h,1} = \Big(h+\frac{1}{4}\Big)\Big(\frac{1}{16} - \sum_\ell a_{2\ell+1}\ell^2\Big).$$

Hence

$$\Xi_1^{h+\frac{1}{2},0} - \Xi_1^{h,1} = \Xi_2^{h+\frac{1}{2},0} - \Xi_2^{h,1},$$

and (B.1) is proved.

Proof of (B.2). By definition,

$$\Xi_3^{h,0} = h\Big[\Big(\sum_\ell a_{2\ell}(h-\ell)\Big)^3 - \sum_\ell a_{2\ell}(h-\ell)^3\Big]$$
$$\Xi_3^{h,1} = h\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^3 - \sum_\ell a_{2\ell+1}(h-\ell)^3\Big].$$

Hence,

$$\begin{aligned}\Xi_3^{h+\frac{1}{2},0} &= \Big(h+\frac{1}{2}\Big)\Big[\Big(\sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)\Big)^3 - \sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)^3\Big]\\ &= \Big(h+\frac{1}{2}\Big)\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^3 - \sum_\ell a_{2\ell+1}(h-\ell)^3\Big].\end{aligned}$$

Therefore,

$$\Xi_3^{h+\frac{1}{2},0} - \Xi_3^{h,1} = \frac{1}{2}\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^3 - \sum_\ell a_{2\ell+1}(h-\ell)^3\Big].$$

Substituting (B.8) and (B.10) into the above equality, we have

$$\Xi_3^{h+\frac{1}{2},0} - \Xi_3^{h,1} = \frac{3}{2}\Big(h+\frac{1}{4}\Big)\Big(\frac{1}{16} - \sum_\ell a_{2\ell+1}\ell^2\Big).$$

Combining with (B.13) yields

$$\Xi_3^{h+\frac{1}{2},0} - \Xi_3^{h,1} = \frac{3}{2}\Big(\Xi_1^{h+\frac{1}{2},0} - \Xi_1^{h,1}\Big),$$

which proves (B.2).

Proof of (B.3). By definition of Υ_i with $k=4$,

$$\Upsilon_1^{h,0} = \frac{1}{2}h\Big[\sum_\ell a_{2\ell}(h-\ell)\sum_\ell a_{2\ell}(h-\ell)(h-\ell-1) - \sum_\ell a_{2\ell}(h-\ell)^2(h-\ell-1)\Big]$$
$$\Upsilon_1^{h,1} = \frac{1}{2}h\Big[\sum_\ell a_{2\ell+1}(h-\ell)\sum_\ell a_{2\ell+1}(h-\ell)(h-\ell-1) - \sum_\ell a_{2\ell+1}(h-\ell)^2(h-\ell-1)\Big]$$

$$\Upsilon_2^{h,0} = \frac{1}{2}h(h-1)\Big[\Big(\sum_\ell a_{2\ell}(h-\ell)\Big)^2 - \sum_\ell a_{2\ell}(h-\ell)^2\Big]$$
$$\Upsilon_2^{h,1} = \frac{1}{2}h(h-1)\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^2\Big].$$

Hence,

$$\begin{aligned}\Upsilon_1^{h+\frac{1}{2},0} &= \frac{1}{2}\Big(h+\frac{1}{2}\Big)\Big[\sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)\sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)(h+\frac{1}{2}-\ell-1) - \sum_\ell a_{2\ell}(h+\frac{1}{2}-\ell)^2(h+\frac{1}{2}-\ell-1)\Big]\\ &= \frac{1}{2}\Big(h+\frac{1}{2}\Big)\Big[\sum_\ell a_{2\ell+1}(h-\ell)\sum_\ell a_{2\ell+1}(h-\ell)(h-\ell-1) - \sum_\ell a_{2\ell+1}(h-\ell)^2(h-\ell-1)\Big].\end{aligned}$$

Therefore,

$$\Upsilon_1^{h+\frac{1}{2},0} - \Upsilon_1^{h,1} = \frac{1}{4}\Big[\sum_\ell a_{2\ell+1}(h-\ell)\sum_\ell a_{2\ell+1}(h-\ell)(h-\ell-1) - \sum_\ell a_{2\ell+1}(h-\ell)^2(h-\ell-1)\Big]. \tag{B.14}$$

Similarly, we have

$$\Upsilon_2^{h+\frac{1}{2},0} - \Upsilon_2^{h,1} = \frac{1}{2}\Big(h-\frac{1}{4}\Big)\Big[\Big(\sum_\ell a_{2\ell+1}(h-\ell)\Big)^2 - \sum_\ell a_{2\ell+1}(h-\ell)^2\Big]. \tag{B.15}$$

Substituting (B.8)-(B.10) into (B.14) and (B.15), yields

$$\Upsilon_1^{h+\frac{1}{2},0} - \Upsilon_1^{h,1} = \frac{1}{2}\big(h - \frac{1}{4}\big)\big(\frac{1}{16} - \sum_\ell a_{2\ell+1}\ell^2\big),$$

$$\Upsilon_2^{h+\frac{1}{2},0} - \Upsilon_2^{h,1} = \frac{1}{2}\big(h - \frac{1}{4}\big)\big(\frac{1}{16} - \sum_\ell a_{2\ell+1}\ell^2\big).$$

Hence,

$$\Upsilon_1^{h+\frac{1}{2},0} - \Upsilon_1^{h,1} = \Upsilon_2^{h+\frac{1}{2},0} - \Upsilon_2^{h,1},$$

and (B.3) is proved. □

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